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AN INEQUALITY FOR CHEBYSHEV CONNECTION COEFFICIENTS

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ABSTRACT. Equivalent conditions are given for the nonnegativity of the coefficients of both the Chebyshev expansions and inversions of the first n polynomials defined by a certain recursion relation. Consequences include sufficient conditions for the coefficients to be positive, bounds on the derivatives of the polynomials, and rates of uniform convergence for the polynomial expansions of power series.

Key words and phrases: Chebyshev coefficient, Ultraspherical polynomial, Polynomial expansion.

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1. INTRODUCTION

In the classical theory of orthogonal polynomials a standard problem is to expand one set of orthogonal polynomials as a linear combination of another with nonnegative coefficients (see [4], [6], [8], [12]). R. Askey [2] and R. Szwarc [13] give general conditions ensuring nonnegativity in terms of underlying recurrence relations, while Askey [1] and W. F. Trench [14] determine when connection coefficients are positive. It is often desirable, especially in numerical analysis (e.g., see [9], [10], [15]), to express orthogonal polynomials in terms of Chebyshev polynomials T_n or cosine polynomials with nonnegative coefficients. In this note, we derive inequalities (2.2) on the connection coefficients of certain Chebyshev expansions that imply positivity. Consequently, we obtain bounds on polynomial derivatives, estimate uniform convergence of polynomial expansions of power series, and illustrate the optimality of Chebyshev polynomials in these contexts.

For given real sequences α_n, β_n and nonzero γ_n we consider the sequence of polynomials P_n defined by $P_{-1} = 0$, $P_0 = 1$, and for $n \geq 0$,

$$xP_n = \gamma_n P_{n+1} + \beta_n P_n + \alpha_n P_{n-1}. \tag{1.1}$$

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For example the Jacobi polynomials $P_n^{(\alpha,\beta)}$ $(\alpha, \beta > -1)$ are defined by (1.1) with

$$\alpha_n = \frac{2(\alpha+n)(\beta+n)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n)}, \qquad \beta_n = \frac{\beta^2-\alpha^2}{(\alpha+\beta+2n+2)(\alpha+\beta+2n)}$$

$$\text{and} \qquad \gamma_n = \frac{2(n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}.$$

In [5] (see also [3]), Askey and G. Gasper characterize those Jacobi polynomials that are combinations of Chebyshev polynomials with nonnegative coefficients. Szwarc ([13, Corollary 1]) gives conditions on sequences of polynomials satisfying (1.1) which imply that their Chebyshev connection coefficients are nonnegative. Our results are based on this work and the following classical expansions of the normalized ultraspherical or Gegenbauer polynomials $P_n^{\{\alpha\}} := \frac{P_n^{(\alpha,\alpha)}}{P_n^{(\alpha,\alpha)}(1)}$. We recall the factorial function $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$ when $n \geq 1$, and $(\alpha)_0 := 1$ for $\alpha \neq 0$. Then $P_n^{\{-\frac{1}{2}\}} = T_n$ and

$$P_n^{\{\alpha\}} = \sum_{m=0}^{n} c(n,m) T_m \qquad \left(\alpha \neq -\frac{1}{2}\right), \tag{1.2}$$

where $c(n,m) = 0$ if $n-m$ is odd, and otherwise

$$c(n,m) = \frac{(2-\delta_{m0})\left(\alpha+\frac{1}{2}\right)_{\frac{n-m}{2}}\left(\alpha+\frac{1}{2}\right)_{\frac{n+m}{2}}}{(2\alpha+1)_n}\binom{n}{\frac{n-m}{2}}.$$

It follows that if $\alpha > -\frac{1}{2}$ and $n-m$ is even, then $c(n,m) > 0$ and the sequence $\langle c(n,n)\rangle$ is monotonically decreasing. Moreover, we have the inversions

$$x^n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{2-\delta_{n,2k}}{2^n}\binom{n}{k} P_{n-2k}^{\{-\frac{1}{2}\}}$$

($[\cdot]$ is the greatest integer function) and

$$x^n = \sum_{m=0}^{n} d(n,m) P_m^{\{\alpha\}} \qquad \left(\alpha \neq -\frac{1}{2}\right), \tag{1.3}$$

where $d(n,m) = 0$ if $n-m$ is odd, and otherwise

$$d(n,m) = \frac{(2\alpha+1+2m)(2\alpha+1)_m\left(\frac{n-m}{2}+1\right)_{\frac{n-m}{2}}}{2^{n+1}\left(\alpha+\frac{1}{2}\right)_{\frac{n+m+2}{2}}}\binom{n}{m}$$

is positive and $\langle d(n,n)\rangle$ is decreasing. The polynomials $P_n^{\{\alpha\}}$ satisfy (1.1) for sequences such that $\alpha_n > 0$ and $\beta_n = 0$. Furthermore, $\alpha > -\frac{1}{2}$ if and only if $\alpha_n < \frac{1}{2}$.

