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CONVOLUTION CONDITIONS FOR SPIRALLIKENESS AND CONVEX SPIRALLIKENESS OF CERTAIN MEROMORPHIC p-VALENT FUNCTIONS

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ABSTRACT. In the present investigation, the authors derive necessary and sufficient conditions for spirallikeness and convex spirallikeness of a suitably normalized meromorphic p-valent function in the punctured unit disk, using convolution. Also we give an application of our result to obtain a convolution condition for a class of meromorphic functions defined by a linear operator.

Key words and phrases: Meromorphic p-valent functions, Analytic functions, Starlike functions, Convex functions, Spirallike functions, Convex Spirallike functions, Hadamard product (or Convolution), Subordination, Linear operator.

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1. INTRODUCTION

Let Σ_p be the class of *meromorphic* functions

$$f(z) = \frac{1}{z^p} + \sum_{n=1-p}^{\infty} a_n z^n \quad (p \in \mathbb{N} := \{1, 2, 3, \dots\}), \tag{1.1}$$

which are *analytic* and p-valent in the punctured unit disk

$$\mathbb{E}^* := \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} = \mathbb{E} \setminus \{0\},$$

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where $\mathbb{E} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$.

For two functions f and g analytic in $\mathbb{E}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{E}$ and write

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in \mathbb{E}),$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{E}$ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1 \quad (z \in \mathbb{E}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{E}). \tag{1.2}$$

In particular, if the function g is univalent in $\mathbb{E}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{E}) \subset g(\mathbb{E}).$$

We define two subclasses of meromorphic p-valent functions in the following:

Definition 1.1. Let $|\lambda| < \frac{\pi}{2}$ and $p \in \mathbb{N}$. Let φ be an analytic function in the unit disk $\mathbb{E}$. We define the classes $S_p^\lambda(\varphi)$ and $C_p^\lambda(\varphi)$ by

$$S_p^\lambda(\varphi) := \left\{ f \in \Sigma_p \ : \ \frac{zf'(z)}{f(z)} \prec -pe^{-i\lambda}\left[\cos\lambda\, \varphi(z) + i\sin\lambda\right] \right\}, \tag{1.3}$$

$$C_p^\lambda(\varphi) := \left\{ f \in \Sigma_p \ : \ 1 + \frac{zf''(z)}{f'(z)} \prec -pe^{-i\lambda}\left[\cos\lambda\, \varphi(z) + i\sin\lambda\right] \right\}. \tag{1.4}$$

Analogous to the well known Alexander equivalence [2], we have

$$f \in C_p^\lambda(\varphi) \Leftrightarrow -\frac{1}{p} zf' \in S_p^\lambda(\varphi) \quad (p \in \mathbb{N}). \tag{1.5}$$

Remark 1.1. For

$$\varphi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \le B < A \le 1), \tag{1.6}$$

we set

$$S_p^\lambda(\varphi) =: S_p^\lambda[A, B] \quad \text{and} \quad C_p^\lambda(\varphi) =: C_p^\lambda[A, B].$$

For $\lambda = 0$, we write

$$S_p^0(\varphi) =: S_p^*(\varphi) \quad \text{and} \quad C_p^0(\varphi) =: C_p(\varphi),$$

$$S_p^0[A, B] =: S_p[A, B] \quad \text{and} \quad C_p^0[A, B] =: C_p[A, B].$$

For $0 \le \alpha < 1$, the classes $S_p^\lambda[1 - 2\alpha, -1]$ and $C_p^\lambda[1 - 2\alpha, -1]$ reduces to the classes $S_p^\lambda(\alpha)$ and $C_p^\lambda(\alpha)$ of *meromorphic p-valently λ-spirallike functions of order α* and *meromorphic p-valently λ-convex spirallike functions of order α* in $\mathbb{E}^*$ respectively:

$$S_p^\lambda(\alpha) := \left\{ f \in \Sigma_p \ : \ \Re\left\{ e^{i\lambda} \frac{zf'(z)}{f(z)} \right\} < -p\alpha\cos\lambda \quad (0 \le \alpha < 1; |\lambda| < \frac{\pi}{2}) \right\},$$

$$C_p^\lambda(\alpha) := \left\{ f \in \Sigma_p \ : \Re\left\{ e^{i\lambda} \left(1 + \frac{zf''(z)}{f'(z)}\right) \right\} < -p\alpha\cos\lambda \quad (0 \le \alpha < 1; |\lambda| < \frac{\pi}{2}) \right\}.$$

The classes $S_p^0[1 - 2\alpha, -1]$ and $C_p^0[1 - 2\alpha, -1]$ reduces to the classes $S_p^*(\alpha)$ and $C_p(\alpha)$ of *meromorphic p-valently starlike functions of order α* and *meromorphic p-valently convex functions of order α* in $\mathbb{E}^*$ respectively.

