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CONVOLUTION OPERATORS WITH HOMOGENEOUS SINGULAR MEASURES ON $\mathbb{R}^3$ OF POLYNOMIAL TYPE. THE REMAINDER CASE.

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ABSTRACT. Let $\varphi(y_1, y_2) = y_2^l P(y_1, y_2)$ where P is a polynomial function of degree l such that $P(1,0) \neq 0$. Let μ_δ be the Borel measure on $\mathbb{R}^3$ defined by $\mu_\delta(E) = \int_{V_\delta} \chi_E(x, \varphi(x))\, dx$ where

$$V_\delta = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 1, \text{and } |x_1| \leq \delta |x_2|\right\}$$

and let T_{μ_δ} be the convolution operator with the measure μ_δ. In this paper we explicitely describe the type set

$$E_{\mu_\delta} := \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \in [0,1] \times [0,1] : \|T_{\mu_\delta}\|_{p,q} < \infty \right\},$$

for δ small enough.

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1. INTRODUCTION

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function of degree $m \geq 2$ and let $D = \{y \in \mathbb{R}^2 : |y| \leq 1\}$. Let μ be the Borel measure on $\mathbb{R}^3$ given by

$$\mu(E) = \int_D \chi_E(y, \varphi(y))\, dy \tag{1.1}$$

and let T_μ be the operator defined, for $f \in S(\mathbb{R}^3)$, by $T_\mu f = \mu * f$. Let E_μ be the set of the pairs $\left(\frac{1}{p}, \frac{1}{q}\right) \in [0,1] \times [0,1]$ such that there exists a positive constant c satisfying $\|Tf\|_q \leq c\|f\|_p$ for all $f \in S(\mathbb{R}^3)$, where the L^p spaces are taken with respect to the Lebesgue measure on $\mathbb{R}^3$.

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For $\left(\frac{1}{p},\frac{1}{q}\right)\in E_{\mu}$, T can be extended to a bounded operator, still denoted by T, from $L^{p}\left(\mathbb{R}^{3}\right)$ into $L^{q}\left(\mathbb{R}^{3}\right)$.

Let $\varphi=\varphi_{1}^{e_{1}}...\varphi_{n}^{e_{n}}$ be a decomposition of φ in irreducible factors with $\varphi_{i}\nmid\varphi_{j}$ for $i\neq j$. In [3] we could give a complete description of the set E_{μ} under the assumption that $e_{i}\neq\frac{m}{2}$ for each φ_{i} of degree 1. If $\det\varphi''\left(y\right)$ is not identically zero and if it vanishes somewhere on $\mathbb{R}^{2}-\{0\}$, the set of the points y where $\det\varphi''\left(y\right)$ vanishes is a finite union of lines $L_{1},...,L_{k}$ through the origin. So, after a possibly linear change of variables, we localized the problem to the x axes and we studied the type set corresponding to measures μ_{δ} defined by

$$\mu_{\delta}\left(E\right)=\int_{V_{\delta}}\chi_{E}\left(y,\varphi\left(y\right)\right)dy,$$

where $V_{\delta}=D\cap\left\{(y_{1},y_{2})\in\mathbb{R}^{2}:\left|y_{2}\right|\leq\delta\left|y_{1}\right|\right\}$ and δ is small enough such that $\det\varphi''\left(y\right)$ only vanishes, on V_{δ}, along the x axes. The only case left was the one corresponding to functions φ of the form $\varphi\left(y_{1},y_{2}\right)=y_{2}^{l}P\left(y_{1},y_{2}\right)$ with $l=\frac{m}{2}$, P being a homogeneous polynomial function of degree l such that $P\left(1,0\right)\neq0$.

In this paper we characterize $E_{\mu_{\delta}}$ in this remainder case.

L^{p} improving properties of convolution operators with singular measures supported on hypersurfaces in $\mathbb{R}^{n}$ have been widely studied in [2], [5], [6]. In particular, in [5], the type set was studied under our actual hypothesis, but the endpoint problem was left open there. Our proof of the main result involves a biparametric family of dilations and will be based on a suitable adaptation of arguments due to M. Christ, developed in [1], where the author studied the type set associated to the two dimensional measure supported on the parabola.

Also, oscillatory integral estimates are involved. A very careful study of this kind of estimate can be found in [4] where the authors study the boundedness of maximal operators associated to mixed homogeneous hypersurfaces.

Throughout this paper c will denote a positive constant, not the same at each occurrence.