2. The Chebyshev Expansion of P_n

The following characterizes those polynomials that satisfy (1.1) and enjoy similar expansions (1.2) and (1.3).

Theorem 2.1. *Let P_n be defined by (1.1) for given sequences α_n, β_n and γ_n, and be normalized by $P_n(1) = 1$. With n fixed, we have that*

$$P_k = \sum_{m=0}^{k} a(k,m) T_m \quad \textit{and} \quad x^k = \sum_{m=0}^{k} b(k,m) P_m \qquad (0 \leq k \leq n) \tag{2.1}$$

for nonnegative coefficients $a(k,m)$ *and* $b(k,m)$ *such that the coefficients* $a(k,k)$ *and* $b(k,k)$ *are monotonically decreasing if and only if* $0 \le \alpha_i \le \frac{1}{2}$ *and* $\beta_i = 0$ *for* $i = 0,...,n-1$. *In this case, the following properties hold.*

(i) $a(k,m) = b(k,m) = 0$ *if* $k-m$ *is odd.*

(ii) *If* $\alpha_1 > 0$, *then* $b(k,m) > 0$ *whenever* $k-m$ *is even.*

(iii) *If* $n \ge 3$ *and* K *is any subset of* $\{0,...,[\frac{n-1}{2}]\}$, *then*

$$\sum_{\frac{n-m}{2}\in K'} a(n-2,m-2) \le \sum_{\frac{n-m}{2}\in K^*} a(n-3,m-1) \tag{2.2}$$

where $K' := \{k \in K : k+1 \in K\} \cup \{\frac{n-2}{2}\}$ *if* $\frac{n-2}{2}$ *is in* K, *and* $K' := \{k \in K : k+1 \in K\}$ *otherwise; and* $K^* := K' \cup \{k \in K : k-1 \in K\}$.

(iv) *If* $n \ge 2$ *and* $\alpha_i < \frac{1}{2}$ *for* $i = 1,...,n-1$, *then* $a(k,m) > 0$ *whenever* $k-m$ *is even.*

(v) *If* $\alpha_i < \frac{1}{2}$ *for* $i = 1,...,n-3$, *then equality holds in (2.2) if and only if either* K *is the whole set* $\{0,...,[\frac{n-1}{2}]\}$ *itself (i.e., (2.2) is just* $P_{n-2}(1) = P_{n-3}(1) = 1$*) or* K' *and* K^* *are the empty sets.*

Proof. Suppose that $P_0,...,P_n$ satisfy (1.1) and (2.1) with nonnegative coefficients such that $P_k(1) = 1$, and $a(k,k)$ and $b(k,k)$ are decreasing ($0 \le k \le n$). It follows that $a(0,0) = a(1,1) = 1$ and

$$\gamma_k a(k+1,j) = -\alpha_k a(k-1,j) - \beta_k a(k,j) + \frac{1}{2}(a(k,j+1) + (1+\delta_{1j})a(k,j-1)) \tag{2.3}$$

for $k = 0,...,n-1$ (We use the convention that entries of vectors or matrices with any negative indices are zero; and $a(i,j) = 0$ if $i < j$. We also assume $\alpha_0 = 0$.). Similarly, by substituting (2.1) into both sides of $x^{k+1} = xx^k$ we have $b(0,0) = 1$ and

$$b(k+1,j) = \gamma_{j-1}b(k,j-1) + \beta_j b(k,j) + \alpha_{j+1}b(k,j+1). \tag{2.4}$$

With $j = k+1$ we conclude γ_k, $a(k,k)$ and $b(k,k)$ are positive; and with $j = k$, $\beta_k = a(k+1,k) = b(k+1,k) = 0$ for $k = 0,...,n-1$. Similarly property (i) follows by induction.

By the normalization we have $\gamma_k + \alpha_k = 1$. Thus by (2.3) and (2.4) with $j = k+1$, it follows that $\frac{1}{2} \le \gamma_k \le 1$, so $0 \le \alpha_k \le \frac{1}{2}$ ($0 \le k \le n-1$), since $a(k,k)$ and $b(k,k)$ are decreasing.

Conversely suppose that $P_0,...,P_n$ are normalized and given by (1.1) for constants such that $0 \le \alpha_i \le \frac{1}{2}$ and $\beta_i = 0$ ($0 \le i \le n-1$). It follows that the degree of P_k is k and thus P_k satisfies (2.1) for coefficients $a(k,m)$ and $b(k,m)$ generated by (2.3) and (2.4) respectively. Since $\gamma_k = 1 - \alpha_k > 0$, the above argument shows that property (i) holds and $b(k,m) \ge 0$; and since $\frac{1}{2} \le \gamma_k \le 1$, $a(k,k)$ and $b(k,k)$ are decreasing for $k = 0,...,n$.

