# Comments on "A New Coastal Wave Model. Part V: Five-Wave Interactions"

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Recent calculations by Lin and Perrie (1997) on the surface wave spectral energy fluxes due to the wave nonlinearity in deep and shallow water appeared after previously published works by Krasitski (1993), Shrira et al. (1996), and Kalmykov (1993, 1995, 1997), while presenting results that are qualitatively different from those obtained previously. This comment is on these obvious differences and why it appears that the conclusions of Lin and Perrie cannot be justified.

Surface waves in deep water as well as in shallow water are very well described by the four-wave kinetic equation as shown by Hasselmann and Hasselmann (1985) and Herterich and Hasselmann (1980). Analogous computation of the five-wave kinetic equation for deep and shallow water show that a five-wave contribution is very small: only 3%–5% of the four-wave one (Kalmykov 1998). Therefore we can conclude that the four-wave kinetic equation still remains dominant for the shallow and deep water wave modeling and that all this discussion is of only academical interest.

During the past 15 years many studies have been made in this area, some of which are not cited by Lin and Perrie (1997), including Parts I and II of the series leading to the present article under discussion (Part III). They present results of their own calculations of the spectral transfer rates in a JONSWAP spectrum, which are qualitatively different from various previously published results, while offering no explanation for the differences found.

The subject of five wave–wave interactions among surface gravity waves is not new. First discussions concerning the energy transfer by five-wave interactions in wave spectra took place at Zakharov's seminar in 1993 in Moscow, where the present author made a report (Kalmykov 1993). In experiments, it was first noticed by Su (1982) in the form of a horseshoe pattern existing in two-dimensional waves. Then McLean (1982a,b) found theoretically that instability II was a possible reason for these three-dimensional patterns. At the same time, this topic was developed by Meiron et al. (1982). They found good agreement between the Su (1982) experiment and the theory and were able to give an exact answer: it was a three-dimensional instability II making these 3D wave patterns. Later, this theme was continued by Stiassnie and Shemer (1984, 1987). They derived the five-wave Zakharov's equation and found that the most unstable 3D wave segments were just those observed by Su (1982) and by other authors. This instability II (3D) was triggered by instability I (2D), or the Benjamen–Feyer instability. Su (1982) observed the wave segments with the scales $L_2 = 2\lambda_0$, where $\lambda_0$ is the wavelength from the spectral peak. McLean (1982a,b) found the coordinate of the most unstable lateral perturbation $\mathbf{k}_1 = (\mathbf{p}, \mathbf{q})$, where $p = 0.5$ and $q = 1.2$; that gives $|k_1| = 1.3$ ($\omega_1 = 1.7$) and angle to the $x$ axis of $\theta_1 = 67°$. Recently, the five-wave Zakharov's equation in Hamiltonian form was derived by Krasitski (1993, 1994). In recent works (Kalmykov 1993, 1995, 1997), numerical estimates of the nonlinear transfer of wave energy by five-wave interactions in the wave spectra were first performed. This transfer was directed from the spectral peak ($\omega_0 = 1$, $\theta_0 = 0$) to the waves of frequency $\omega_1 = 1.7$–$2.0$ and angles $\theta_1 = \pm(60°$–$70°)$ from the main direction for different depths. As can be seen in each of the works cited above except for Lin and Perrie (1997), we have energy flux from low to high frequency, but not an inverse cascade. In some sense this is evident because five-wave interactions do not conserve action, only energy. The most recent paper devoted to this theme is that of Shrira et al. (1996), where energy by five-wave interactions is also transferred from low wavenumbers to the higher ones.

Now let us see the equation in more detail. For the estimation of the nonlinear transfer of energy by five-wave resonant interactions, we will use the kinetic equation derived by Krasitski (1993):

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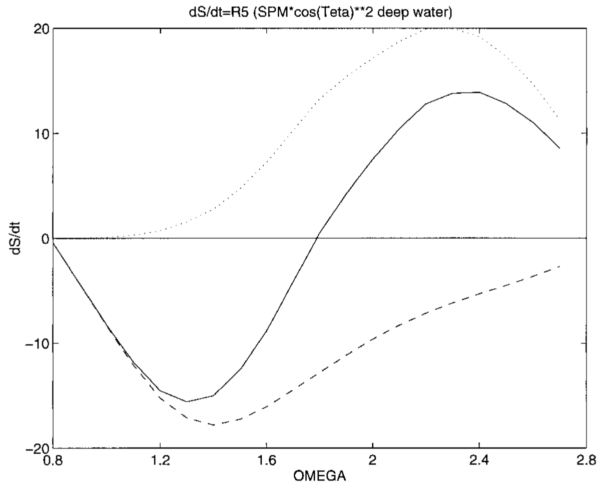