2. THE MAIN RESULT

We assume $\varphi\left(y_{1},y_{2}\right)=y_{2}^{l}P\left(y_{1},y_{2}\right)$, where $l=\frac{m}{2}$ and P is a homogeneous polynomial function of degree l such that $P\left(1,0\right)\neq0$. We take $\delta_{1}>0$ such that, for $y\in V_{\delta_{1}}$ such that $y_{2}\neq0$, $\det\varphi''\left(y\right)\neq0$. Moreover, since $P\left(1,0\right)\neq0$ we can assume that $P\left(y\right)\neq0$ and $P_{1}\left(y\right)\neq0$ for all $y\in V_{\delta_{1}}$. Now, if $\max_{V_{\delta_{1}}}\left|P_{2}\left(y_{1},y_{2}\right)\right|\neq0$, we choose $\delta<\min\left(\frac{l\min_{V_{\delta_{1}}}\left|P(y_{1},y_{2})\right|}{2\max_{V_{\delta_{1}}}\left|P_{2}(y_{1},y_{2})\right|},\delta_{1}\right)$. In the other case we take $\delta=\delta_{1}$.

The main result we prove is the following.

Theorem 2.1. *Let $\varphi\left(y_{1},y_{2}\right)=y_{2}^{l}P\left(y_{1},y_{2}\right)$ where $l=\frac{m}{2}$ and P is a homogeneous polynomial function of degree l such that $P\left(1,0\right)\neq0$ and $y_{2}\nmid P\left(y_{1},y_{2}\right)$. Let V_{δ} be defined as above and let $E_{V_{\delta}}$ be the corresponding type set. Then $E_{V_{\delta}}$ is the closed polygonal region with vertices $\left(0,0\right),\left(1,1\right),\left(\frac{2l+1}{2l+2},\frac{2l-1}{2l+2}\right)$ and $\left(\frac{3}{2l+2},\frac{1}{2l+2}\right)$.*

Standard arguments (see, for example Lemma 2 and Lemma 3 in [3]) imply the following result.

Lemma 2.2. *If $\left(\frac{1}{p},\frac{1}{q}\right)\in E_{\mu_{\delta}}$ then $\frac{1}{q}\leq\frac{1}{p}$, $\frac{1}{q}\geq\frac{3}{p}-2$ and $\frac{1}{q}\geq\frac{1}{p}-\frac{1}{l+1}$.*

So, since $\left\|T_{\mu_{\delta}}\right\|_{1,1}<\infty$, by duality arguments it only remains to prove that

$$\left\|T_{\mu_{\delta}}\right\|_{\frac{2l+2}{2l+1},\frac{2l+2}{2l-1}}<\infty. \tag{2.1}$$

We set $Q_0 = \left[\frac{1}{4}, 2\right] \times \left[\frac{\delta}{64}, \frac{\delta}{8}\right]$. We take a truncation function $\theta \in C^\infty(\mathbb{R}^2)$, $\theta(y_1, y_2) \geq 0$, $\operatorname{supp}\theta \subset Q_0$ and $\theta(y_1, y_2) = 1$ on $\left[\frac{1}{2}, 1\right] \times \left[\frac{\delta}{32}, \frac{\delta}{16}\right]$. We define, for $\varepsilon, \gamma > 0$, the biparametric family of dilations on $\mathbb{R}^2$ and $\mathbb{R}^3$ given by $(\varepsilon, \gamma) \circ (y_1, y_2) = (\varepsilon y_1, \gamma y_2)$ and $(\varepsilon, \gamma) \circ (y_1, y_2, y_3) = \left(\varepsilon y_1, \gamma y_2, \varepsilon^l \gamma^l y_3\right)$ repectively. Also, for $j, k \geq 0$, we set $Q_{j,k} = \left(2^{-j}, 2^{-k}\right) \circ Q_0$.