We will show that $a(k,m) \ge 0$ simultaneously with inequality (2.2) by induction on n. Now $a(0,0) = a(1,1) = 1$, $a(2,0) = \frac{1-2\alpha_1}{2\gamma_1}$, $a(2,2) = \frac{1}{2\gamma_1}$, $\gamma_2 a(3,1) = \left(-\alpha_2 + \frac{1}{2}\right) + \frac{1}{2}a(2,0)$ and $a(3,3) = \frac{1}{4\gamma_1\gamma_2}$. Hence $a(k,m) \ge 0$ when n is either 2 or 3. Also (2.2) is easily checked when $n = 3$.

Henceforth let $n > 3$ and suppose that $a(i, i-2k) \ge 0$ ($0 \le i \le n-1$; $0 \le k \le [\frac{i}{2}]$) and that (2.2) is true for all integers n' such that $3 \le n' < n$. Let j be given, $0 \le j \le [\frac{n}{2}]$, and let K be a subset of $\{0,...,[\frac{n-1}{2}]\}$. We show that $a(n,n-2j) \ge 0$ and that

$$\sum_{k\in K'} a(n-2,n-2k-2) \le \sum_{k\in K^*} a(n-3,n-2k-1) \tag{2.5}$$

beginning with the latter.

We may assume that K is not the whole set $\{0,...,[\frac{n-1}{2}]\}$ since in the contrary case (2.5) reduces to $P_{n-2}(1) = 1 \le P_{n-3}(1) = 1$. Observe that K may be written as a disjoint union of sets of the form $\{k, ..., k+m\}$ such that there is at least one integer between any two such sets. Since K' and K^* are then corresponding disjoint unions of subsets of these sets, and both sides of (2.5) may be separated into sums over these subsets accordingly, we may assume $K = \{k, ..., k+m\}$.

Thus we wish to show

$$\sum_{i=0}^{m} \delta_{k+i,\frac{n-2}{2}} a(n-2, n-2(k+i)-2) \leq \sum_{i=0}^{m} a(n-3, n-2(k+i)-1). \tag{2.6}$$

By (2.3) we have

$$\begin{aligned}\gamma_{n-3} a(n-2, n-2(k+i)-2) &\\ = -\alpha_{n-3} a(n-4, n-2(k+i)-2) &+ \frac{1}{2}(a(n-3, n-2(k+i)-1) \\ &+ (1+\delta_{3,n-2(k+i)}) a(n-3, n-2(k+i)-3)),\end{aligned}$$

where $\delta_{3,n-2(k+i)} = 1$ if and only if n is odd and $k+i+1 = \frac{n-1}{2} = k+m$ $(\in K)$. Moreover, $a(n-3, n-2(k+i)-3) = 0$ if n is even and $\frac{n-2}{2} = k+m$. Hence the left side of (2.6) may be combined as follows:

$$\begin{aligned}&\sum_{i=0}^{m} \delta_{k+i,\frac{n-2}{2}} a(n-2, n-2(k+i)-2) \\ &= -\frac{\alpha_{n-3}}{\gamma_{n-3}} \sum_{i=0}^{m} \delta_{k+i,\frac{n-2}{2}} a(n-4, n-2(k+i)-2) \\ &\qquad + \frac{1}{\gamma_{n-3}} \sum_{i=1}^{m-1} a(n-3, n-2(k+i)-1) \\ &\qquad + \frac{1}{2\gamma_{n-3}} \left(a(n-3, n-2k-1) + \left(1+\delta_{k+m,[\frac{n-1}{2}]}\right) a(n-3, n-2(k+m)-1)\right).\end{aligned}$$

Therefore (2.6) becomes

$$\begin{aligned}&-\frac{\alpha_{n-3}}{\gamma_{n-3}} \left(\sum_{i=0}^{m} \delta_{k+i,\frac{n-2}{2}} a(n-4, n-2(k+i)-2) - \sum_{i=1}^{m-1} a(n-3, n-2(k+i)-1)\right) \\ &\qquad + \frac{1}{2\gamma_{n-3}} \left(1+\delta_{k+m,[\frac{n-1}{2}]}\right) a(n-3, n-2(k+m)-1) \\ &\qquad\qquad \leq \frac{1-2\alpha_{n-3}}{2\gamma_{n-3}} a(n-3, n-2k-1) + a(n-3, n-2(k+m)-1).\end{aligned} \tag{2.7}$$

