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# INEQUALITIES FOR A SUM OF EXPONENTIAL FUNCTIONS

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ABSTRACT. We generalize the result $\min_{x>0} \frac{e^{\tau x}}{x} = \tau e$, $(\tau > 0)$, to a function in which the numerator is the sum $\sum_{i=1}^{n} p_i e^{\tau_i x}$. Upper and lower estimates are close to the exact result when $\frac{\min_{1\le i\le n} \tau_i}{\max_{1\le i\le n} \tau_i}$ is not far from unity. Computational results are given to verify the main results.

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## 1. INTRODUCTION

If we let $x = e^y$ in the inequalities

$$x^a - ax + a - 1 \ge 0, \qquad a \ge 1,$$
$$x^a - ax + a - 1 \le 0, \quad 0 < a \le 1,$$

which hold for $x > 0$ [1], we have the following inequalities for the exponential function which hold for all $y$

$$e^{ay} - ae^y + a - 1 \ge 0, \qquad a \ge 1, \tag{1.1}$$

$$e^{ay} - ae^y + a - 1 \le 0, \quad 0 < a \le 1. \tag{1.2}$$

The above results were used in [2] to find some sufficient conditions for the oscillation of a delay differential equation. Inequalities for exponential functions play an important role in the theory of delay equations since the characteristic equation associated with a delay differential equation contains, in general, a sum of exponential functions. For $\tau > 0$ the result

$$\min_{x>0} \frac{e^{\tau x}}{x} = \tau e \tag{1.3}$$

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is frequently used [3]. Li has employed the inequality

$$e^{rx} \geq x + \frac{\ln(er)}{r} \quad \text{for} \quad r > 0,$$

to find a sufficient condition for the oscillation of a non-autonomous delay equation. An equivalent result, but more suitable for our purpose, is the following inequality which holds for $a > 0$,

$$e^x \geq ax + a(1 - \ln a). \tag{1.4}$$

In this paper we wish to generalize (1.3). Consider

$$s = \min_{x>0} \frac{\sum_{i=1}^n p_i e^{\tau_i x}}{x}, \tag{1.5}$$

where $p_i, \tau_i \geq 0$, for $i = 1, \dots, n$. The result

$$\min_{x>0}\{f(x) + g(x)\} \geq \min_{x>0} f(x) + \min_{x>0} g(x)$$

and a repeated use of (1.3) gives a lower estimate for $s$. The case when all but one of the $\tau_i$ vanishes is treated first and this is used to find an upper estimate for $s$.

## 2. MAIN RESULTS

Our main results are contained in the following theorems.

**Theorem 2.1.** *Let* $p > 0, \tau > 0, q \geq 0$ *then*

$$p\tau \exp\left(1 + \frac{q}{pe} e^{-\sqrt{\frac{q}{pe^2}}}\right) \leq \min_{x>0} \frac{pe^{\tau x} + q}{x} \leq p\tau \exp\left(1 + \frac{q}{pe}\right).$$

**Theorem 2.2.** *Let* $p_i > 0,\ \tau_i \geq 0, i = 1, \dots, n;\ 0 < \tau = \max_{1 \leq i \leq n} \tau_i$, *then*

$$e \sum_{i=1}^n p_i \tau_i \leq s \leq \left(\sum_{i=1}^n p_i \tau_i\right) \exp\left(1 + \frac{\sum_{i=1}^n p_i(\tau - \tau_i)}{e \sum_{i=1}^n p_i \tau_i}\right).$$

We shall prove a lemma before taking up the proofs of the theorems.

**Lemma 2.3.** *Let* $a > 0$ *and* $u_0$ *be the unique root of the equation*

$$u = ae^{-u},$$

*then*

$$a \exp\left(-\sqrt{\frac{a}{e}}\right) \leq u_0 \leq \sqrt{\frac{a}{e}}.$$

*Proof.* It is obvious that $u_0$ is positive. Since we can re-write the equation as

$$a = ue^u,$$

we have, on using (1.3)

$$a \geq eu_0^2,$$

or

$$u_0 \leq \sqrt{\frac{a}{e}}. \tag{2.1}$$

Now, since $u_0 = ae^{-u_0}$, we make use of (2.1) on the right hand side to obtain

$$u_0 \geq a \exp\left(-\sqrt{\frac{a}{e}}\right). \tag{2.2}$$

Combining (2.1) and (2.1), we get the inequality of the lemma. □

*Proof of Theorem 2.1.* Define $y = \tau x$ then, for $x > 0$,

$$\frac{pe^{\tau x}+q}{x} = \frac{p\tau e^{y}+q\tau}{y}$$
$$\geq \frac{p\tau\{ay+a(1-\ln a)\}+q\tau}{y}, \tag{2.3}$$

