Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 3, Issue 3, Article 47, 2002

BOUNDS ON CERTAIN INTEGRAL INEQUALITIES

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Received 19 January, 2002; accepted 16 February, 2002.
Communicated by D. Bainov

ABSTRACT. The main objective of this paper is to establish explicit bounds on certain integral inequalities and their discrete analogues which can be used as tools in the study of some classes of integral and sum-difference equations.

Key words and phrases: Integral inequalities, Discrete inequalities, Integral equations, Partial derivatives, Hyperbolic partial integrodifferential equation, Sum-difference equation.

2000 *Mathematics Subject Classification.* 26D10, 26D20.

1. INTRODUCTION

In [2], J.A. Oguntuase obtained a bound on the following integral inequality

$$u(t) \leq c + \int_a^t f(s) \left(u(s) + \int_a^s k(s,\sigma) u(\sigma) d\sigma \right) ds, \tag{1.1}$$

for $a \leq \sigma \leq s \leq t \leq b$ in the form

$$u(t) \leq c \left[1 + \int_a^t f(s) \exp\left(\int_a^s [f(\sigma) + k(\sigma,\sigma)] d\sigma \right) ds \right], \tag{1.2}$$

under some suitable conditions on the functions and the constant c involved in (1.1) and also the bound on the inequality of the form (1.1) when the function $u(\sigma)$ in the inner integral on the right side of (1.1) is replaced by $u^p(\sigma)$ for $0 \leq p < 1$. In [2], the author tried to obtain the generalizations of the inequalities in [3] and did not succeed, because of his incorrect proofs. Indeed, in the proof of Theorem 2.1, the inequality below (2.7) on page 2 and in the proof of Theorem 2.7, the inequality below (2.19) on page 4 given in [2] are not correct. The aim of the present paper is to correct the explicit bound obtained in (1.2) and also obtain a bound on the general version of (1.1). The two independent variable generalisations of the main results, discrete analogues and some applications are also given.

ISSN (electronic): 1443-5756

012-02

2. STATEMENT OF RESULTS

In what follows, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$, $N_0 = \{0, 1, 2, \dots\}$ are the given subsets of $\mathbb{R}$. The partial derivatives of a function $v(x, y)$, $x, y \in \mathbb{R}$ with respect to x, y and xy are denoted by $D_1 v(x, y)$, $D_2 v(x, y)$ and $D_1 D_2 v(x, y) = D_2 D_1 v(x, y)$ respectively. For the functions $w(m)$, $z(m, n)$, $m, n \in N_0$, we define the operators Δ, Δ_1, Δ_2 by

$$\begin{aligned} \Delta w(m) &= w(m+1) - w(m), \\ \Delta_1 z(m, n) &= z(m+1, n) - z(m, n), \\ \Delta_2 z(m, n) &= z(m, n+1) - z(m, n) \end{aligned}$$

respectively and

$$\Delta_2 \Delta_1 z(m, n) = \Delta_2 (\Delta_1 z(m, n)).$$

We denote by

$$\begin{aligned} G_1 &= \left\{(t, s) \in \mathbb{R}_+^2 : 0 \le s \le t < \infty\right\}, \\ G_2 &= \left\{(x, y, s, t) \in \mathbb{R}_+^4 : 0 \le s \le x < \infty,\ 0 \le t \le y < \infty\right\}, \\ H_1 &= \left\{(m, n) \in N_0^2 : 0 \le n \le m < \infty\right\}, \\ H_2 &= \left\{(x, y, m, n) \in N_0^4 : 0 \le m \le x < \infty,\ 0 \le n \le y < \infty\right\}. \end{aligned}$$

Let $C(G, H)$ denote the class of continuous functions from G to H. We use the usual conventions that the empty sums and products are taken to be 0 and 1 respectively. Throughout, all the functions which appear in the inequalities are assumed to be real-valued and all the integrals, sums and products involved exist on the respective domains of their definitions.

Our main results on integral inequalities are established in the following theorems.

Theorem 2.1. *Let $u(t), f(t), a(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$, $k(t, s), D_1 k(t, s) \in C(G_1, \mathbb{R}_+)$ and c be a nonnegative constant.*

(a_1) *If*

$$u(t) \le c + \int_0^t f(s) \left[u(s) + \int_0^s k(s, \sigma) u(\sigma) d\sigma\right] ds, \tag{2.1}$$

for $t \in \mathbb{R}_+$, then

$$u(t) \le c \left[1 + \int_0^t f(s) \exp\left(\int_0^s [f(\sigma) + A(\sigma)] d\sigma\right) ds\right], \tag{2.2}$$

for $t \in \mathbb{R}_+$, where

$$A(t) = k(t, t) + \int_0^t D_1 k(t, \tau) d\tau, \tag{2.3}$$

for $t \in \mathbb{R}_+$.