Let us suppose first that $k+m \neq [\frac{n-1}{2}]$. Then (2.7) may be rearranged as follows:

$$\begin{aligned}&-\alpha_{n-3} \left(\sum_{i=0}^{m-1} a(n-4, n-2(k+i)-2) - \sum_{i=1}^{m-1} a(n-3, n-2(k+i)-1)\right) \\ &\qquad \leq \left(\frac{1}{2} - \alpha_{n-3}\right) (a(n-3, n-2k-1) + a(n-3, n-2(k+m)-1)),\end{aligned} \tag{2.8}$$

which is clearly true if $\alpha_{n-3} = 0$ so we assume $\alpha_{n-3} \neq 0$. With $n' = n-1$, replacing i by $i+1$ in the second sum, we seek to show

$$\sum_{i=0}^{m-2} a(n'-2, n'-2(k+i)-2) \leq \sum_{i=0}^{m-1} a(n'-3, n'-2(k+i)-1). \tag{2.9}$$

Since $k+m < [\frac{n-1}{2}]$, we have $K' = \left\{k, ..., k+m-1\right\} \subseteq \left\{0, ..., [\frac{n'-1}{2}]\right\}$. Moreover, $(K')' = \left\{k, ..., k+m-2\right\}$ where the second prime is with respect to n'. Hence (2.9) follows from the induction hypothesis

$$\sum_{k\in(K')'} a(n'-2, n'-2k-2) \leq \sum_{k\in(K')^*} a(n'-3, n-2k-1). \tag{2.10}$$

Finally suppose that $k+m = [\frac{n-1}{2}]$ so that (2.7) becomes

$$\begin{aligned} - \alpha_{n-3} \left(\sum_{i=0}^{m} \delta_{k+i, \frac{n-2}{2}} a(n-4, n-2(k+i)-2) - \sum_{i=1}^{m} a(n-3, n-2(k+i)-1) \right) \\ \leq \left(\frac{1}{2} - \alpha_{n-3}\right) a(n-3, n-2k-1). \end{aligned} \tag{2.11}$$

As above we may assume $\alpha_{n-3} \neq 0$ and wish to show

$$\sum_{i=0}^{m-1} a(n'-2, n'-2(k+i)-2) \leq \sum_{i=0}^{m} \delta_{k+i, \frac{n-2}{2}} a(n'-3, n'-2(k+i)-1). \tag{2.12}$$

If n is even then n' is odd and $(K')'$ (with respect to n') is $\{k, ..., k+m-1\}$. Thus (2.12) is a consequence of the induction hypothesis as above. On the other hand, if n is odd, then $(K')' = \{k, ..., k+m-1\}$ and (2.12) again reduces to the hypothesis.

Next we verify $a(n, n-2j) \geq 0$. Since $a(n,n) = \prod_{j=2}^{n} \frac{1}{2\gamma_{n-j+1}} > 0$ suppose further that $j \geq 1$. Let

$$K_0 := \left\{k : k \text{ integer, } 0 \leq k \leq \left[\frac{n-1}{2}\right],\ k \neq j-1, j\right\}.$$

If n is even and $j = \frac{n-2}{2}$, define $K_1 := K_0$; otherwise let

$$K_1 := \left\{k : k \text{ integer, } 0 \leq k \leq \left[\frac{n-2}{2}\right],\ k \neq j-1\right\}.$$

Note that $a(n-2, n-2k-2) = 0$ if $k = [\frac{n-1}{2}] > [\frac{n-2}{2}]$. Furthermore, the sum $\sum_{\substack{k\notin K_0 \\ k+1\in K_0}} a(n-2, n-2k-2)$ is zero unless $j+1$ is in K_0 (in which case the sum is $a(n-2, n-2j-2)$). Solving the equations $P_{n-2}(1) = 1$ and $P_{n-1}(1) = 1$ for $a(n-2, n-2j)$ and $a(n-1, n-2j+1) + a(n-1, n-2j-1)$ respectively, and substituting them into $\gamma_{n-1} a(n, n-2j)$ as given by

(2.3), we have the following identities.

