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A Class of Nonlinear Markov Iterated Function System Attractors

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Abstract: In this article the theory of NMIFS (Nonlinear Markov Iterated Function System) and the construction method of an NMIFS are presented. The balanced vector measure and the recursive calculation of the "moment" of a class of NMIFS attractors are discussed, and the structure characteristics are analyzed. The result shows that we can calculate the moments $\hat{M}^{(i)}$ $(i = 1, 2, \dots)$ for MIFS, but for NMIFS, we cannot calculate $\hat{M}^{(i)}$ directly because the calculation of $\hat{M}^{(i)}$depends on the value of $\hat{M}^{(j)}(j \geq i)$. So only the approximated value of $\hat{M}^{(i)}$ could be obtained.

Key words: NMIFS attractor; balanced vector measure; moment.
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1. Introduction

The iterated function system (IFS) has become a powerful tool for the construction as well as the analysis of typically fractal sets. The idea of constructing such sets was first put forward by Hutchinson[1], while its complete theory was set up by Barnsley and others[2,3]. So far the IFS attractor has been studied more deeper than before and the theory of fractal is greatly enriched in [4–6]. In 1982, MIFS was introduced by Dekking. In the years that followed, a great deal of work has been done by Barnsley, Elton, Grigorescu, Vrscay, Lasotaa and Stenflo, etc. Among the various aspects studied are Hausdorff-Besicovitch dimension, probability measures and their moments, ergodic theory, dynamical systems, etc[7−12]. After 1991, Vrscay and the author have made some research on the NIFS attractor[4,13]. This article extends the previous work, mainly discussing the recursive computation of the balanced vector measures and moments, and analyzing the characteristic of the NMIFS structure.

2. Theorem and methods

The Markov character was first put forward by the Russian mathematician Mapkob in 1906. From the simple Markov character to the abstract concept of Markov process, decades of years have passed[14]. Nowadays the Markov process has been widely studied and applied in modern

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physics, biology, mathematics and many other subjects and fields[15−17].

Definition 1 *Let I denote a state space of random process $\{X(t), t \in T\}$. For n arbitrary time values $t_1 < t_2 < \cdots < t_n$, $n \geq 3$, $t_i \in T$, if the conditional distribution function of $X(t_n)$ under condition $X(t_i) = x_i$, $x_i \in I$, $i = 1, 2, \ldots, n-1$ equals to the one under condition $X(t_{n-1}) = x_{n-1}$, i.e.*

$$P\{X(t_n) \leq x_n \,|X(t_1) = x_1, X(t_2) = x_2\ , \ldots, X(t_{n-1}) = x_{n-1}\}$$
$$= P\{X(t_n) \leq x_n \,|X(t_{n-1}) = x_{n-1}\},\ x_n \in R$$

or

$$F_{t_n|t_1\ldots t_{n-1}}(x_n, t_n\,|x_1\ , x_2, \ldots, x_{n-1}; t_1, t_2, \ldots, t_{n-1}) = F_{t_n|t_{n-1}}(x_n, t_n\,|x_{n-1}\ , t_{n-1}),$$

then we call the process $\{X(t), t \in T\}$ has the Markov character, or call this process the Markov process.

Let (X, ρ) be metric space and $F(X)$ denote sets of the nonempty compact subset $S = (S_1, \ldots, S_N)$ of X where $S_i \subset w_i(X)$. Then $F(X)$ is the complete metric space with metric

$$h_\rho(A, B) = \max_i h(A_i, B_i),$$

where $h(A, B) = \sup_{x\in A} \inf_{y\in B} d(x, y) + \sup_{x\in B} \inf_{y\in A} d(x, y)$ is the Hausdorff distance with $d(x, y) = \|x - y\|$.

Definition 2 *A contractive NMIFS is constituted of a class of contractive mapping $w = (w_i : i = 1, 2, \ldots, N)$ on (X, ρ). Let*

$$w_i(x) = a_i x^n + b_i, \tag{1}$$

where $a_i, b_i \in R$ or C, n is an integer, and $n \neq 1$. For every $w_i : X \to X$, if it satisfies

$$\|w_i(x) - w_i(y)\| \leq c_i \|x - y\| \quad (\forall x, y \in X),$$

where $0 \leq c_i < 1$, then we call $c = \max_{1\leq i\leq N}\{c_i\}$ the contraction ratio of the contractive NMIFS. If a Markov transition probability matrix $P = [p_{ij}]_{N\times N}$ satisfies

$$p_{ij} \geq 0\ (i, j = 1, 2, \ldots, N)\ \text{ and } \sum_{j=1}^{N} p_{ij} = 1\ \ (i = 1, 2, \ldots, N),$$

then we will refer the integers $1, 2, \ldots, N$ as states and call $\{X, w, P\}$ a nonlinear Markov iterated function system, abbreviated as NMIFS.