For $f \in S(\mathbb{R}^3)$, we define

(2.2) $$T_{j,k} f(x_1, x_2, x_3) = \int f(x_1 - y_1, x_2 - y_2, x_3 - \varphi(y_1, y_2))\, \theta\left(2^j y_1, 2^k y_2\right) dy_1 dy_2$$

so for $f \geq 0$,

(2.3) $$T_{\mu_{\frac{\delta}{8}}} f \leq c \sum_{0 \leq j \leq k} T_{j,k} f.$$

To study $\sum\limits_{0 \leq j \leq k} T_{j,k} f$, we will adapt the argument developed by M. Christ (see [1]) to the setting of biparametric dilations. First of all, we prove the following

Proposition 2.3. *There exists a positive constant* $c > 0$ *such that for* $0 \leq j \leq k$,

$$\|T_{j,k}\|_{\frac{2l+2}{2l+1}, \frac{2l+2}{2l-1}} \leq c.$$

Proof.

$$\begin{aligned} &T_{j,k} f(x_1, x_2, x_3) \\ &= \int f(x_1 - y_1, x_2 - y_2, x_3 - \varphi(y_1, y_2))\, \theta\left(2^j y_1, 2^k y_2\right) dy_1 dy_2 \\ &= 2^{-(j+k)} \int f\left(x_1 - 2^{-j} y_1, x_2 - 2^{-k} y_2, x_3 - \varphi\left(2^{-j} y_1, 2^{-k} y_2\right)\right) \theta(y_1, y_2)\, dy_1 dy_2 \\ &= 2^{-(j+k)} T^{(j-k)} f_{j,k}\left(2^j x_1, 2^k x_2, 2^{(j+k)l} x_3\right), \end{aligned}$$

where we denote

$$T^{(j)} f(x_1, x_2, x_3) = \int f\left(x_1 - y_1, x_2 - y_2, x_3 - y_2^l P\left(y_1, 2^j y_2\right)\right) \theta(y_1, y_2)\, dy_1 dy_2$$

and

$$f_{j,k}(x_1, x_2, x_3) = f\left(\left(2^{-j}, 2^{-k}\right) \circ (x_1, x_2, x_3)\right).$$

So

(2.4) $$\|T_{j,k} f(x_1, x_2, x_3)\|_q = 2^{(j+k)\left(\frac{1+l}{p} - \frac{1+l}{q} - 1\right)} \left\|T^{(j-k)}\right\|_{p,q} \|f\|_p .$$

Now,

$$\det\left(y_2^l P\left(y_1, 2^{j-k} y_2\right)\right)'' = 2^{(2-2l)(j-k)} \det(\varphi)''\left(y_1, 2^{j-k} y_2\right).$$

so as in the proof of Lemma 4 in [3] we obtain that there exists $c > 0$ such that $\left\|T^{(j-k)}\right\|_{\frac{2l+2}{2l+1}, \frac{2l+2}{2l-1}} \leq c$ for $0 \leq j \leq k$, and the proposition follows. □

We take $0 \leq j \leq k$, and denote by $\mu_{j,k}$ and $\mu^{(j)}$ the measures associated to $T_{j,k}$ and $T^{(j)}$ respectively. For $\xi = (\xi_1, \xi_2, \xi_3)$,

$$\widehat{\mu^{(j-k)}}(\xi) = \int e^{-i\left(\xi_1 y_1 + \xi_2 y_2 + \xi_3 y_2^l P\left(y_1, 2^{j-k} y_2\right)\right)} \theta(y_1, y_2)\, dy_1 dy_2.$$

If for some ξ on the unit sphere, $\Omega_\xi^{(j-k)}(y_1, y_2) = \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_2^l P\left(y_1, 2^{j-k} y_2\right)$ has a critical point belonging to the $\operatorname{supp}\theta$, then

$$\xi_1 + \xi_3 y_2^l P_1\left(y_1, 2^{j-k} y_2\right) = 0$$

and

$$\xi_2+\xi_3\left(2^{j-k}y_2^lP_2\left(y_1,2^{j-k}y_2\right)+ly_2^{l-1}P\left(y_1,2^{j-k}y_2\right)\right)=0,$$

but then, since $P_1(y)\neq 0$ for $y\in V_{\delta_1}$, from the first equation we obtain that there exist constants $a,b\in\mathbb{Z}$ with $a<b$ such $2^a|\xi_3|\leq|\xi_1|\leq 2^b|\xi_3|$, and, from the second one and the choice of δ we obtain constants $c,d\in\mathbb{Z}^2$ with $c<d$ such that $2^c|\xi_3|\leq|\xi_2|\leq 2^d|\xi_3|$. So ξ belongs to the cone

$$C_0=\left\{\xi\in\mathbb{R}^3:2^a|\xi_3|<|\xi_1|<2^b|\xi_3|\,,\;2^c|\xi_3|<|\xi_2|<2^d|\xi_3|\right\}.$$