$$\begin{aligned}
&\gamma_{n-1}a(n,n-2j)\\
&=\frac{1}{2}-\alpha_{n-1}+\sum_{k\in K_1}\left(\alpha_{n-1}-\frac{1}{2}\left(1-\frac{1-2\alpha_{n-2}}{2\gamma_{n-2}}\right)\right)a(n-2,n-2k-2)\\
&\quad+\alpha_{n-1}\delta_{n,2j+2}a(n-2,0)+\frac{\alpha_{n-2}}{2\gamma_{n-2}}\sum_{k\in K_0}a(n-3,n-2k-1)\\
&\quad-\frac{1}{2\gamma_{n-2}}\left(\alpha_{n-2}+\frac{1-2\alpha_{n-2}}{2}\right)\sum_{k\in K_0'}a(n-2,n-2k-2)\\
&\quad+\frac{1}{2}\delta_{1,n-2j}a(n-1,n-2j-1)\\
&=\left(\frac{1}{2}-\alpha_{n-1}\right)\sum_{k\notin K_1}a(n-2,n-2k-2)\\
&\quad+\frac{1}{2\gamma_{n-2}}\left(\frac{1}{2}-\alpha_{n-2}\right)\sum_{\substack{k\in K_1\\ k\notin K_0'}}a(n-2,n-2k-2)\\
&\quad+\alpha_{n-1}\delta_{n,2j+2}a(n-2,0)+\frac{1}{2}\delta_{1,n-2j}a(n-1,n-2j-1)\\
&\quad+\frac{\alpha_{n-2}}{2\gamma_{n-2}}\left(\sum_{k\in K_0}a(n-3,n-2k-1)-\sum_{k\in K_0'}a(n-2,n-2k-2)\right).
\end{aligned}$$

Each of the terms in the last expression is nonnegative, the final one a result of the induction assumption on (2.2). Therefore $a(n,n-2j)\geq 0$.

Property (i) has already been shown so for (ii), assume $\alpha_1>0$. Since $b(0,0)=1$ and $b(k+1,0)=\alpha_1\gamma_0 b(k-1,0)+\alpha_1\alpha_2 b(k-1,2)$, we have that $b(k+1,0)>0$ for $k+1$ even. But also $b(k+1,0)=\alpha_1 b(k,1)$ so $b(k,1)>0$ for odd k. Hence (ii) is now straightforward from (2.4).

For property (iv), suppose $\alpha_i<\frac{1}{2}$ $(i=1,...,n-1)$. By the induction argument above, the first term of the last identity for $\gamma_{n-1}a(n,n-2j)$ is positive since $k=j-1\notin K_1$. Since the other terms are nonnegative, $a(n,n-2j)>0$.

Next let $\alpha_i<\frac{1}{2}$ $(i=1,...,n-3)$ and suppose that equality holds in (2.2) for some subset K of $\{0,...,[\frac{n-1}{2}]\}$. As above, it follows that equality holds in (2.2) over each subset of K of the form $\{k,...,k+m\}$. Hence assume $K=\{k,...,k+m\}$. If $m=0$ and $k\neq\frac{n-2}{2}$ then K' and K^* are empty. If $K=\{\frac{n-2}{2}\}$, then by equality in (2.11) we have $-\alpha_{n-3}a(n-4,0)=\left(\frac{1}{2}-\alpha_{n-3}\right)a(n-3,1)$ which is impossible since the left side is nonpositive and the right side is positive by property (iv).

Therefore, let $m\geq 1$. If $k+m\neq[\frac{n-1}{2}]$ then equality holds in (2.8), and by (2.9) and (2.10) we have

$$\begin{aligned}
&-\alpha_{n-3}\left(\sum_{k\in(K')^*}a(n'-3,n'-2k-1)-\sum_{k\in(K')'}a(n'-2,n'-2k-2)\right)\\
&\qquad=\left(\frac{1}{2}-\alpha_{n-3}\right)(a(n-3,n-2k-1)+a(n-3,n-2(k+m)-1)).
\end{aligned}$$

This is impossible since both sides must be zero which implies $a(n-3,n-2(k+m)-1)=0$.

Finally, if $k+m=[\frac{n-1}{2}]$ then equality holds in (2.11) and a similar argument shows $a(n-3,n-2k-1)=0$ which by property (iv) implies $k=0$ and thus K must be the whole set $\{0,...,[\frac{n-1}{2}]\}$. □

3. BOUNDS ON POLYNOMIAL DERIVATIVES

It is well known [9] that T_n is bounded by one and for $1\le k\le n$

$$\left|T_n^{(k)}(x)\right|\le T_n^{(k)}(1)=\frac{n^2(n^2-1)(n^2-4)\cdots(n^2-(k-1)^2)}{1\cdot3\cdot5\cdots(2k-1)}$$

when $-1\le x\le 1$; and if $1\le k<n$ and $\left|T_n^{(k)}(x)\right|=T_n^{(k)}(1)$, then $x=\pm1$. More generally, it follows from a result of R.J. Duffin and A.C. Schaeffer ([7], [9, Thm. 2.24]) that if P_n is any polynomial of degree n that is bounded by one in $[-1,1]$ and $1\le k<n$, then $\left|P_n^{(k)}(x)\right|\le T_n^{(k)}(1)$ with equality holding only when $P_n=\pm T_n$ and $x=\pm1$. For the polynomials in Theorem 2.1, we may be more precise.