Theorem 1 *Let $D(r, z_0)$ denote a circle with center z_0 and radius r in the complex plane C. For complex mappings $T(z) = a \cdot z^n + b$, we have*

(i) $T(z) = a$ is a contractive mapping on $D(r, z_0)$, where

$$r^{n-1} = \frac{1}{n\,\|a\|}; \tag{2}$$

(ii) If $\|a\| \cdot r^n + \|b\| \leq r$, *then* $T(D(r,0)) \subset D(r,0)$, *and*

$$\|b\| \leq \frac{n-1}{n} \cdot \frac{1}{(n\,\|a\|)^{1/(n-1)}}. \tag{3}$$

Proof Since $\frac{\mathrm{d}}{\mathrm{d}z}T(z) = n \cdot a \cdot z^{n-1}$, $T(z)$ is a contractive mapping if $\|z\|^{n-1} < \frac{1}{n\|a\|}$. Substituting r, the radius of the disc$D(r,0)$, for $||z||$ leads to Equation (2).

For part (ii), observe $T(D(r,0)) = D(\|a\| \cdot r^n, b) \subset D(\|a\| \cdot r^n + \|b\|\,,\ 0)$, so $T(D(r,0)) \subset D(r,0)$ if $\|a\| \cdot r^n + \|b\| \leq r$. By substituting equation (3) into $\|a\| \cdot r^n + \|b\| \leq r$ and solving for $\|b\|$, Equation (3) is obtained.

Theorem 2 *For NIFS:* $\{T_i(z), i = 1, \ldots, N\}$, $T_i(z) = a_i z^n + b_i$, *if*

$$\|a_{\max}\| \cdot r_{\max}^{n_{\min}} + \|b_{\max}\| \leq r_{\min} \tag{4}$$

holds, then $\{T_i(z), i = 1, \ldots, N\}$ *is a contractive NIFS on* $D(r_{\min}, 0)$, *where* $r_{\min} = \min\{r_1, \ldots, r_N\}$ *and* $r_i < 1$.

Proof Since $T_m(D(r_i,0)) \subset D(\|a_m\| \cdot r_i^{n_m} + \|b_m\|\,, 0)$, if Equation (4) holds when $r_i < 1$, then

$$T_m(D(r_i,0)) \subset D(r_{\min},0) \quad (1 \leq i,\ m \leq N).$$

So Theorem 2 holds.

Corollary for $\|a_m\| = A$, $\|b_m\| = B$ and $n_m = n$, it can be infered from Equation (2) that $r_m = r$. So by Equation (4) we can get

$$A \cdot r^n + B \leq r. \tag{5}$$

By substituting Equation (2) into Equation (5), the relation between A and B will be:

$$A \geq \frac{(n-1)^{n-1}}{n^n} \cdot \frac{1}{B^{n-1}},\ \text{if } n < 1; \quad A < \frac{(n-1)^{n-1}}{n^n} \cdot \frac{1}{B^{n-1}},\ \text{if } n > 1.$$

Theorem 3 *Let* $\{X, w, P\}$ *be contractive NMIFS on the complete metric space* (X, ρ) *with contraction ratio c. Now we define* $W : F(X) \to F(X)$ *by* $W(B) = S$, *where*

$$S_j = \bigcup_{i, p_{ij} \neq 0} w_j(B_i).$$

Then W *is a contractive map on* $(F(X), h_\rho)$ *with contraction ratio c, i.e.*

$$h_\rho(W(B), W(C)) \leq c h_\rho(B, C) \quad (\forall B, C \in F(X)),$$

and there exists a unique fixed point $A = (A_1, \ldots, A_N), A \in F\,(X)$, *satisfying*

$$A = W(A) = \bigcup_{n=1}^{N} w_n(A),$$

and for $\forall B \in F(X)$,

$$A = \lim_{n \to \infty} W^n(B).$$

Proof We can get from Equation (1) that

$$h_\rho(W(B),W(C))=\max_j h(\bigcup_{i,p_{ij}\neq 0} w_j(B_i),\bigcup_{i,p_{ij}\neq 0} w_j(C_i)),$$

and obviously,

$$h(\bigcup_k B_k,\bigcup_k C_k)\leq \max_k h(B_k,C_k).$$

So we get

$$h_\rho(W(B),W(C))\leq \max_j \max_{i,p_{ij}\neq 0} h(w_j(B_i),w_j(C_i))\leq c\max_i h(B_i,C_i)\leq ch_\rho(B,C). \tag{6}$$

From Equation (6) we can see that if $W(A)=A$ and $W(B)=B$, then A and B are invariant sets. When $h_\rho(A,B)=0$, we have $A=B$, namely, the invariant set is unique. Because

$$h_\rho(W(B),A)=h_\rho(W(B),W(A))\leq ch_\rho(B,A)\quad (\forall B\in F(X)),$$

we can get

$$h_\rho(W^n(B),A)\leq c^n h_\rho(B,A).$$

So $A=\lim_{n\to\infty} W^n(B)$. The proof is complete.