For two functions $f(z)$ given by (1.1) and

$$g(z) = \frac{1}{z^p} + \sum_{n=1-p}^{\infty} b_n z^n \quad (p \in \mathbb{N}), \tag{1.7}$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := \frac{1}{z^p} + \sum_{n=1-p}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{1.8}$$

Many important properties of certain subclasses of meromorphic p-valent functions were studied by several authors including Aouf and Srivastava [1], Joshi and Srivastava [3], Liu and Srivastava [4], Liu and Owa [5], Liu and Srivastava [6], Owa *et al.* [7] and Srivastava *et al.* [9]. Motivated by the works of Silverman *et al.* [8], Liu and Owa [5] have obtained the following Theorem 1.2 with $\lambda = 0$ for the class $S_p^*(\alpha)$ and Liu and Srivastava [6] have obtained it for the classes $S_p^\lambda(\alpha)$ (with a slightly different definition of the class).

Theorem 1.2. [6, Theorem 1, p. 14] *Let $f(z) \in \Sigma_p$. Then*

$$f \in S_p^\lambda(\alpha) \quad (0 \leq \alpha < 1; |\lambda| < \frac{\pi}{2}; p \in \mathbb{N})$$

if and only if

$$f(z) * \left[\frac{1 - \Omega z}{z^p(1-z)^2}\right] \neq 0 \quad (z \in \mathbb{E}^*),$$

where

$$\Omega := \frac{1 + x + 2p(1-\alpha)\cos\lambda e^{-i\lambda}}{2p(1-\alpha)\cos\lambda e^{-i\lambda}}, \quad |x| = 1.$$

In the present investigation, we extend the Theorem 1.2 for the above defined class $S_p^\lambda(\varphi)$. As a consequence, we obtain a convolution condition for the functions in the class $C_p^\lambda(\varphi)$. Also we apply our result to obtain a convolution condition for a class of meromorphic functions defined by a linear operator.

2. CONVOLUTION CONDITION FOR THE CLASS $S_p^\lambda(\varphi)$

We begin with the following result for the general class $S_p^\lambda(\varphi)$:

Theorem 2.1. *Let φ be analytic in $\mathbb{E}$ and be defined on $\partial\mathbb{E} := \{z \in \mathbb{C} : |z| = 1\}$. The function $f \in \Sigma_p$ is in the class $S_p^\lambda(\varphi)$ if and only if*

$$f(z) * \frac{1 - \Psi z}{z^p(1-z)^2} \neq 0 \quad (z \in \mathbb{E}^*) \tag{2.1}$$

where

$$\Psi := \frac{1 + p\left\{1 - e^{-i\lambda}[\cos\lambda\,\varphi(x) + i\sin\lambda]\right\}}{p\left\{1 - e^{-i\lambda}[\cos\lambda\,\varphi(x) + i\sin\lambda]\right\}} \quad (|x| = 1; |\lambda| < \frac{\pi}{2}). \tag{2.2}$$

Proof. In view of (1.3), $f(z) \in S_p^\lambda(\varphi)$ if and only if

$$\frac{zf'(z)}{f(z)} \neq -pe^{-i\lambda}[\cos\lambda\,\varphi(x) + i\sin\lambda] \quad (z \in \mathbb{E}^*; |x| = 1; |\lambda| < \frac{\pi}{2})$$

or

$$zf'(z) + pe^{-i\lambda}[\cos\lambda\,\varphi(x) + i\sin\lambda]f(z) \neq 0 \quad (z \in \mathbb{E}^*; |x| = 1; |\lambda| < \frac{\pi}{2}). \tag{2.3}$$

For $f \in \Sigma_p$ given by (1.1), we have

$$f(z) = f(z) * \frac{1}{z^p(1-z)} \quad (z \in \mathbb{E}^*) \tag{2.4}$$

and

$$zf'(z) = f(z) * \left[\frac{1}{z^p(1-z)^2} - \frac{p+1}{z^p(1-z)}\right] \quad (z \in \mathbb{E}^*). \tag{2.5}$$

By making use of the convolutions (2.5) and (2.4) in (2.3), we have

$$f(z) * \left[\frac{1}{z^p(1-z)^2} - \frac{p+1}{z^p(1-z)} + \frac{pe^{-i\lambda}[\cos\lambda\,\varphi(x) + i\sin\lambda]}{z^p(1-z)}\right] \neq 0$$
$$(z \in \mathbb{E}^*; |x| = 1; |\lambda| < \frac{\pi}{2})$$

or

$$f(z) * \left[\frac{p\left\{e^{-i\lambda}[\cos\lambda\varphi(x) + i\sin\lambda] - 1\right\}}{z^p(1-z)^2}\right.$$
$$\left. + \frac{\left[1 + p\left\{1 - e^{-i\lambda}[\cos\lambda\varphi(x) + i\sin\lambda]\right\}\right] z}{z^p(1-z)^2}\right] \neq 0$$
$$(z \in \mathbb{E}^*; |x| = 1; |\lambda| < \frac{\pi}{2}),$$

which yields the desired convolution condition (2.1) of Theorem 2.1. □

By taking $\lambda = 0$ in the Theorem 2.1, we obtain the following result for the class $S_p^*(\varphi)$.

