Article ID: 1000-341X(2006)04-0708-07 **Document code**: A

Existence of Global Attractor for LS Type Equations

ZHANG Rui-feng
(College of Mathematics and Information Science, Henan University, Kaifeng 475001, China)
(E-mail: zrf615@henu.edu.cn)

Abstract: The global attractor problem of the long-short wave equations with periodic boundary condition was studied. The existence of global attractor of this problem was proved by means of a uniform a priori estimate for time.

Key words: long-short wave equations; a priori estimate; global attractor.
MSC(2000): 35B40, 35Q20
CLC number: O175.2

1. Introduction

The LS type equations

$$iS_t + S_{xx} - LS = 0,$$

$$L_t + |S|^2_x = 0$$

were first derived by Djordjevic and Redekopp in [1], where S is the envelope of the short wave, and L is the amplitude of long wave and is real. This system describes the resonance interaction between the long wave and the short wave. Furthermore, Benney[2], Kawahara and Sugimoto[3] derived the following system

$$iS_t + S_{xx} - \alpha LS = \gamma|S|^2S,$$

$$L_t + lLL_x = m|S|^2_x,$$

where α, γ, m and l are real constants. This system describes the general theory of water wave interaction in a nonlinear medium[2−3]. Guo obtained the existence of global solution for long-short wave equations in [4]. Zhang and Guo have studied the longtime behavior of the solution for generalized long-short wave equations[5]. Guo and Wang have studied the approximation inertial manifolds for LS type equations[6].

In this paper, we consider the existence of a global attractor for the class of the generalized LS type equations:

$$iu_t + u_{xx} - nu + i\alpha u + g(|u|^2)u + h_1(x) = i\beta u_{xx}, \tag{1}$$

$$n_t + |u|^2_x + lnn_x + \delta n + f(|u|^2) + h_2(x) = \gamma n_{xx}, \tag{2}$$

Received date: 2004-05-09
Foundation item: the Natural Science Foundation of Henan Educational Committee (2003110005)

with the initial conditions

$$u|_{t=0} = u_0(x), \quad n|_{t=0} = n_0(x), \quad x \in \Omega = (-D, D), D > 0 \tag{3}$$

and the periodic boundary conditions

$$u(x+2D,t) = u(x,t), \quad n(x+2D,t) = n(x,t), \tag{4}$$

where $\alpha, \beta, \gamma, \delta$ and l are positive constants. $u(x,t)$ is an unknown complex valued vector, $n(x,t)$ is an unknown real valued function, and $h_1(x)$and $h_2(x)$ are given in $L^2(\Omega)$, $g(s)$ and $f(s)(0 \le s < \infty)$ are smooth real valued functions which satisfy

$$|g(s)| \le c_1 s^{2-\sigma} + c_2, \quad s \ge 0, \tag{5}$$

$$|f(s)| \le c_3 s^{\frac{3}{2}-\nu} + c_4, \quad s \ge 0, \tag{6}$$

where σ, ν and $c_i (i = 1,2,3,4)$ are positive constants. By means of a uniform a priori estimates for time, and Temam's[7] methods, we obtain the above results.

Throughout this paper, c will be used to indicate generic constants and dependent of data $(\alpha, \beta, \gamma, \delta, l, f, g, h_1, h_2, R)$. We denote by $\|\cdot\|$ the norm of $H = L^2_{\text{per}}(\Omega)$ (real or complex space) with the corresponding inner product $(\cdot,\cdot)$, by $\|\cdot\|_p$ the norm of $L^p_{\text{per}}(\Omega)$ for all $1 \le p \le \infty (\|\cdot\|_2 = \|\cdot\|)$, and by $\|\cdot\|_X$ the norm of any Banach space X.

2. The global smooth solution

In this section, we derive uniform a priori estimates for time which enable us to show the existence of the global smooth solution.

Lemma 1 *If $u_0, h_1 \in L^2(\Omega)$, then for the solution (u,n) of problem (1)-(4) we have*

$$\|u\|^2 \le c_0, \quad \forall t \ge t_0, \tag{7}$$

where c_0 and t_0 depend on the data (α, h_1, R) and R when $\|u_0\| \le R$.