The fixed point A of Theorem 3 is called the attractor of the NMIFS. In general the attractor of NMIFS is fractal, and always called deterministic fractal. With Theorem 3 we could set up the algorithm constructing the NMIFS attractor as follows: Let $\{X,w,P\}$ be a contractive NMIFS with $w=(w_i:i=1,2,\ldots,N)$ and the Markov transition probability matrix $P=[p_{ij}]_{N\times N}$. First select an initial point $x_0\in X$ and an initial state $i_0\in\{1,2,\ldots,N\}$, then map x_0 by a mapping w_{i_1} chosen from the set $\{w_i:p_{i_oi}>0\}$, the choice being weighted according to the associated probabilities $p_{i_0i_1}$ but rather random, to obtain $x_1=w_{i_1}(x_0)$. Next, a mapping w_{i_2} is chosen in the same manner, subject to $p_{i_1i_2}>0$, to obtain $x_2=w_{i_2}(x_1)$. This process is continued, and a sequence $\{x_m\}$ is produced. With an integer $M_{\max}$ big enough the sequence $\{x_m, m\geq M_{\max}\}$ will converge to the NMIFS attractor A. However, limited to the actual resolution of computer display, the number of iteration m should not be larger than a certain value. When m is larger than this value, it will produce no better effects. But how is the difference between the fractal set E and the NMIFS attractor A when m gets to this value? The following theorem which is called collage theorem tells the evaluation under Hausdorff metric.

Theorem 4 *Let (X,ρ) be a complete metric space. $\{X,w,P\}$ is a contractive NMIFS with contraction ratio c and $w=(w_i:i=1,2,\ldots,N)$, if its fixed point (invariant set) is A, then*

$$h_\rho(E,A)\leq (1-c)^{-1}h_\rho(E,\bigcup_{n=1,0}^{N} w_n(E))\quad (\forall E\in F(x)). \tag{7}$$

Proof Because A is invariant set, we have $A=W(A)$, then

$$\begin{aligned}h_\rho(W(E),A)&=\max_j h(\bigcup_{i,p_{ij}\neq 0} w_j(E_i),\bigcup_{i,p_{ij}\neq 0} w_j(A_i))\leq \max_j \max_{i,p_{ij}\neq 0} h(w_j(E_i),w_j(A_i))\\&\leq c\max_i h(E_i,A_i)\leq ch_\rho(E,A).\end{aligned}$$

According to triangle inequality of the Hausdorff distance we can obtain

$$h_\rho(E,A) \le h_\rho(E, \bigcup_{n=1}^{N} w_n(E)) + h_\rho(\bigcup_{n=1}^{N} w_n(E), A) \le h_\rho(E, \bigcup_{n=1}^{N} w_n(E)) + ch_\rho(E,A), \quad (8)$$

Equation (7) is an immediate result of Equation (8).

The collage theorem ensures that the produced image on computer is an approximation to the attractor A. The Hausdorff distance between the two sets may be evaluated by the Hausdorff distance between sets and its image. So the collage theorem gives a theoretical base to the construction of the NMIFS attractors.

3. Experiment and result

3.1. Balanced vector measures

Let $\{X, w, P\}$ be an NMIFS and let $B \in B(X)$, where $B(X)$ is Borel subsets of X. Associated with each MIFS is a unique stationary probability measure $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ whose support that is the attractor A of the NMIFS, and for all Borel subsets B of X, it satisfies[3]

$$\mu(B) = \int_X P(x,B)\mathrm{d}\mu(x).$$

Notice that the discrete Markov transition probability matrix $P = [p_{ij}]_{N\times N}$ in the above expression satisfies

$$p_{ij}(x,B) = p_{ij}\delta_{w_j(x)}(B),$$

where $p_{ij}(x,B)$ is the transform probability from $x \in X$ to the Borel subset B under the mapping w_j, and for some $y \in X$ there exists $x = w_i(y)$. $\delta_z(B)$ satisfies the following condition:

$$\delta_z(B) = \begin{cases} 1, & \text{if } z \in B \\ 0, & \text{if } z \notin B \end{cases}.$$