Corollary 2.2. *Let φ be analytic in $\mathbb{E}$ and be defined on $\partial\mathbb{E}$. The function $f \in \Sigma_p$ is in the class $f \in S_p^*(\varphi)$ if and only if*

$$f(z) * \frac{1 - \Upsilon z}{z^p(1-z)^2} \neq 0 \quad (z \in \mathbb{E}^*) \tag{2.6}$$

where

$$\Upsilon := \frac{1 + p\,(1 - \varphi(x))}{p\,(1 - \varphi(x))} \quad (|x| = 1). \tag{2.7}$$

By taking $\varphi(z) = (1 + Az)/(1 + Bz)$, $-1 \leq B < A \leq 1$ in Theorem 2.1, we obtain the following result for the class $S_p^\lambda[A, B]$.

Corollary 2.3. *The function $f \in \Sigma_p$ is in the class $S_p^\lambda[A, B]$ if and only if*

$$f(z) * \frac{1 - \Upsilon z}{z^p(1-z)^2} \neq 0 \quad (z \in \mathbb{E}^*) \tag{2.8}$$

where

$$\Upsilon := \frac{x - B + p(A - B)\cos\lambda e^{-i\lambda}}{p(A - B)\cos\lambda e^{-i\lambda}} \quad (|x| = 1). \tag{2.9}$$

Remark 2.4. By taking $A = 1 - 2\alpha$, $B = -1$ in the above Corollary 2.3, we obtain Theorem 1.2 of Liu and Srivastava [6].

3. CONVOLUTION CONDITION FOR THE CLASS $C_p^\lambda(\varphi)$

By making use of Theorem 2.1, we obtain a convolution condition for functions in the class $C_p^\lambda(\varphi)$ in the following:

Theorem 3.1. *Let φ be analytic in $\mathbb{E}$ and be defined on $\partial\mathbb{E}$. The function $f \in \Sigma_p$ is in the class $C_p^\lambda(\varphi)$ if and only if*

$$f(z) * \frac{p - [2 + p + (p-1)\Psi]z + (p+1)\Psi z^2}{z^p(1-z)^2} \neq 0 \quad (z \in \mathbb{E}^*) \tag{3.1}$$

where Ψ is given by (2.2).

Proof. In view of the Alexander-type equivalence (1.5), we find from Theorem 2.1 that $f \in C_p^\lambda(\varphi)$ if and only if

$$zf'(z) * \frac{1 - \Psi z}{z^p(1-z)^2} = f(z) * z\left(\frac{1 - \Psi z}{z^p(1-z)^2}\right)' \neq 0 \quad (z \in \mathbb{E}^*)$$

which readily yields the desired assertion (3.1) of Theorem 3.1. □

By taking $\lambda = 0$ in the Theorem 3.1, we obtain the following result of the class $C_p(\varphi)$.

Corollary 3.2. *Let φ be analytic in $\mathbb{E}$ and be defined on $\partial\mathbb{E}$. The function $f \in \Sigma_p$ is in the class $f \in C_p(\varphi)$ if and only if*

$$f(z) * \frac{p - [2 + p + (p-1)\Upsilon]z + (p+1)\Upsilon z^2}{z^p(1-z)^2} \neq 0 \quad (z \in \mathbb{E}^*) \tag{3.2}$$

where Υ is given by (2.7).