Proof Taking the inner product of (1) with u in H, we obtain

$$(iu_t + u_{xx} - nu + i\alpha u + g(|u|^2)u + h_1(x) - i\beta u_{xx}, u) = 0. \tag{8}$$

Taking the imaginary part of (8), we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \alpha\|u\|^2 + \beta\|u_x\|^2 + \mathrm{Im}(h_1, u) = 0.$$

By Young inequality and Gronwall lemma we have Lemma 1. Furthermore,

$$\overline{\lim}_{t\to\infty}\|u(\cdot,t)\|^2 \le \frac{1}{\alpha}\|h_1\|^2 \equiv E_0. \tag{9}$$

Lemma 2 *Suppose that (5) and (6) hold. Then we have*

$$\|u_x(t)\|^2 + \|n(t)\|^2 \le c_1, \quad \forall t \ge t_1, \tag{10}$$

where c_1 and t_1 depend on the data $(\alpha,\beta,\gamma,\delta,f,g,h_1,h_2)$ and R when $\|u_0\|_{H^1_{per}}\leq R$ and $\|n_0\|\leq R$.

Proof Taking the inner product of (2) with n in H, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|n\|^2+2\delta\|n\|^2+2\gamma\|n_x\|^2=2\int_\Omega n_x|u|^2\mathrm{d}x-2\int_\Omega f(|u|^2)n\mathrm{d}x-2\int_\Omega h_2n\mathrm{d}x. \tag{11}$$

Taking the imaginary part of the inner product of (1) with u_{xx} in H, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|^2+2\alpha\|u_x\|^2+2\beta\|u_{xx}\|^2=2\mathrm{Im}(gu,u_{xx})-2\mathrm{Im}(nu,u_{xx})+2\mathrm{Im}(h_1,u_{xx}). \tag{12}$$

In the following, $\forall\rho>0$, when t is large enough, we have

$$|\int_\Omega n_x|u|^2\mathrm{d}x|\leq\|n_x\|\cdot\|u\|^2_{L^4}\leq c\|n_x\|\|u_x\|^{\frac12}\|u\|^{\frac32}\leq\frac{\rho}{2}\|n_x\|^2+\frac{\rho}{2}\|u_{xx}\|^2+c(\rho),$$

$$|\mathrm{Im}(g(|u|^2)u,u_{xx})|\leq c\|u_{xx}\|^{\frac34}\cdot\|u\|^{\frac54}\|g'\|_{L^2}\leq c\|u_{xx}\|^{\frac{5-2\sigma}{4}}\|u\|^{\frac{15-2\sigma}{4}}\leq\frac{\rho}{2}\|u_{xx}\|^2+c(\rho),$$

$$|\mathrm{Im}(nu,u_{xx})|\leq c\|n_x\|\|u_{xx}\|^{\frac34}\|u\|^{\frac54}\leq\frac{\rho}{2}\|n_x\|^2+\frac{\rho}{2}\|u_{xx}\|^2+c(\rho).$$

Combining (11) and (12), we get that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|n\|^2+\|u_x\|^2)+2\alpha\|u_x\|^2+2\beta\|u_{xx}\|^2+2\delta\|n\|^2+2\gamma\|n_x\|^2\\&\quad\leq 2\rho\|n\|^2+\rho\|u_x\|^2+2\rho\|n_x\|^2+4\rho\|u_{xx}\|^2+\frac{1}{\rho}(\|h_1\|^2+\|h_2\|^2)+c(\rho).\end{aligned}$$

Let $E=\|n\|^2+\|u_x\|^2$. Assume that $\alpha\leq\delta$, then we see that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}E+\alpha E+2\beta\|u_{xx}\|^2+(2\delta-\alpha)\|n\|^2+2\gamma\|n_x\|^2+\alpha\|u_x\|^2\\&\quad\leq 2\rho\|n\|^2+\rho\|u_x\|^2+2\rho\|n_x\|^2+4\rho\|u_{xx}\|^2+\frac{1}{\rho}(\|h_1\|^2+\|h_2\|^2)+c(\rho).\end{aligned}$$

Let $\rho\leq\min\{\alpha,\gamma,\frac{\beta}{2},\delta-\frac{\alpha}{2}\}$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}E+\alpha E\leq\frac{1}{\rho}(\|h_1\|^2+\|h_2\|^2)+c(\rho),\quad\forall t\geq t_0. \tag{13}$$

By Gronwall lemma we see that

$$E(t)\leq E(t_0)e^{-\alpha(t-t_0)}+\frac{1}{\alpha}(1-e^{-\alpha(t-t_0)})(\frac{1}{\rho}\|h_1\|^2+\frac{1}{\rho}\|h_2\|^2+c),\quad\forall t\geq t_0.$$