Lemma 2.4. *Suppose C_0 is as above. Then the family of cones $\left\{\left(2^j,2^k\right)\circ C_0\right\}_{j,k\in\mathbb{Z}}$ has finite overlapping (i.e., $\#\left\{(j,k)\in\mathbb{Z}^2:C_0\cap\left(\left(2^j,2^k\right)\circ C_0\right)\neq\emptyset\right\}<\infty$).*

Proof. We suppose $\xi\in C_0$ and $\left(2^j,2^k\right)\circ\xi\in C_0$, then

$$2^a|\xi_3|<|\xi_1|<2^b|\xi_3|\,,\qquad 2^c|\xi_3|<|\xi_2|<2^d|\xi_3|$$

and

$$2^{(j+k)l+a}|\xi_3|<2^j|\xi_1|<2^{(j+k)l+b}|\xi_3|\,,$$
$$2^{(j+k)l+c}|\xi_3|<2^k|\xi_2|<2^{(j+k)l+d}|\xi_3|$$

so

$$2^j|\xi_1|<2^{(j+k)l+b}|\xi_3|<2^{(j+k)l+b-a}|\xi_1|$$

and

$$2^b|\xi_3|>|\xi_1|>2^{-j}2^{(j+k)l+a}|\xi_3|\,,$$

so

$$a-b-kl<j(l-1)<b-a-kl,$$

analogously we obtain

$$c-d-jl<k(l-1)<d-c-jl,$$

thus

$$\frac{(c-d)(l-1)+(a-b)l}{l^2-(l-1)^2}<k<\frac{(d-c)(l-1)+(b-a)l}{l^2-(l-1)^2}$$

and so

$$\frac{a-b}{l-1}-l\frac{(d-c)(l-1)+(b-a)l}{\left(l^2-(l-1)^2\right)(l-1)}<j<\frac{(b-a)}{l-1}+l\frac{(d-c)(l-1)+(b-a)l}{\left(l^2-(l-1)^2\right)(l-1)}.$$

□

We define $m_0(\xi)=n(\xi_1,\xi_3)\,r(\xi_2,\xi_3)$ where n and r belong to $C^\infty\left(R^2-\{0\}\right)$, are homogeneous of degree zero with respect to the isotropic dilations,

$$\operatorname{supp}n\subset\left\{(\xi_1,\xi_3):2^{a-1}|\xi_3|<|\xi_1|<2^{b+1}|\xi_3|\right\}$$

$n\geq 0$ and $n\equiv 1$ on $\left\{(\xi_1,\xi_3):2^a|\xi_3|<|\xi_1|<2^b|\xi_3|\right\}$,

$$\operatorname{supp}r\subset\left\{(\xi_2,\xi_3):2^{c-1}|\xi_3|<|\xi_2|<2^{d+1}|\xi_3|\right\},$$

$r\geq 0$ and $r\equiv 1$ on $\left\{(\xi_2,\xi_3):2^c|\xi_3|<|\xi_2|<2^d|\xi_3|\right\}$, so m_0 is homogeneous of degree zero with respect to the isotropic dilations, it belongs to C^∞ on each octant of $\mathbb{R}^3$, $m_0\geq 0$, $m_0\equiv 1$ on C_0 and

$$\operatorname{supp}m_0\subset\widetilde{C_0}=\left\{\xi\in\mathbb{R}^3:2^{a-1}|\xi_3|<|\xi_1|<2^{b+1}|\xi_3|\,,\;2^{c-1}|\xi_3|<|\xi_2|<2^{d+1}|\xi_3|\right\}.$$

For $(j,k)\in\mathbb{Z}^2$, we define $m_{j,k}(\xi)=m_0\left(\left(2^{-j},2^{-k}\right)\circ\xi\right)$ and $\mathfrak{Q}_{j,k}$ the operator with multiplier $m_{j,k}$. If ξ belongs to an open octant of $\mathbb{R}^3$ then ξ belongs to $\left(2^j,2^k\right)\circ C_0$ for some $(j,k)\in\mathbb{Z}^2$ (indeed $2^{-k}\sim\frac{|\xi_1|}{|\xi_3|}$ and $2^{-j}\sim\frac{|\xi_2|}{|\xi_3|}$) and from the previous lemma, it belongs to a finite number of them (independent of ξ). So $\sum_{(j,k)\in\mathbb{Z}^2} m_{j,k}(\xi) \le c$. Now it is easy to check that, for $1<p<\infty$, there exists $A_p>0$ such that for $f\in L^2\cap L^p$ and any choice of $\varepsilon_{j,k}=\pm 1$,

$$\left\| \sum_{(j,k)\in\mathbb{Z}^2} \varepsilon_{j,k}\mathfrak{Q}_{j,k} f \right\|_p \le A_p \|f\|_p . \tag{2.5}$$

Indeed, we now show that

$$m(\xi) = \sum_{(j,k)\in\mathbb{Z}^2} \varepsilon_{j,k} m_{j,k}(\xi)$$

satisfies the hypothesis of the Marcinkiewicz Theorem, as stated in Theorem 6' in [7].