Corollary 3.1. *Let $P_0,...,P_n$ be defined by (1.1) with $0\le\alpha_i\le\frac12$, $\beta_i=0$, and $\gamma_i=1-\alpha_i$ for $i=0,...,n-1$. Then*

(a) $|P_n(x)|\le1=P_n(1)=T_n(1)$*; and if $n\ge1$, $|P_n(x)|=1$, and $\alpha_i<\frac12$ for $i=1,...,n-1$, then $x=\pm1$.*

(b) *If $1\le k\le n$, then*

$$\left|P_n^{(k)}(x)\right|\le P_n^{(k)}(1)\le T_n^{(k)}(1)$$

for all x in $[-1,1]$. Moreover in this case:

(i) *If $k<n$ and $\left|P_n^{(k)}(x)\right|=P_n^{(k)}(1)$, then $x=\pm1$.*

(ii) *If $P_n^{(k)}(1)=T_n^{(k)}(1)$, then $P_n=T_n$. In particular, if $n\ge2$ and $\alpha_i<\frac12$ $(i=1,...,n-1)$, then $P_n^{(k)}(1)<T_n^{(k)}(1)$.*

Proof. By Theorem 2.1, $P_n=\sum_{m=0}^n a(n,m)T_m$ where $a(n,m)\ge0$. Therefore

$$|P_n(x)|\le\sum_{m=0}^n a(n,m)\,|T_m(x)|\le P_n(1)=1.$$

Suppose that $n\ge1$ and $|P_n(x)|=1$. If $n=1$, then $x=\pm1$, so assume $n\ge2$ and $\alpha_i<\frac12$ $(i=1,...,n-1)$. Then $P_n(x)=\pm P_n(1)$ and hence $a(n,m)(1\pm T_m(x))=0$ for all m. Since $T_0=1$, $T_1=x$ and $T_2=2x^2-1$, property (iv) of Theorem 2.1 implies that $x=\pm1$.

Next assume $k\ge1$. Since $\left\langle T_m^{(k)}(1)\right\rangle_m$ is increasing,

$$\begin{aligned}\left|P_n^{(k)}(x)\right|&=\left|\sum_{m=k}^n a(n,m)T_m^{(k)}(x)\right|\\&\le\sum_{m=k}^n a(n,m)\left|T_m^{(k)}(x)\right|\\&\le\sum_{m=k}^n a(n,m)T_m^{(k)}(1)=P_n^{(k)}(1)\\&\le T_n^{(k)}(1)\sum_{m=k}^n a(n,m)\le T_n^{(k)}(1).\end{aligned}$$

Assume $1 \leq k < n$ and $\left|P_n^{(k)}(x)\right| = P_n^{(k)}(1)$. Then

$$0 = \sum_{m=k}^{n} a(n,m) \left(T_m^{(k)}(1) - \left|T_m^{(k)}(x)\right|\right),$$

where each term is nonnegative. Since $a(n,n) > 0$ we have that $\left|T_n^{(k)}(x)\right| = T_n^{(k)}(1)$ so $x = \pm 1$.

Finally suppose that $k \geq 1$ and $P_n^{(k)}(1) = T_n^{(k)}(1)$. Then $\sum_{m=k}^{n} a(n,m) = 1$ so $a(n,m) = 0$ for $m < k$. Also $a(n,m)T_m^{(k)}(1) = a(n,m)T_n^{(k)}(1)$ so $a(n,m) = 0$ for $m = k, ..., n-1$. Therefore $P_n = a(n,n)T_n$ and thus $a(n,n) = 1$.

However the previous case is impossible if $n \geq 2$ and $\alpha_i < \frac{1}{2}$ $(i = 1, ..., n-1)$ since by property (iv) of Theorem 2.1 we would have $a(n,0) > 0$ or $a(n,1) > 0$. □

Remark 3.2. For fixed k the sequence $P_n^{(k)}(1)$ of bounds is increasing. In fact by (1.1), $P_n^{(k)}(1)$ may be generated recursively as follows: Initially we have $P_1^{(1)}(1) = 1$, $P_k^{(k)}(1) = \frac{k}{1-\alpha_{k-1}} P_{k-1}^{(k-1)}(1)$ $(k \geq 2)$ and set $e_{kk} := \frac{k+\alpha_k}{1-\alpha_k} P_k^{(k)}(1)$. Then for $n \geq 1$,

$$P_{n+1}^{(k)}(1) = P_n^{(k)}(1) + e_{nk} \geq P_n^{(k)}(1) \geq 0,$$

where the differences e_{nk} are defined by

$$e_{nk} := \frac{\alpha_n}{1-\alpha_n} e_{n-1,k} + \frac{k}{1-\alpha_n} P_n^{(k-1)}(1).$$

Ultraspherical polynomials $y = P_n^{\{\alpha\}}$ with $\alpha \geq -\frac{1}{2}$ satisfy the differential equation

$$(1-x^2)y'' - 2(\alpha+1)xy' + n(n+2\alpha+1)y = 0$$

and thus a closed form for $P_n^{(k)}(1)$ is possible in this case since

$$P_n^{(k)}(1) = \frac{n(n+2\alpha+1) - (k-1)(2\alpha+1) - (k-1)^2}{2(\alpha+k)} P_n^{(k-1)}(1).$$

This extends known Chebyshev and Legendre identities ([9, p. 33], [11, p. 251]).