Note

$$|E(t_0)|=\|n(t_0)\|^2+\|u_x(t_0)\|^2\leq 2R^2,$$

where R satisfies $\|u_0\|_{H^1}\leq R,\|n_0\|\leq R$. Then, we infer that

$$E(t)\leq\frac{2}{\alpha}(\frac{1}{\rho}\|h_1\|^2+\frac{1}{\rho}\|h_2\|^2+c),\quad\forall t\geq t_1, \tag{14}$$

where $t_1=\max\{t_0,t_0+\frac{1}{\alpha}\ln\frac{2\alpha R^2-\frac{1}{\rho}\|h_1\|^2-\frac{1}{\rho}\|h_2\|^2+c}{\frac{1}{\rho}\|h_1\|^2+\frac{1}{\rho}\|h_2\|^2+c}\}$.
By (14) it follows that

$$\|u_x(t)\|^2+\|n(t)\|^2\le c_1,\quad \forall t\ge t_1,$$

which concludes Lemma 2. If $\alpha\ge\delta$, we can also deduce the same result. Furthermore,

$$\overline{\lim}_{t\to\infty}(\|u_x(\cdot,t)\|^2+\|n(\cdot,t)\|^2)\le\frac{2}{\alpha}(\frac{1}{\rho}\|h_1\|^2+\frac{1}{\rho}\|h_2\|^2+\max_{t\ge t_1}c)\equiv E_1. \tag{15}$$

By Agmon inequality, Lemmas 1 and 2 imply that

$$\|u(t)\|_{H^1_{per}}+\|u(t)\|_{L^\infty}\le c,\quad \forall t\ge t_1. \tag{16}$$

Lemma 3 *Suppose that Lemma 2 holds and that $h_2,h_1\in H^1_{per}(\Omega)$. Then we have*

$$\|u_{xx}(t)\|+\|n_x(t)\|\le c_2,\quad \forall t\ge t_2, \tag{17}$$

where c_2 and t_2 depend on the data $(\alpha,\beta,\gamma,\delta,l,f,g,\ h_1,h_2)$ and R when $\|u_0\|_{H^2_{\rm per}}\le R$ and $\|n_0\|_{H^1_{\rm per}}\le R$.

Proof Taking the imaginary part of the inner product of (1) with u_{xxxx} in H, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_{xx}\|^2+2\alpha\|u_{xx}\|^2+2\beta\|u_{xxx}\|^2=-2\mathrm{Im}(gu,u_{x^4})+2\mathrm{Im}(nu,u_{x^4})-2\mathrm{Im}(h_1,u_{x^4}).$$

Taking the inner product of (2) with n_{xx} in H, we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n_x\|^2+\delta\|n_x\|^2+\gamma\|n_{xx}\|^2=\int_\Omega|u|^2_x n_{xx}\mathrm{d}x+\int_\Omega f(|u|^2)n_{xx}\mathrm{d}x+l\|n_x\|^3_{L^3}+\int_\Omega h_2n_{xx}\mathrm{d}x.$$

Similar to the method of Lemma 2, we calculate and derive that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|n_x\|^2+\|u_{xx}\|^2)+2\alpha\|u_{xx}\|^2+2\beta\|u_{xxx}\|^2+2\delta\|n_x\|^2+2\gamma\|n_{xx}\|^2\\&\quad\le\rho\|n_x\|^2+3\rho\|u_{xx}\|^2+4\rho\|n_{xx}\|^2+6\rho\|u_{xxx}\|^2+\frac{1}{\rho}(\|h_{1x}\|^2+\|h_{2x}\|^2)+c(\rho).\end{aligned}$$

And we have

$$\|u_{xx}(t)\|^2+\|n_x(t)\|^2\le c_2,\quad \forall t\ge t_2.$$

Furthermore,

$$\overline{\lim}_{t\to\infty}(\|u_{xx}(\cdot,t)\|^2+\|n_x(\cdot,t)\|^2)\le\frac{2}{\alpha}(\frac{1}{\rho}\|h_{1x}\|^2+\frac{1}{\rho}\|h_{2x}\|^2+\max_{t\ge t_2}c)\equiv E_2. \tag{18}$$

The proof of Lemma 3 is completed.