FIG. 1. Nonlinear transfer of energy (two-dimensional) (3) for Pierson-Moscovitz spectrum ($\gamma = 1$, $m = 2$); units are nondinemsional ($g = k_0 = \omega_0 = 1$): $I_{51}$ dotted line, $I_{52}$ dashed line, and $I_{51} + I_{52}$ solid line.

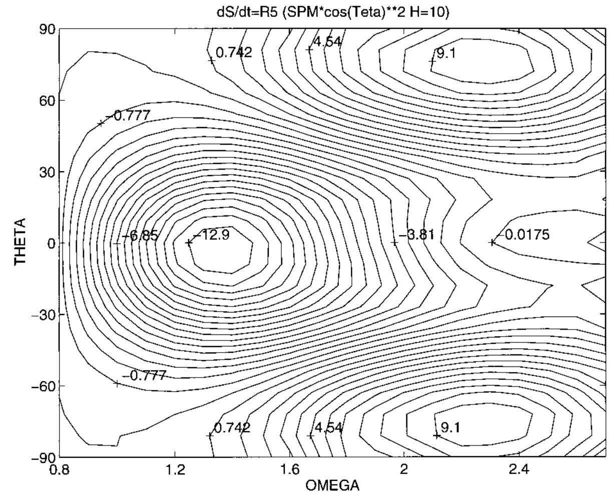


FIG. 2. Nonlinear transfer of energy (three-dimensional) (3).

$$
\begin{aligned}
\frac{\partial n_1}{\partial t} = 12\pi \int_{-\infty}^{\infty} & W_{1,2,3,4,5}^2 n_1 n_2 n_3 n_4 n_5 \left( \frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} - \frac{1}{n_5} \right) \\
& \times \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4 - \omega_5) \\
& \times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4 - \mathbf{k}_5)\, \mathbf{dk}_2\mathbf{k}_3\mathbf{k}_4\mathbf{k}_5 \\
+ 18\pi \int_{-\infty}^{\infty} & W_{5,4,3,2,1}^2 n_1 n_2 n_3 n_4 n_5 \left( \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} - \frac{1}{n_4} - \frac{1}{n_5} \right) \\
& \times \delta(\omega_1 + \omega_2 + \omega_3 - \omega_4 - \omega_5) \\
& \times \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4 - \mathbf{k}_5)\, \mathbf{dk}_2\mathbf{k}_3\mathbf{k}_4\mathbf{k}_5,
\end{aligned}
\tag{1}
$$

in which $\mathbf{k}$ is the wavenumber vector, and $k = |\mathbf{k}|$; $\omega = [gk \tanh(kh)]^{1/2}$ is the dispersion relation; $g = 9.81$ m s$^{-2}$ is the gravitational acceleration; $h$ is the depth; $W_{1,2,3,4,5} = W(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4, \mathbf{k}_5)$ is the kernel function for arbitrary depth; $\delta(\cdots)$ is the Dirac delta function;

$$
n_1 = n(\mathbf{k}_1) = \mathbf{S}(\mathbf{k}_1)\frac{4\pi^2\mathbf{g}}{\omega_1} \tag{2}
$$

is the spectrum of wave action; and $S(\mathbf{k})$ is the wavenumber spectrum of the surface waves. In order to simplify our presentation, we split the integral in Eq. (1) into two parts as follows:

$$
\frac{\partial n_1}{\partial t} = I_5 = I_{51} + I_{52}, \tag{3}
$$

where $I_{51}$ is the first term in Eq. (1) and $I_{52}$ is the second one. All computations are performed in nondimensional form. Thus without loss of generality, we take $g = k_0 = \omega_0 = 1$, with $k_0$ and $\omega_0$ as the wavenumber and the frequency at the spectral peak in deep water [in shallow water $g = \omega_0 = 1$, $k_0 = k_0(\omega_0(h))$].