We have just observed that

$$|m(\xi)| \le \sum_{(j,k)\in\mathbb{Z}^2} m_{j,k}(\xi) \le c.$$

Now we want to estimate $\left|\frac{\partial}{\partial\xi_1} m(\xi)\right|$. We recall that $\frac{\partial}{\partial\xi_1} m_0$ is homogeneous of degree -1. We pick ξ in an open octant. In a small neighborhood of ξ only finitely many $(j,k)\in\mathbb{Z}^2$ (independent of ξ) are involved. For each one of them,

$$\begin{aligned}\frac{\partial}{\partial\xi_1} m_{j,k}(\xi) &= 2^{-j}\frac{\partial}{\partial\xi_1} m_0\left(2^{-j}\xi_1, 2^{-k}\xi_2, 2^{-(j+k)l}\xi_3\right) \\ &\le c2^{-j}\left|2^{-j}\xi_1, 2^{-k}\xi_2, 2^{-(j+k)l}\xi_3\right|^{-1} \le c2^{-j}\left|2^{-j}\xi_1\right|^{-1},\end{aligned}$$

so

$$\sup_{\xi_2,\xi_3} \int_{2^s}^{2^{s+1}} \left|\frac{\partial}{\partial\xi_1} m(\xi)\right| d\xi_1 \le c,$$

and in a similar way (using the homogeneity of the derivatives of $m_{j,k}$) we obtain that for each $0<k\le 3$,

$$\sup_{\xi_{k+1},\dots,\xi_3} \int_\rho \left|\frac{\partial^k}{\partial\xi_1 \dots \partial\xi_k} m(\xi)\right| d\xi_1 \le c,$$

as ρ ranges over dyadic rectangles of $\mathbb{R}^k$ and that this inequality holds for every one of the six pemutations of the variables ξ_1,ξ_2,ξ_3.

We now define $h(\xi)\in C^\infty(\mathbb{R}^3)$, $h\ge 0$, $h\equiv 1$ on the unit ball of $\mathbb{R}^3$, $h_{j,k}(\xi) = h\left(\left(2^{-j},2^{-k}\right)\circ\xi\right)$ and $R_{j,k}$ the operators with multipliers $h_{j,k}$.

Lemma 2.5. *There exists a constant $C>0$, independent of K, such that*

$$\left\| \sum_{0\le j\le k\le K} T_{j,k} R_{j,k} \right\|_{\frac{2l+2}{2l+1},\frac{2l+2}{2l-1}} \le C.$$

Proof. Let $K_{j,k}$ be the kernel of $T_{j,k}R_{j,k}$. A computation shows that,

$$K_{j,k}(x) = 2^{(j+k)l}\left(\mu^{(j-k)} * h^{\wedge\vee}\right)\left(\left(2^j,2^k\right)\circ x\right).$$

Thus

$$\sum_{0\le j\le k\le K} |K_{j,k}(\xi)| \le \sum_{0\le j\le k} 2^{(j+k)l}\left|G^{(j,k)}\left(\left(2^j,2^k\right)\circ\xi\right)\right|$$

with $G^{(j,k)}$ defined by $\left(G^{(j,k)}\right)^{\wedge} = \left(\mu^{(j-k)}\right)^{\wedge} h$. Since $j-k \le 0$, as in Lemma 7 in [3] we obtain that $\left(G^{(j,k)}\right)^{\wedge} \in S\left(\mathbb{R}^3\right)$ with each seminorm bounded on j,k, it follows that the same holds for $G^{(j,k)}$. Now

$$\sum_{0\le j\le k} 2^{(j+k)l} \left|G^{(j,k)}\left(\left(2^j,2^k\right)\circ\xi\right)\right| \le \sum_{j,k,h\ge 0} 2^{ja+ka+ha}\left|G^{(j,k,h)}\left(2^j\xi_1,2^k\xi_2,2^h\xi_3\right)\right|$$