4. POLYNOMIAL EXPANSIONS OF POWER SERIES

A standard application of the theory of orthogonal polynomials is the least squares or uniform approximation of functions by partial sums of generalized Fourier expansions in terms of orthogonal polynomials, especially Chebyshev polynomials. The coefficients of the expansion are given by an inner product used in generating the polynomials. In our case, we may define Fourier coefficients for expansions of power series in terms of the polynomials that satisfy (1.1): Let $\sum a_i x^i$ be a convergent power series on $(-1,1)$, and for every n let P_n be a polynomial of degree n. Then $x^n = \sum_{m=0}^{n} b(n,m)P_m$ for some numbers $b(n,m)$; and we define the Fourier coefficient c_j of $\sum a_i x^i$ with respect to the sequence $\langle P_n \rangle$ by

$$c_j := \sum_i a_i b(i,j)$$

whenever this sum converges. Note that $c_{nj} := \sum_{i=0}^{n} a_i b(i,j)$ is then the jth coefficient in the expansion of the partial sum $\sum_{i=0}^{n} a_i x^i$: since $b(i,j) = 0$ for $j > i$,

$$\sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} a_i \sum_{j=0}^{n} b(i,j)P_j = \sum_{j=0}^{n} c_{nj} P_j.$$

We have the following estimate where $\|f\|$ denotes the uniform norm $\max\{|f(x)| : -1 \leq x \leq 1\}$. The optimal property of Chebyshev expansions extends a result of T.J. Rivlin and M.W. Wilson ([10], [9, Thm. 3.17]).

Corollary 4.1. *Let $\langle P_n \rangle$ be given by (1.1) with $\langle \alpha_n \rangle$, $\langle \beta_n \rangle$ and $\langle \gamma_n \rangle$ nonnegative, and suppose that $P_n(1) = 1$ for all n. If $\sum a_i$ converges absolutely, then the coefficient c_j of $\sum a_i x^i$ with respect to $\langle P_n \rangle$ exists for every j and*

$$\left| c_j - \sum_{i=0}^{n} a_i b(i,j) \right| \leq \sum_{\substack{i>n \\ i-j \text{ even}}} |a_i| .$$

Moreover, if $\sum i a_i$ converges absolutely, then

$$\left| \sum a_i x^i - \sum_{j=0}^{n} c_j P_j(x) \right| \leq \sum_{i>n} (|a_i| + |i a_i|)$$

for all x in $[-1,1]$. In this case, if $\alpha_i \leq \frac{1}{2}$ and $\beta_i = 0$ for $i = 0, ..., n-1$, and if $a_i \geq 0$ for all i and d_k is the kth Chebyshev coefficient of $\sum a_i x^i$, then

$$\left\| \sum a_i x^i - \sum_{j=0}^{n} c_j P_j \right\| \geq \left\| \sum a_i x^i - \sum_{k=0}^{n} d_k T_k \right\| . \tag{4.1}$$

In addition, if $\alpha_i < \frac{1}{2}$ ($i = 1, ..., n-1$), then equality holds in (4.1) if and only if $\sum a_i x^i$ is a polynomial of degree at most n.

Proof. Asume that $\langle \alpha_n \rangle$, $\langle \beta_n \rangle$ and $\langle \gamma_n \rangle$ are nonnegative, and $P_n(1) = 1$ for all n. By (1.1) the degree of P_n is n for all n. Thus $x^n = \sum_{m=0}^{n} b(n,m) P_m$ where $b(n,m) \geq 0$ by (2.4), and $b(n,m) \leq 1$ by the normalization since we have $1^n = \sum_{m=0}^{n} b(n,m)$.