Theorem 5 *For all the Borel subsets B of X, there exists a probability measure* $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ *which satisfies*

$$\mu_j(B) = \sum_{i=1}^{N} p_{ij} \int_X \delta_{w_j(x)}(B)\mathrm{d}\mu_i(x) \quad (j = 1, 2, \ldots, N). \quad (9)$$

Proof Define the operator in the Banach space

$$(T_j f)(x) = \sum_{i=1}^{N} p_{ij} f(w_j(x)).$$

It can be drawn from the Schauder fixed point theorem that there exists a fixed point μ_j with the operator T^*[18]. And because

$$(T_j^* v)(B) = \sum_{i=1}^{N} p_{ij}(w_j^{\#} \circ v)(B) = \sum_{i=1}^{N} p_{ij} \int_X \delta_{w_j(x)}(B)\mathrm{d}v(x),$$

where $(w_j^{\#} \circ v)(B) = v(w_i^{-1}(B))$, the fixed point μ_j satisfies

$$\mu_j(B) = \sum_{i=1}^{N} p_{ij} \int_X \delta_{w_j(x)}(B) \mathrm{d}\mu_j(x) = \int_X P(x, B) \mathrm{d}\mu_j(x).$$

Definition 3 *The probability measure $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ defined in Equation (9) is called the balanced vector measure of $\{X, w, P\}$.*

3.2. Recursive calculation of moments

If $\{X, w, P\}$ is a contractive MIFS in R^d with attractor$A$, then we define the associated invariant measure by the Lebesgue integrals[3]

$$g_m = \int_A x^m \mathrm{d}\mu. \tag{10}$$

For convenience, the measure is assumed to be normalized, i.e., $g_0 = \int_A \mathrm{d}\mu = 1$.

It can be seen from Equation (9) that if $C(X)$ denotes the continuous function space on X, then we can get the following equation when $f \in C(X)$,

$$\int_C f(z) \mathrm{d}\mu_i(z) = \sum_{j=1}^{N} p_{ji} \int_C f(w_i(z)) \mathrm{d}\mu_j(z) \quad (j = 1, 2, \ldots, N). \tag{11}$$

We define the mth moment $\hat{M}^{(m)}$ by

$$\hat{M}^{(m)} = (g_1^{(m)}, g_2^{(m)}, \ldots, g_N^{(m)}) = (M_1^{(m)}, M_2^{(m)}, \ldots, M_N^{(m)}), \tag{12}$$

where

$$g_i^{(m)} = M_i^{(m)} = \int_C z^m \mathrm{d}\mu_i(z).$$

And so, by Equation (11)

$$M_i^{(m)} = \sum_{j=1}^{N} p_{ji} \int_C (w_i(z))^m \mathrm{d}\mu_j(z).$$

For Equation (11), we can get

$$M_i^{(m)} = \sum_{j=1}^{N} p_{ji} \int_C (a_i z^n + b_i)^m \mathrm{d}\mu_j(z). \tag{13}$$

In case of $n = 1$, we have

$$M_i^{(m)} = \sum_{j=1}^{N} p_{ji} \int_C (a_i z + b_i)^m \mathrm{d}\mu_j(z) \quad (i = 1, 2, \ldots, N). \tag{14}$$

Theorem 6 *Let $\{X, w, P\}$ be an MIFS, with X a compact subset of C and $w_i = a_i z + b_i$ $(i = 1, 2, \ldots, N)$. Define the matrix $A^{(n,k)}$ as follows*

$$A^{(m,k)} = \left[a_{ij}^{(m,k)}\right] = \left[\begin{pmatrix} m \\ k \end{pmatrix} p_{ji} a_i^k b_i^{m-k}\right] \quad (m = 1, 2, \ldots; k = 0, 1, 2, \ldots, m).$$

Given $\hat{M}^{(0)}$, the moments $\hat{M}^{(m)}$ can be calculated recursively as follows

$$(I - A^{(m,m)})\hat{M}^{(m)} = \sum_{k=0}^{m-1} A^{(m,k)}\hat{M}^{(k)}. \tag{15}$$

Proof From Equation (14), we can obtain

$$M_i^{(m)} = \sum_{j=1}^{N} p_{ji} \sum_{k=0}^{m} a_i^k b_i^{m-k} \binom{m}{k} \int_C z^k \mathrm{d}\mu_j(z) = \sum_{k=0}^{m} \left(\sum_{j=1}^{N} (\binom{m}{k}) p_{ji} a_i^k b_i^{m-k} \right) M_j^{(k)}$$
$$= \sum_{k=0}^{m} \sum_{j=1}^{N} a_{ij}^{(m,k)} M_j^{(k)}.$$

Or, in matrix form,

$$\hat{M}^{(m)} = \sum_{k=0}^{m} A^{(m,k)}\hat{M}^{(k)},$$

from which Equation (15) holds.