Lemma 4 *Suppose that Lemma 3 holds, and that $h_1\in H^2_{per}(\Omega),h_2\in H^1_{per}(\Omega)$. Then we have*

$$\|u_{xxx}(t)\|^2+\|n_{xx}(t)\|^2\le\frac{c}{t},\quad \forall t>0, \tag{19}$$

where the constant c depends on the data $(\alpha,\beta,\gamma,\delta,l,f,g)$, $\|h_1\|_{H^2}$, $\|h_2\|_{H^1}$, $\|u_0\|_{H^2_{per}}$, $\|n_0\|_{H^1_{per}}$ and T.

Proof Taking the imaginary part of the inner product of (1) with $t^2\frac{\partial^6 u}{\partial x^6}$ in H, we see that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|tu_{xxx}\|^2+\alpha\|tu_{xxx}\|^2+\beta\|tu_{xxxx}\|^2\\&\quad=t\|u_{xxx}\|^2+\mathrm{Im}(gu,t^2\frac{\partial^6 u}{\partial x^6})-\mathrm{Im}(nu,t^2\frac{\partial^6 u}{\partial x^6})+\mathrm{Im}(h_1,t^2\frac{\partial^6 u}{\partial x^6}).\end{aligned}\tag{20}$$

Taking the inner product of (2) with t^2n_{xxxx} in H, we find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|tn_{xx}\|^2+\delta\|tn_{xx}\|^2+\gamma\|tn_{xxx}\|^2\\&\quad=t\|n_{xx}\|^2-(|u|_x^2,t^2n_{xxxx})-l(nn_x,t^2n_{x^4})-(f(|u|^2),t^2n_{xxxx})-(h_2,t^2n_{xxxx})\end{aligned}\tag{21}$$

Similar to the method of Lemma 2, we calculate every term in (20) and (21) and note that

$$t\|u_{xxx}\|^2\le\frac{\beta}{8}\|tu_{xxxx}\|^2+c,\quad t\|n_{xx}\|^2\le\frac{\gamma}{8}\|tn_{xxx}\|^2+c,$$

where the constant c depends on data $(\alpha,\beta,\gamma,\delta,l,f,g),\|u_o\|_{H^2},\|n_0\|_{H^1}$ and T. We find that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|tn_{xx}\|^2+\|tu_{xxx}\|^2)+2\alpha\|tu_{xxx}\|^2+\beta\|tu_{xxxx}\|^2+2\delta\|tn_{xx}\|^2+\gamma\|tn_{xxx}\|^2\\&\quad\le c(\|tn_{xx}\|^2+\|tu_{xxx}\|^2+\|h_{1xx}\|^2+\|h_{2x}\|^2).\end{aligned}$$

And then

$$\|u_{xxx}(t)\|^2+\|n_{xx}(t)\|^2\le\frac{c}{t},\quad\forall t>0.$$

By an induction argument, Galerkin method and the techniques used in [4], we can easily obtain the main results:

Lemma 5 *Suppose that (5) and (6) hold,* $h_1\in H^2_{\mathrm{per}}(\Omega)$, $h_2\in H^1_{\mathrm{per}}(\Omega)$ *and* $u_0(x)\in H^3_{\mathrm{per}}(\Omega),n_0(x)\in H^2_{\mathrm{per}}(\Omega)$. *Then we have*

$$\sup_{0\le t\le T}\|u_N(\cdot,t)\|^2_{H^3(\Omega)}\le c,\tag{22}$$

$$\sup_{0\le t\le T}\|n_N(\cdot,t)\|^2_{H^2(\Omega)}\le c,\tag{23}$$

where $u_N(x,t),n_N(x,t)$ *is the approximate solution of the problems (1)–(4), and* c *depends on the data* $(\alpha,\beta,\gamma,\delta,l,f,g),\|h_1\|_{H^2},\|h_2\|_{H^1},\|u_0\|_{H^3}$ *and* $\|n_0\|_{H^2}$, *and is independent of* N.

Theorem 1 *If the conditions in Lemma 5 are satisfied, then there exists a unique global solution* $u(x,t),n(x,t)$ *for the initial value problems (1)–(4):*

$$u(x,t)\in L^\infty(0,T;H^3(\Omega)),\quad n(x,t)\in L^\infty(0,T;H^2(\Omega)).\tag{24}$$

3. The global attractor

In order to prove the existence of global attractor of problems (1)–(4), we need the following result.