We treat the wave vector in polar coordinates; thus, $\mathbf{k} = (\omega^2, \theta)$ is a vector with amplitude $\omega^2$ and angle $\theta$ to the $x$ axis. According to Eq. (1), it is necessary to satisfy the following resonant conditions for five vectors and frequencies:

$$
\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = \mathbf{k}_4 + \mathbf{k}_5, \tag{4a}
$$

$$
\omega_1 + \omega_2 + \omega_3 = \omega_4 + \omega_5. \tag{4b}
$$

We adopt the JONSWAP wave spectrum here as the frequency function and angular distribution in the form of $K(\theta) \sim \cos^m(\theta)$.

Figure 1 shows the two-dimensional energy transfer from (3) versus frequency. One can see that energy cascades from the spectral peak to higher frequencies. Here the first term from (3) is positive and the second term is negative, while their sum gives the energy flux from the spectral peak to the high frequency: a direct cascade. Figure 2 shows the same three-dimensional energy transfer versus frequency and angle.

It follows that when the wave energy is transferred by five-wave interactions, it is from the spectral maximum to the higher harmonics propagating at angles $\pm 70°$ from the main direction. For any odd number of wave interactions, the energy transfer is a direct cascade from low frequency to high, while for any even number we get an inverse cascade (Zakharov 1997). The only exception is the case when the waves are almostly colinear (Dychenko et al. 1994), in which case the energy transfer could be represented by an inverse cascade. Those cases can only be realized for swell propagation, but they are unstable to any cross-wave perturbations. It follows from the previous discussion where it was concluded that five-wave interactions are rather three-dimensional than colinear (four-wave interactions). For example, let us take three of the five participating waves to be at the spectral peak with frequency and wavenumber as $\omega_0$ and $k_0$; a perturbation consequently will be $\omega_1$ and $k_1$. Then, from (4), we have $3\omega_0 = 2\omega_1$;

therefore $\omega_1 = (3/2)\omega_0$, a direct cascade. Furthermore, $3k_0 = 2k_1 \cos(\beta)$, thus $\beta$ will have to be around 50°, an off-main direction propagation. This is very close to what we have obtained and the results of other authors cited above. The energy transfer rate for the five-wave interactions will only consist of a few percent of four-wave interactions because $\alpha = I_5/I_4 = \epsilon^2$, where $\epsilon = k_0 a_0$ is the wave steepness, and $I_4$, $I_5$ are four- and five-wave interactions, respectively. It is reasonable to conclude that four-wave interactions remains dominant except maybe for very steep breaking waves or for extremely shallow depth, where the kinetic equation (3) is not valid and the KdV equation must be used. As a function of the depth, it is easy to find that $\alpha = \epsilon_\infty^2[1 + 2x/\sinh(2x)]^{-1} \tanh^{-2}(x) \sim \epsilon_\infty^2/2x^2$, where $x = kh \ll 1$ and $\epsilon_\infty$ is wave steepness in deep water. For $x = 3.6$ (practically deep water) from Lin and Perrie (1997), wave steepness $\epsilon_\infty = 0.3$ (for the sea is typical $\epsilon_\infty \leq 0.1$), and $\alpha \sim 0.10$, only 10%. It contradicts the conclusion of Lin and Perrie (1997) that five-wave interactions have the same value or even dominate over four-wave ones. For example, take $x = 0.5$ from Kalmykov (1995) and assume $\epsilon_\infty = 0.3$, $\alpha \sim 0.2$. So, even for such large wave steepness of 0.3 and small depth 1.4 m [for $f_0 = 0.3$ Hz (Lin and Perrie 1997) and $kh = 0.5$ (Kalmykov 1995)] five-wave interactions consist of only 20% of the four-wave ones. That five-wave interactions are much smaller than four-wave ones is reasonable and is in accordance with the perturbation theory where the higher order terms are less than lower ones. If somewhere higher-order terms become equal or greater than lower ones, it means that the equation in this case is not valid. A few words about units: The method of computing Eq. (3) is the same as the one in Masuda (1980) for four-wave energy transfer. To get units of meters squared for $dS/dt$ one has to multiply nondimensional values of the five-wave energy transfer on the figures by the factor:

$$c_5 = S^4(\omega_0)\omega_0^{16}g^{-6}, \quad (5)$$

and do the same for four-wave case (Masuda 1980):

$$c_4 = S^3(\omega_0)\omega_0^{11}g^{-4}; \quad (6)$$

that is why $\alpha = c_5/c_4 = \epsilon^2$. Furthermore, values of $dS/dt$ for different $\gamma$ are *different*, as, for example in Masuda (1980). Figure 2 shows $dS/dt$ as a symmetrical and smooth function due to the fine grid used later in computation and the contribution of singular points, contrary to the first 1993–95 its estimates on coarse grid and without singularities, but the general outlook and orders of values remains the same. The complete version of this paper is not cited because it is in review (submitted to *Global Atmosphere and Ocean System*). In conclusion, it must be said that it appears that the results presented by Lin and Perrie (1997) cannot be justified.

## REFERENCES


Dyachenko, A. I., Y. V. Lvov, and V. E. Zakharov, 1994: Five-wave interaction on the surface of deep fluid. *Physica D.,* **87,** 233–261.

Hasselmann, S., and K. Hasselmann, 1985: Computation and parametrizations of the nonlinear energy transfer in a gravity wave spectrum. Part I: A new method for efficient computation of the exact nonlinear transfer integral. *J. Phys. Oceanogr.,* **15,** 1369–1377.

Herterich, K., and K. Hasselmann, 1980: A similarity relation for the nonlinear transfer in a finite depth gravity-wave spectrum. *J. Fluid Mech.,* **97,** 215–224.

Kalmykov, V. A., 1993: Numerical calculation of the nonlinear transfer of energy in the spectra of surface gravity waves by 5 wave resonant interactions. *Dokl. Akad. Nauk Ukraini,* **8,** 101–104.

——, 1995: Estimates of energy transfer in the spectra of surface gravity waves by nonlinearity of high order. *Dokl. Akad. Nauk Ukraini,* **11,** 87–89.

——, 1997: Numerical calculation of nonlinear transfer of energy in spectra of surface gravity waves from the five wave resonant interactions. *The Air-Sea Interface Radio and Acoustic Sensing, Turbulence and Wave Dynamics,* M. A. Donelan, W. H. Hui, and W. J. Plant, Eds., University of Miami, 161–166.

——, 1998: Energy transfer in the spectrum of surface gravity waves by resonance five wave–wave interactions. *Amer. Math. Soc. Transl.,* **182,** 83–94.

Krasitski, V. P., 1993: 5 wave kinetic equation for surface gravity waves. *Mar. Hydrophys. J.* **6,** 17–25.

——, 1994: On reduced equations in the Hamiltonian theory of weakly nonlinear surface waves. *J. Fluid Mech.,* **272,** 1–20.

Lin, R., and W. Perrie, 1997: A new coastal wave model. Part V: Five-wave interactions. *J. Phys Oceanogr.,* **27,** 2169–2186.

Masuda, A., 1980: Nonlinear energy transfer between wind waves. *J. Phys Oceanogr.,* **10,** 2082–2093.

McLean, J. W., 1982a: Instabilities of finite amplitude gravity waves. *J. Fluid Mech.,* **114,** 315–330.

——, 1982b: Instabilities of finite amplitude gravity waves on water of finite depth. *J. Fluid Mech.,* **114,** 331–341.

Meiron, D. I., P. G., Saffman, and H. C. Yuen, 1982: Calculation of the steady three-dimensional deep water waves. *J. Fluid Mech.,* **124,** 109–121.

Shrira, V. I., S. I. Badulin, and C. Khariff, 1996: A model of water wave ''horse-shoe'' patterns. *J. Fluid Mech.,* **318,** 375–404.

Stiassnie, M., and L. Shemer, 1984: On modification of Zakharov's equation of surface gravity waves. *J. Fluid Mech.,* **143,** 47–67.

——, and ——, 1987: Energy computations for evolution of class I and II instabilities of Stokes waves. *J. Fluid Mech.,* **174,** 299–312.

Su, M. Y., 1982: Three-dimensional deep-water waves, Part 1. Experimental measurement of skew and symmetric wave patterns. *J. Fluid Mech.,* **124,** 73–108.

Zakharov, V. E., 1997: The statistical theory of shallow water waves. *Amer. Math. Soc. Transl.,* **182,** 167–197.