When $n \neq 1$, with Equations (10) and (11) we can obtain

$$g_i^{(m)} = \sum_{j=1}^{d} p_{ji} \int_A (a_i z^n + b_i)^m \mathrm{d}\mu_j(z). \tag{16}$$

Setting $g_i^{(0)} = 1$, the first three equations corresponding to $n = 1, 2, 3$ in Equation (16) are

$$g_i^{(1)} = g_i^{(n)} \sum_j p_{ji} a_i + \sum_j p_{ji} b_i, \tag{17}$$

$$g_i^{(2)} = g_i^{(2n)} \sum_j p_{ji} a_i^2 + 2g_i^{(n)} \sum_j p_{ji} a_i b_i + \sum_j p_{ji} b_i^2, \tag{18}$$

$$g_i^{(3)} = g_i^{(3n)} \sum_j p_{ji} a_i^3 + 3g_i^{(2n)} \sum_j p_{ji} a_i^2 b_i + 3g_i^{(n)} \sum_j p_{ji} a_i b_i^2 + \sum_j p_{ji} b_i^3. \tag{19}$$

Define matrix $A^{(s,t)} = [\alpha_{ij} a_i^s b_i^t] = \sum_i \sum_j p_{ij} a_i^s b_j^t$ $(i, j = 1, 2, \ldots, d)$, then we can obtain the following three equations according to Equations (17), (18) and (19).

$$g^{(1)} = A^{(0,0)} g^{(n)} + A^{(0,1)}, \tag{20}$$

$$g^{(2)} = A^{(2,0)} g^{(2n)} + 2A^{(1,1)} g^{(n)} + A^{(0,2)}, \tag{21}$$

$$g^{(3)} = A^{(3,0)} g^{(3n)} + 3A^{(2,1)} g^{(2n)} + 3A^{(1,2)} g^{(n)} + A^{(0,3)}. \tag{22}$$

It is obvious that Equations (20), (21) and (22) do not satisfy the recursive calculation of moments. In fact, we can obtain from Equation (16) that

$$g_i^{(m)} = \sum_{j=1}^{d} p_{ji} \int_A (a_i z^n + b_i)^m \mathrm{d}\mu_j(z) = \sum_{j=1}^{d} p_{ji} \int_A \binom{m}{k} a_i^k z^{kn} b_i^{(m-k)} \mathrm{d}\mu_j(z)$$
$$= \sum_{j=1}^{d} p_{ji} \binom{m}{k} a_i^k b_i^{(m-k)} \int_A z^{kn} \mathrm{d}\mu_j(z),$$

namely,

$$g^{(m)} = \begin{pmatrix} m \\ k \end{pmatrix} A^{(k,m-k)} g^{(kn)}. \tag{23}$$

It can be seen from Equation (23) that a new set of moments g_i^k or $g^{(k)}$ is involved for every m $(m+1 \leq k \leq mn)$. So we can not calculate moments recursively. However, the following can be obtained after analysis that

$$g^{(p)} = G(g^{(1)}, \ldots, g^{(m)}) \quad (m = 1, 3, \ldots, [p/n] \text{ and } m \text{ is not the multiple of } n), \tag{24}$$

where $[x]$ denotes the integer part of x. Equation (24) means that $g^{(p)}$ is a linear combination of $g^{(1)}, \ldots, g^{(m)}$. According to Ref. [13], the sequence of unknown odd moments $g^{(i)}$ $(i = 1, 2, \ldots)$ will be referred as missing moments. We now proceed to find approximations to these unknown variables.

Theorem 7 *Let $g^{(n)}$ $(n = 0, 1, 2, \ldots)$ denote an infinite sequence of real numbers. A necessary and sufficient condition for the existence of a unique invariant measure μ on $[0, 1]$ such that*

$$g^{(n)} = \int_0^1 z^n \mathrm{d}\mu$$

is that the $g^{(n)}$ satisfies the following inequalities (The equality holds only when μ consists of point masses at $z = 0$ and / or 1)

$$I(m, n) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^k g^{(m+k)} \geq 0 \quad (m, n = 0, 1, 2, \ldots). \tag{25}$$

Theorem 7 has been proved in Ref. [13] for IFS. For NMIFS, the theorem can also be proved easily and thus the proof is omitted. Theorem 7 shows that $g^{(n)}$ $(n = 0, 1, 2, \ldots)$ is nonincreasing. Now we only consider the first n $(n > 0)$ missing moments. Denote these missing moments as $\hat{x}$ with form:

$$\hat{x} = (x^{(1)}, \ldots, x^{(m)})^{\mathrm{T}} = (g^{(1)}, \ldots, g^{(m)})^{\mathrm{T}}.$$

For Equation (1), the vector define a unique time series K_j $(j = 0, 1, \ldots, nm)$.