Lemma 6[7] *Let E be a Banach Space. Let $\{S_t, t \geq 0\}$ be a set of semi-group operators, i.e. $S_t : E \to E$ satisfies*

$$S_t S_\tau = S_{t+\tau}, S_0 = I,$$

where I is the identity operator. We also assume the following.

(i) Operator S_t is bounded, i.e. for each $R > 0$, there exists a constant $c(R)$ such that $\|u\|_E \leq R$ implies

$$\|S_t u\|_E \leq c(R), \quad t \in [0, \infty);$$

(ii) There is a bounded absorbing set $B_0 \subset E$, i.e. for any bounded set $B \subset E$, there exists a constant T such that

$$S_t B \subset B_0, \quad t \geq T;$$

(iii) S_t is a completely continuous operator for $T > 0$.

Then the operator semi-group S_t has a compact global attractor.

Theorem 2 *Suppose that problems (1)–(4) have a global smooth solution and that the conditions in Lemma 4 are satisfied, then there exists a global attractor A of the periodic initial value problem (1)–(4), i.e. there is a set A, such that*

(i) $S_t A = A$, $t \in R^+$;

(ii) $\lim_{t\to\infty} \mathrm{dist}(S_t B, A) = 0$, for any bounded set $B \subset E$,

where

$$\mathrm{dist}(X, Y) = \sup_{x\in X} \inf_{y\in Y} \|x - y\|_E,$$

and S_t is a semi-group operator generated by problems (1)–(4).

Proof On account of the result of Lemma 6, we shall prove this theorem by checking the conditions in Lemma 6. Under the assumptions of the theorem, we know that there exists an operator semi-group generated by problems (1)–(4). We set the Banach Space

$$E = H^2_{\mathrm{per}}(\Omega) \times H^1_{\mathrm{per}}(\Omega), \;\; (u, n) \in E, \;\; \|(u, n)\|^2_E = \|u\|^2_{H^2} + \|n\|^2_{H^1}$$

and $S_t : E \to E$. Using the results of Lemmas 1–3, and assuming that $B \subset E$ belongs to the ball $\{\|(u, n)\|_E \leq R\}$, we have

$$\begin{aligned}\|S_t(u_0, n_0)\|^2_E &= \|(u, n)\|^2_E = \|u\|^2_{H^2} + \|n\|^2_{H^1}\\ &\leq C(R^2, \|h_1\|^2_{H^1} + \|h_2\|^2_{H^1}), \quad t \geq 0, (u_0, n_0) \in B.\end{aligned}$$

This means that $\{S_t\}$ is uniformly bounded in E. Furthermore, from the results of the Lemmas 1–3 we see that

$$\|S_t(u_0, n_0)\|^2_E = \|u\|^2_{H^2} + \|n\|^2_{H^1} \leq 2(E_0 + E_1 + E_2), \quad t \geq t_0(R, \|h_1\|^2_{H^1} + \|h_2\|^2_{H^1}).$$

Hence,

$$\bar{A} = \{(u(\cdot, t), n(\cdot, t)) \in E, \|(u, n)\|^2_E \leq 2(E_0 + E_1 + E_2)\}$$

is a bounded absorbing set of the operator semi-group S_t. From Lemma 4 we know that

$$\|u_{xxx}\|^2 + \|n_{xx}\|^2 \le \frac{c}{t} \quad \forall t > 0, \|(u_0, n_0)\|_E \le R.$$

By the compact imbedding $H^3 \times H^2 \hookrightarrow H^2 \times H^1$, the operator semi-group $S_t : E \to E$ for $t > 0$ is completely continuous. The proof of the theorem is now completed.

Remark Just as the remarks in [7], the attractor A obtained in Theorem 2 is the ω-limit set of the absorbing set $\bar{A}$, i.e.

$$A = \omega(\bar{A}) = \bigcap_{\tau \ge 0} \overline{\cup_{t \ge \tau} S_t \bar{A}}.$$

Acknowledgments The author would like to thank Academician GUO Bo-ling for guidance regarding this paper.

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一类长短波方程组的整体吸引子的存在性

张 瑞 凤

(河南大学数学与信息科学学院，河南 开封 475001)

摘要: 本文应用对时间的一致先验估计，证明了一类具有周期边值条件的长短波方程组的整体吸引子的存在性.

关键词: 长短波方程；先验估计；整体吸引子.