with $a=\frac{l}{l+1}$, $G^{(j,k,h)} = G^{(j,k)}$ for $h = l\,(j+k)$ and $G^{(j,k,h)} = 0$ otherwise. It is well known that from the uniform boundedness properties of $G^{(j,k,h)}$ it follows that

$$\sum_{j,k,h\ge 0} 2^{ja+ka+ha}\left|G^{(j,k,h)}\left(2^j\xi_1,2^k\xi_2,2^h\xi_3\right)\right| \le \frac{c}{|\xi_1|^a\,|\xi_2|^a\,|\xi_3|^a},$$

so

$$\sum_{0\le j\le k\le K}\left|K_{j,k}(\xi)\right| \le \frac{c}{|\xi_1|^{\frac{l}{l+1}}\,|\xi_2|^{\frac{l}{l+1}}\,|\xi_3|^{\frac{l}{l+1}}},$$

so $\sum\limits_{0\le j\le k\le K} T_{j,k}R_{j,k}$ convolves $L^p\left(\mathbb{R}^3\right)$ into $L^q\left(\mathbb{R}^3\right)$ for $\frac{1}{q}=\frac{1}{p}-\frac{1}{l+1}$ with bounds independent of K. □

Lemma 2.6. *There exists a constant $C>0$, independent of K, such that*

$$\left\|\sum_{1\le j\le k\le K} T_{j,k}\left(I-P_{j,k}\right)\left(I-\mathfrak{Q}_{j,k}\right)\right\|_{\frac{2l+2}{2l+1},\frac{2l+2}{2l-1}} \le C.$$

Proof. The kernel $H_{j,k}$ of

$$\sum_{1\le j\le k\le K} T_{j,k}\left(I-P_{j,k}\right)\left(I-\mathfrak{Q}_{j,k}\right)$$

satisfies

$$\sum_{1\le j\le k\le K}\left|H_{j,k}(\xi)\right| \le \sum_{0\le j\le k} 2^{(j+k)l}\left|g^{(j,k)}\left(\left(2^j,2^k\right)\circ\xi\right)\right|$$

with $g^{(j,k)}$ defined by $\left(g^{(j,k)}\right)^{\wedge} = \left(\mu^{(j-k)}\right)^{\wedge}(1-h)(1-m_0)$.

Observe that, from Lemma 7 in [3], we have $\left(\mu^{(j-k)}\right)^{\wedge}(1-h)(1-m_0)\in S(\mathbb{R}^3)$ with each seminorm bounded on j,k. From this fact the proof follows as in the previous lemma. □

Proof of the theorem. We have just observed that it is enough to prove (2.1). Since we can suppose $f\ge 0$, by (2.3), we need only check that there exists $C>0$, independent of K such that

$$\left\|\sum_{0\le j\le k\le K} T_{j,k}\right\|_{\frac{2l+2}{2l+1},\frac{2l+2}{2l-1}} \le C,$$

where $T_{j,k}$ are defined by (2.2). For a constant $c_0>0$, we define $\mathfrak{Q}'_{j,k} = \sum\limits_{|i-j|\le c_0}\mathfrak{Q}_{i,k}$. So $\mathfrak{Q}'_{j,k}$ have the same properties as $\mathfrak{Q}_{j,k}$ and $\mathfrak{Q}'_{j,k}\circ\mathfrak{Q}_{j,k} = \mathfrak{Q}_{j,k}$ thus we have that (2.5) holds for $\mathfrak{Q}'_{j,k}$. Then, for $1<p<\infty$ and

$$F=\{f_{j,k}\}_{j,k\ge 0}\in L^p\left(l^2\right), \qquad \left\|\sum_{j,k\ge 0}\mathfrak{Q}'_{j,k}f_{j,k}\right\|_p \le c_p\,\|F\|_{L^p(l^2)}.$$

We decompose

$$\sum_{0\le j\le k\le K} T_{j,k} f = \sum_{0\le j\le k\le K} T_{j,k}\left(I-P_{j,k}\right)\left(I-\mathfrak{Q}'_{j,k}\right) f + \sum_{0\le j\le k\le K} T_{j,k} P_{j,k} f + \sum_{0\le j\le k\le K} T_{j,k}\mathfrak{Q}'_{j,k}\left(I-P_{j,k}\right) f.$$

Now, proceeding as in [1], the theorem follows from Proposition 2.3, Lemmas 2.5 and 2.6 and the remarks in [8, p. 85] concerning the multiparameter maximal function. □

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