Example 1 Consider NMIFS

$$w_1(x) = \frac{1}{2}x^2, \quad w_2(x) = \frac{1}{2}x^2 + \frac{1}{2}, \quad P = (p_{ij}) = \begin{pmatrix} 0.5 & 0.5 \\ 1 & 0 \end{pmatrix}.$$

With the means put forward, it can be seen that the sequence of odd moments can be considered as independent variables while all the even moments may be written as linear functions of the odd ones. The first five even moments are given by (including $g^{(1)}$)

$$g^{(2)} = g^{(1)} - \frac{1}{4}, \quad g^{(4)} = \frac{3}{2}g^{(1)} - \frac{5}{8}, \quad g^{(6)} = 4g^{(3)} - \frac{15}{8}g^{(1)} + \frac{13}{32}, \quad g^{(8)} = -4g^{(3)} + \frac{85}{8}g^{(1)} + \frac{143}{32}.$$

m \ n	1	2	3	4
0	$1-g^{(1)}$	$\frac{1}{2}$	$-\frac{1}{2}+3g^{(1)}-g^{(3)}$	$-4+14g^{(1)}-4g^{(3)}$
1	$\frac{1}{2}-g^{(1)}$	$1-3g^{(1)}+g^{(3)}$	$-\frac{7}{2}-11g^{(1)}+3g^{(3)}$	
2	$-\frac{1}{2}+2g^{(1)}-g^{(3)}$	$-\frac{5}{2}+8g^{(1)}-2g^{(3)}$		
3	$2-6g^{(1)}+g^{(3)}$			

Table 1. Hausdorff inequalities $I(m,n)$ of Equation (14) for $0 \le m \le 3$, $1 \le n \le 4$, in terms of the missing moments $g^{(1)}$ and $g^{(3)}$

Some values are calculated and presented in Table 1. From Theorem 7 we can see that $g^{(1)} > g^{(2)} > g^{(4)}$. It can be easily computed that

$$\frac{5}{12} < g^{(1)} < \frac{3}{4}. \tag{26}$$

So the upper limit and the lower limit of $g^{(i)}$ $(i = 1, 2, \ldots, 8)$ can be obtained by Equations (25), (26) and Table 1.

According to the work of Bessis and Demko on $\int \sqrt{x}\mathrm{d}\mu$ of attractor A[19], we can see that

$$T^{(n)}f(x) \to \int f\mathrm{d}\mu_i, \quad x \in \mu_i \tag{27}$$

holds. Thus the somewhat accurate approximation of $g^{(2k-1)}$ $(k = 1, 2, \ldots)$ can be calculated with Equation (27). Notice that μ in Equation (27) is the invariant measure of the NMIFS $\{X, w, P\}$, and the operator $T : C(X) \to C(X)$ is defined as

$$T(f)(x) = \sum_i \sum_{j=1}^{N} p_{ij}(f \circ w_j)(x), \tag{28}$$

where $C(X)$ denotes the continuous functions space on X. With Equation (13) we can deduce that

$$T^{(n)}f(x) = \sum_i \cdots \sum_i \left(\sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} p_{ij_1} \cdots p_{ij_n} f(w_{j_1} \circ \cdots \circ w_{j_n})(x) \right). \tag{29}$$

Given $f(x) = x^{2k-1}$, a better result of $g^{(2k-1)}$ can be calculated with Equation (29).

3.3. Construction of a class NMIFS

Based on the algorithm of constructing NMIFS attractor, the proper transition probability matrix $P = [p_{ij}]$ is selected listed in table 2, and proper $w = (w_i : i = 1, 2, \ldots, N)$ are selected as follows according to Equation (1).