4. CONVOLUTION CONDITIONS FOR A CLASS OF FUNCTION DEFINED BY LINEAR OPERATOR

We begin this section by defining a class $\mathcal{T}_{n+p-1}(\varphi)$. First of all for a function $f(z) \in \Sigma_p$, define $D^{n+p-1}f(z)$ by

$$\begin{aligned} D^{n+p-1}f(z) &= f(z) * \left[\frac{1}{z^p(1-z)^{n+p}}\right] \\ &= \frac{\left(z^{n+2p-1}f(z)\right)^{(n+p-1)}}{(n+p-1)!z^p} \\ &= \frac{1}{z^p} + \sum_{m=1-p}^{\infty} \frac{(m+n+2p-1)!}{(n+p-1)!(m+p)!} a_m z^m. \end{aligned}$$

By making use of the operator $D^{n+p-1}f(z)$, we define the class $\mathcal{T}_{n+p-1}(\varphi)$ by

$$\mathcal{T}_{n+p-1}(\varphi) = \left\{ f(z) \in \Sigma_p \ : \ \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} \prec \varphi(z) \right\}.$$

When

$$\varphi(z) = \frac{1 + (1-2\gamma)z}{1-z}$$

where

$$\gamma = \frac{n + p(2-\alpha)}{n+p} \quad (0 \leq \alpha < 1),$$

the class $\mathcal{T}_{n+p-1}(\varphi)$ reduces to the following class $\mathcal{T}_{n+p-1}(\alpha)$ studied by Liu and Owa [5]:

$$\mathcal{T}_{n+p-1}(\alpha) = \left\{ f(z) \in \Sigma_p \ : \ \Re\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - \frac{n+2p}{n+p} \right) < -\frac{p\alpha}{n+p} \right\}.$$

By making use of Corollary 2.2, we prove the following result for the class $\mathcal{T}_{n+p-1}(\varphi)$:

Theorem 4.1. *The function $f(z) \in \Sigma_p$ is in the class $\mathcal{T}_{n+p-1}(\varphi)$ if and only if*

$$f(z) * \frac{1 + [(n+p)(1-\Omega) - 1]z}{z^p(1-z)^{n+p+1}} \neq 0 \quad (z \in E^*; |x| = 1), \tag{4.1}$$

where Ω is given by

$$\Omega := \frac{1 + (n+p)(\varphi(x) - 1)}{(n+p)(\varphi(x) - 1)} \quad (|x| = 1). \tag{4.2}$$

Proof. By making use of the familiar identity

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - (n+2p)D^{n+p-1}f(z),$$

we have

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = (n+p)\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - (n+2p),$$

and therefore, by using the definition of the class $\mathcal{T}_{n+p-1}(\varphi)$, we see that $f(z) \in \mathcal{T}_{n+p-1}(\varphi)$ if and only if

$$D^{n+p-1}f(z) \in S_p^*\left(\frac{n+2p}{p} - \frac{n+p}{p}\varphi(z) \right).$$

Then, by applying Corollary 2.2 for the function $D^{n+p-1}f(z)$, we have

$$D^{n+p-1}f(z) * \frac{1-\Omega z}{z^p(1-z)^2} \neq 0, \tag{4.3}$$

where

$$\begin{aligned}\Omega &= \frac{1 + p\left[1 - \left\{\frac{n+2p}{p} - \frac{n+p}{p}\varphi(x)\right\}\right]}{p\left[1 - \left\{\frac{n+2p}{p} - \frac{n+p}{p}\varphi(x)\right\}\right]} \\ &= \frac{1 + (n+p)\left(\varphi(x) - 1\right)}{(n+p)\left(\varphi(x) - 1\right)}, \qquad |x| = 1.\end{aligned}$$

Since

$$D^{n+p-1}f(z) = f(z) * \left[\frac{1}{z^p(1-z)^{n+p}}\right],$$

the condition (4.3) becomes

$$f(z) * \left(g(z) * \frac{1-\Omega z}{z^p(1-z)^2} \right) \neq 0 \tag{4.4}$$

where

$$g(z) = \frac{1}{z^p(1-z)^{n+p}}.$$

By making use of the convolutions (2.5) and (2.4), it is fairly straight forward to show that

$$g(z) * \frac{1-\Omega z}{z^p(1-z)^2} = \frac{1 + [(n+p)(1-\Omega) - 1]z}{z^p(1-z)^{n+p+1}}. \tag{4.5}$$

By using (4.5) in (4.4), we see that the assertion in (4.1) follows and thus the proof of our Theorem 4.1 is completed. □

By taking

$$\varphi(z) = \frac{1 + (1 - 2\gamma)z}{1 - z}$$

where

$$\gamma = \frac{n + p(2 - \alpha)}{n + p} \quad (0 \leq \alpha < 1)$$

in our Theorem 4.1, we obtain the following result of Liu and Owa [5]:

Corollary 4.2. *The function $f(z) \in \Sigma_p$ is in the class $\mathcal{T}_{n+p-1}(\alpha)$ if and only if*

$$f(z) * \frac{1 + [(n + p)(1 - \Omega) - 1]z}{z^p(1 - z)^{n+p+1}} \neq 0 \quad (z \in E^*; |x| = 1),$$

where Ω is given by

$$\Omega = \frac{1 + x + 2p(1 - \alpha)}{2p(1 - \alpha)} \quad (|x| = 1).$$

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