where we have used (1.4).We choose $a$ such that $a(1-\ln a) = \frac{-q}{p}$. Note that this equation possesses a root $a_0 \geq e$. Set $a = e^{1+b}$, then $b$ will satisfy

$$u = \frac{q}{pe}e^{-u}$$

and (2.3) reduces to

$$\frac{pe^{\tau x}+q}{x} \geq p\tau e^{1+b}. \tag{2.4}$$

Now use the lemma to obtain the left side of the inequality of Theorem 2.1. In order to prove the other side, let $f(x) = pe^{\tau x}+q$. The tangent to the curve $y = f(x)$, with slope $m$ will have the equation

$$y - \left(\frac{m}{\tau}+q\right) = m\left(x - \frac{1}{\tau}\ln\left(\frac{m}{p\tau}\right)\right).$$

This line will pass through the origin if the slope satisfies the equation

$$m - p\tau e^{1+\frac{q\tau}{m}} = 0. \tag{2.5}$$

Let $g(m)$ denote the left side of (2.5). The case of $q = 0$ is covered by (1.3), therefore we consider $q > 0$. Since $g(0+) = -\infty$, $g(\infty) = \infty$, and $g(m)$ is an increasing function on $(0,\infty)$, it follows that (2.5) has a unique positive root say, $m_0$. Hence for $x > 0$ we have

$$pe^{\tau x}+q \geq m_0 x,$$

or

$$\min_{x>0}\frac{pe^{\tau x}+q}{x} = m_0. \tag{2.6}$$

It is obvious that

$$m_0 \geq \min_{x>0}\frac{pe^{\tau x}}{x} = p\tau e.$$

Using this in (2.5), we get

$$m_0 = p\tau e^{1+\frac{q\tau}{m_0}} \leq p\tau e^{1+\frac{q}{pe}}$$

Combining the above result with (2.6), we get the right side of the inequality of Theorem 2.1. □

*Proof of Theorem 2.2.* The left side of the inequality is obtained by using (1.3) separately for each exponential function and applying the result

$$s \geq \sum_{i=1}^{n}\min_{x>0}\frac{p_i e^{\tau_i x}}{x},$$

where $s$ is defined by (1.5). In order to prove the right hand side of the inequality, define $y = \tau x$. Then

$$\min_{x>0}\frac{\sum_{i=1}^{n} p_i e^{\tau_i x}}{x} = \min_{y>0}\frac{\tau\sum_{i=1}^{n} p_i e^{\frac{\tau_i}{\tau}y}}{y}. \tag{2.7}$$

Since

$$\frac{\tau_i}{\tau} \leq 1, \text{ for } i = 1,\dots,n,$$

we have, on using (1.2)

$$e^{\frac{\tau_i}{\tau}y} \leq \frac{\tau_i}{\tau}e^y + 1 - \frac{\tau_i}{\tau}, \text{ for } i = 1, \ldots, n.$$

If we make use of the above inequality in (2.7), we get

$$\min_{x>0} \frac{\sum_{i=1}^n p_i e^{\tau_i x}}{x} \leq \min_{y>0} \frac{\left(\sum_{i=1}^n p_i \tau_i\right) e^y + \sum_{i=1}^n p_i(\tau - \tau_i)}{y}. \tag{2.8}$$

Now an application of Theorem 2.1 gives the right hand side of the inequality of the theorem. □

## 3. COMPUTATIONAL RESULTS

In this section we present some numerical results to verify the accuracy of various approximate results. If we let $p = 3, q = 2$ and $\tau = 1$, then the exact value of $\min_{x>0} \frac{pe^{\tau x}+q}{x}$ is 9.96696 which occurs at $x = 1.2007$. For these values of the parameters, the number on the left of the inequality of Theorem 2.1 is 9.7785 while the number on the right hand side is 10.4214. It is obvious that the lower as well as the upper estimate will come closer to the exact value if $q$ and/or $\tau$ are decreased.

To verify the inequality given by Theorem 2.2, we let $p, q$ and $\tau$ retain their values of the last example and let $\sigma$ take successive values of 0.3, 0.9 and 0.98. The results are given in the following table.

**Table**

| $\sigma$ | Left side | Exact value | Right side |
|---|---|---|---|
| 0.3 | 9.7858 | 10.6885 | 11.2909 |
| 0.9 | 13.0478 | 13.0646 | 13.2493 |
| 0.98 | 13.4827 | 13.4833 | 13.5227 |

The inequality of Theorem 2.2.

In the table, the left side and right side respectively refer to the left and the right hand sides of the inequality of Theorem 2.2, while the exact value is the value of $s$ defined by (1.5). It is clear that as the difference between $\tau$ and $\sigma$ decreases the gap between the approximate and the exact values steadily decreases.

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