Figure number	$M=(m_{ij})_{4\times4}$	Figure number	$M=(m_{ij})_{4\times4}$	Figure number	$M=(m_{ij})_{4\times4}$	Figure number	$M=(m_{ij})_{4\times4}$
1(a)	$\begin{bmatrix}1&0&1&1\\1&1&1&1\\1&1&1&0\\0&1&0&1\end{bmatrix}$	1(b)	$\begin{bmatrix}1&1&0&1\\1&1&1&1\\1&1&0&1\\0&1&0&0\end{bmatrix}$	1(c)	$\begin{bmatrix}0&1&1&1\\0&0&1&1\\1&0&0&1\\1&1&1&1\end{bmatrix}$	1(d)	$\begin{bmatrix}0&0&1&0\\1&1&0&0\\1&1&1&1\\0&1&1&1\end{bmatrix}$
2(a)	$\begin{bmatrix}1&1&1&0\\0&0&1&1\\1&1&1&1\\1&1&0&0\end{bmatrix}$	2(b)	$\begin{bmatrix}1&1&0&1\\0&1&1&0\\0&1&1&1\\1&0&0&1\end{bmatrix}$	2(c)	$\begin{bmatrix}1&1&1&0\\1&1&1&1\\1&1&0&0\\1&0&0&1\end{bmatrix}$	2(d)	$\begin{bmatrix}0&1&0&1\\1&1&1&1\\1&0&1&1\\1&1&1&0\end{bmatrix}$
3(a)	$\begin{bmatrix}1&1&1&0\\0&0&1&1\\1&1&1&1\\1&1&0&0\end{bmatrix}$	3(b)	$\begin{bmatrix}1&1&0&1\\0&1&3&0\\0&1&1&1\\3&1&0&0\end{bmatrix}$	3(c)	$\begin{bmatrix}1&1&1&0\\1&1&1&1\\0&1&1&0\\1&0&1&0\end{bmatrix}$	3(d)	$\begin{bmatrix}0&1&0&1\\1&1&1&1\\1&0&1&1\\1&1&0&0\end{bmatrix}$

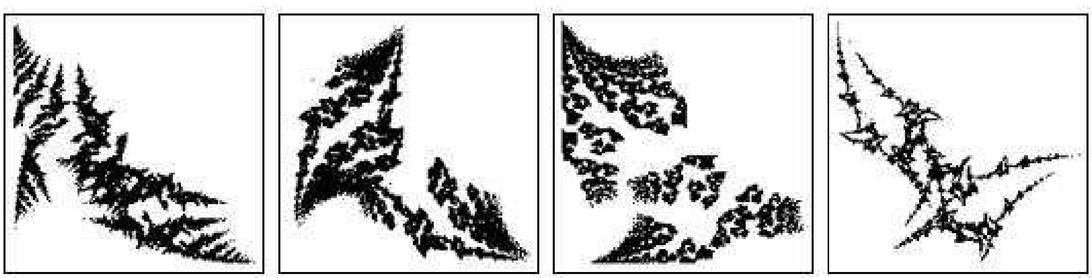
Table 2. The transition probability matrix of the NMIFS$P=[p_{ij}]$
(the relation between P and M is $p_{ij}=m_{ij}/n_i$, $n_i=\sum_{j=1}^{4}m_{ij}$)

$$w=\left(\frac{1}{2}z^2+\frac{1}{2},\frac{i}{2}z^2+\frac{2}{5}+\frac{1}{2}i,-\frac{1}{2}z^2+\frac{1}{2}+\frac{2}{5}i,-\frac{i}{2}z^2+\frac{i}{2}\right), \tag{30}$$

$$w=\left(\frac{3}{5}z^2,\frac{3i}{5}z^2+\frac{1}{2},\frac{1}{2}z^2+\frac{1}{2},\frac{i}{2}z^2+\frac{1}{2}+\frac{i}{2}\right), \tag{31}$$

$$w=\left(-\frac{i}{2}z^2,\frac{1}{2}z^2+\frac{1}{2},\frac{1}{2}z^2+\frac{i}{2},-\frac{1}{2}z^2+\frac{1}{2}+\frac{i}{2}\right). \tag{32}$$

Depending on the choice for the initial point the first few points produced during the iteration may not yet be close to the attractor. Therefore they should be eliminated from the approximating point set. As an example, the first 1000 points in our computation are abandoned and the following iteration lasts 500000 times. Many interesting images are produced in this way. Fig.1-Fig.3 are some representative ones.