Suppose that $\sum |a_i|$ converges. Then $\sum_i a_i b(i,j)$ converges absolutely by the comparison test so c_j exists and

$$\left| \sum_{i>n} a_i b(i,j) \right| \leq \sum_{\substack{i>n \\ i-j \text{ even}}} |a_i| .$$

Next assume $\sum |i a_i| < \infty$. Then $\sum |a_i| < \infty$ so c_j exists for every j. Thus by Corollary 3.1 we have

$$\begin{aligned} \left| \sum a_i x^i - \sum_{j=0}^{n} c_j P_j(x) \right| &\leq \left| \sum_{i>n} a_i x^i \right| + \left| \sum_{j=0}^{n} (c_j - c_{nj}) P_j(x) \right| \\ &\leq \sum_{i>n} |a_i| + \sum_{j=0}^{n} |c_j - c_{nj}| , \end{aligned}$$

where

$$\sum_{j=0}^{n} |c_j - c_{nj}| = \sum_{j=0}^{n} \left| \sum_{i>n} a_i b(i,j) \right| \leq \sum_{j=0}^{n} \sum_{i \geq n+1} \frac{1}{i} |i a_i| \leq \frac{n+1}{n+1} \sum_{i>n} |i a_i| .$$

Suppose further that $\alpha_i \leq \frac{1}{2}$, $\beta_i = 0$ ($i = 0, ..., n-1$) and a_i is nonnegative for all i. Then $c_j \geq 0$ for all j and by Theorem 2.1, $P_j = \sum_{k=0}^{j} a(j,k) T_k$ where $a(j,k) \geq 0$. Since we also have $\sum a_i x^i = \sum c_j P_j$ uniformly on $[-1, 1]$ in this case, it follows that

$$\left\| \sum a_i x^i - \sum_{j=0}^{n} c_j P_j \right\| = \left\| \sum_{j>n} c_j P_j \right\| = \sum_{j>n} c_j$$

and similarly

$$\left\| \sum a_i x^i - \sum_{k=0}^{n} d_k T_k \right\| = \sum_{k>n} d_k.$$

But

$$\begin{aligned} \sum d_k T_k &= \sum a_i x^i \\ &= \sum c_j P_j \\ &= \sum_j c_j \sum_{k=0}^{\infty} a(j,k) T_k \\ &= \sum_k \left(\sum_{j \geq k} c_j a(j,k) \right) T_k \end{aligned}$$

since

$$\left| \sum_{j \geq k} c_j a(j,k) \right| \leq \sum_{j \geq k} c_j \leq \sum c_j P_j(1) = \sum a_i < \infty.$$

Since the coefficients in a uniformly convergent Chebyshev expansion are unique, $d_k = \sum_{j \geq k} c_j a(j,k)$. Therefore

$$\begin{aligned} \sum_{k>n} d_k &= \sum_{j>n} c_j \sum_{k>n} a(j,k) \\ &= \sum_{j>n} c_j \left(\sum_{k=0}^{j} a(j,k) - \sum_{k=0}^{n} a(j,k) \right) \\ &= \sum_{j>n} c_j - \sum_{k=0}^{n} \sum_{j>n} c_j a(j,k) \\ &\leq \sum_{j>n} c_j. \end{aligned}$$

Finally, assume that $\alpha_i < \frac{1}{2}$ $(i = 1, ..., n-1)$. By property (iv) of Theorem 2.1, $a(j, 0) + a(j, 1) > 0$ for $j > n$ so if equality holds in the last inequality then $c_j = 0$ for all $j > n$. Thus $\sum a_i x^i = \sum c_j P_j$ is a polynomial of degree at most n. □

Remark 4.2. If $\langle P_n \rangle$ is defined as in Corollary 4.1 and, more generally, $\sum (ia_i)^2$ converges, then by the Schwarz inequality it follows that

$$\sum_{i>n} |a_i| \leq \left(\sum_{i>n} (ia_i)^2 \right)^{\frac{1}{2}} \left(\sum_{i>n} \frac{1}{i^2} \right)^{\frac{1}{2}}$$

so $\sum |a_i| < \infty$ and c_j exists for every j. Moreover in this case we have

$$\begin{aligned}
&\left|\sum a_i x^i - \sum_{j=0}^{n} c_j P_j(x)\right| \\
&\leq \left|\sum_{i>n} a_i x^i\right| + \sum_{j=0}^{n}\left|\sum_{i>n} a_i b(i,j)\right| \\
&\leq \left(\sum_{i>n}(ia_i)^2\right)^{\frac{1}{2}}\left(\sum_{i>n}\left(\frac{x^i}{i}\right)^2\right)^{\frac{1}{2}} + \sum_{j=0}^{n}\left(\sum_{i>n}(ia_i)^2\right)^{\frac{1}{2}}\left(\sum_{i>n}\left(\frac{b(i,j)}{i}\right)^2\right)^{\frac{1}{2}} \\
&\leq (n+2)\left(\sum_{i>n}\frac{1}{i^2}\right)^{\frac{1}{2}}\left(\sum_{i>n}(ia_i)^2\right)^{\frac{1}{2}} \leq \frac{n+2}{\sqrt{n}}\left(\sum_{i>n}(ia_i)^2\right)^{\frac{1}{2}}
\end{aligned}$$

where the last inequality follows from the proof of the integral test.

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