(a) (b) (c) (d)

Figure 1: The NMIFS attractor of Equation (30)

From Fig.1–Fig.3 we can see that different transition probability matrices may cause different images. In addition, we can set up the continuous dependent relation between the contractive

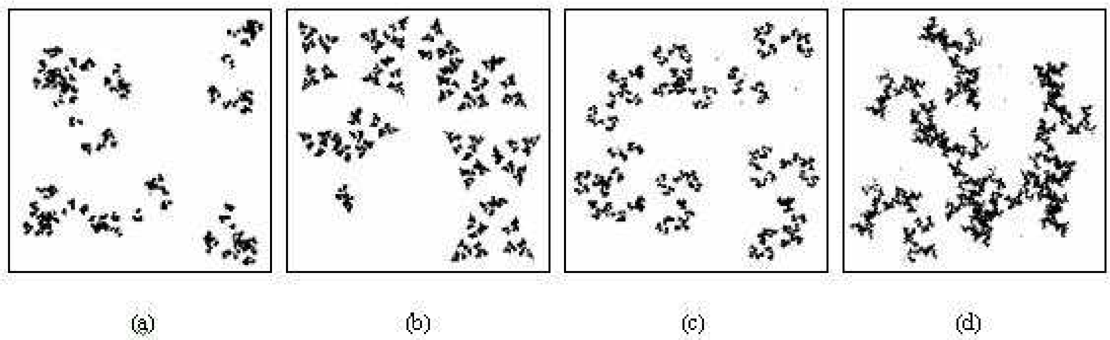


Figure 2: The NMIFS attractor of Equation (31)

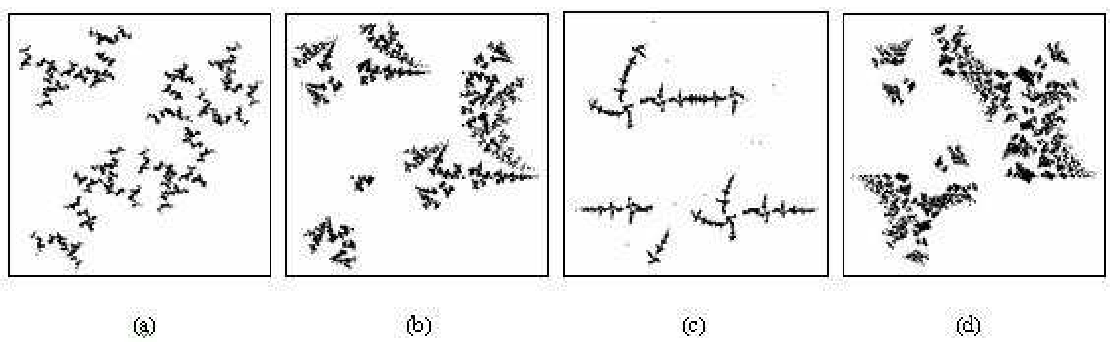


Figure 3: The NMIFS attractor of Equation (32)

NMIFS attractor and its parameters, that is, for a contractive NMIFS, a small change of the parameters will cause a small change of the attractor's structure. This is important because of its practical application. For example, we could adjust the parameters continuously to control the NMIFS attractor in image compression. Meanwhile, this makes it possible to interpolate to the attractor which is fairly favourable to the computer simulation.

4. Conclusion

In this article, the theory of NMIFS, the balanced vector measure and the computation of moments are discussed. The NMIFS attractor is also introduced and some images are presented. The major point of MIFS is that every point in the orbit generated by IFS is associated with a "state" and the change of the state is controlled by the Markov process. The balanced vector measures and the moments can be treated as vectors. It has been showed in our work that the moments $\hat{M}^{(i)}$ $(i = 1, 2, \ldots)$ can be calculated by recursion for MIFS, while for a NMIFS, it is another thing. For NMIFS, the value of $\hat{M}^{(i)}$ depends on the value of $\hat{M}^{(j)}$ $(j \geq i)$. So we cannot calculate its value directly but only its approximate value can be gotten. Here we only discussed the map $w_i(x) = a_i x^n + b_i$ $(i = 1, 2, \ldots, N)$, but it can be analogously reasoned to other kinds easily like $w_i(z) = a_{i_n} z^n + a_{i_{n-1}} z^{n-1} + \cdots + a_{i_1} z + b_i$, so the methods we adopted and the conclusions we took are of general meanings.

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一类 NMIFS 吸引子

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摘要: 本文给出了 NMIFS (Nonlinear Markov Iterated Function System) 理论与构造 NMIFS 吸引子的方法，讨论了一类 NMIFS 吸引子的平衡向量测度和 “矩” 的递归计算，分析了 NMIFS 吸引子的结构特征. 研究结果表明：对于 MIFS, 可以通过递归方法来计算矩 $\hat{M}^{(i)}(i = 1, 2, \ldots)$; 而对于 NMIFS, 因 $\hat{M}^{(i)}$ 的计算依赖于 $\hat{M}^{(j)}(j \geq i)$, 故不能直接计算 $\hat{M}^{(i)}$, 而只能计算其近似值.

关键词: NMIFS 吸引子；平衡向量测度；矩.