(a_2) *If*

$$u(t) \le a(t) + \int_0^t f(s) \left[u(s) + \int_0^s k(s, \sigma) u(\sigma) d\sigma\right] ds, \tag{2.4}$$

for $t \in \mathbb{R}_+$, then

$$u(t) \le a(t) + e(t) \left[1 + \int_0^t f(s) \exp\left(\int_0^s [f(\sigma) + A(\sigma)] d\sigma\right) ds\right], \tag{2.5}$$

for $t \in \mathbb{R}_+$, *where*

$$e(t) = \int_0^t f(s)\left[a(s) + \int_0^s k(s,\sigma)\, a(\sigma)\, d\sigma\right] ds, \tag{2.6}$$

for $t \in \mathbb{R}_+$, *and* $A(t)$ *is defined by (2.3).*

Remark 2.2. We note that the bound obtained in (2.2) is the corrected version of the bound given in (1.2) and the inequality established in (a_2) is a further generalization of the inequality given in (a_1). In the special case when $k(t,s) = k(s)$, the inequality given in (a_1) reduces to the inequality established earlier by Pachpatte in [3, Theorem 1] (see, also [4, Theorem 1.7.1, p. 33]).

The following theorem deals with two independent variable versions of the inequalities established in Theorem 2.1 which can be used in certain situations.

Theorem 2.3. *Let* $u(x,y),\ f(x,y),\ a(x,y) \in C(\mathbb{R}^2_+, \mathbb{R}_+),\ k(x,y,s,t),\ D_1k(x,y,s,t),$ $D_2k(x,y,s,t),\ D_1D_2k(x,y,s,t) \in C(G_2, \mathbb{R}_+)$ *and* c *be a nonnegative constant.*

(b_1) *If*

$$u(x,y) \le c + \int_0^x \int_0^y f(s,t)\left[u(s,t) + \int_0^s \int_0^t k(s,t,\sigma,\xi)\, u(\sigma,\xi)\, d\xi d\sigma\right] dtds, \tag{2.7}$$

for $x, y \in \mathbb{R}_+$, *then*

$$u(x,y) \le c\left[1 + \int_0^x \int_0^y f(s,t) \exp\left(\int_0^s \int_0^t [f(\sigma,\xi) + A(\sigma,\xi)]\, d\xi d\sigma\right) dtds\right], \tag{2.8}$$

for $x, y \in \mathbb{R}_+$, *where*

$$\begin{aligned} A(x,y) = {} & k(x,y,x,y) + \int_0^x D_1k(x,y,\tau,y)\, d\tau \\ & + \int_0^y D_2k(x,y,x,\eta)\, d\eta + \int_0^x \int_0^y D_1D_2k(x,y,\tau,\eta)\, d\eta d\tau, \end{aligned} \tag{2.9}$$

for $x, y \in \mathbb{R}_+$.

(b_2) *If*

$$u(x,y) \le a(x,y) + \int_0^x \int_0^y f(s,t)\left[u(s,t) + \int_0^s \int_0^t k(s,t,\sigma,\xi)\, u(\sigma,\xi)\, d\xi d\sigma\right] dtds, \tag{2.10}$$

for $x, y \in \mathbb{R}_+$, *then*

$$\begin{aligned} u(x,y) \le a(x,y) + e(x,y) & \left[1 + \int_0^x \int_0^y f(s,t)\right. \\ & \left. \times \exp\left(\int_0^s \int_0^t [f(\sigma,\xi) + A(\sigma,\xi)]\, d\xi d\sigma\right) dtds\right], \end{aligned} \tag{2.11}$$

for $x, y \in \mathbb{R}_+$, *where*

$$e(x,y) = \int_0^x \int_0^y f(s,t)\left[a(s,t) + \int_0^s \int_0^t k(s,t,\sigma,\xi)\, a(\sigma,\xi)\, d\xi d\sigma\right] dtds, \tag{2.12}$$

for $x, y \in \mathbb{R}_+$ *and* $A(x,y)$ *is defined by (2.9).*

Remark 2.4. By taking $k(x,y,s,t) = k(s,t)$, the inequality given in (b_1) reduces to the inequality given in [4, Remark 4.4.1] and the inequality in (b_2) can be considered as a further generalization of the inequality given in [4, Theorem 4.4.2].

The discrete analogues of the inequalities in Theorems 2.1 and 2.3 are given in the following theorems.

Theorem 2.5. *Let* $u(n), f(n), a(n)$ *be nonnegative functions defined on* N_0, $k(n,s), \Delta_1 k(n,s)$, $0 \le s \le n < \infty$, $n, s \in N_0$ *be nonnegative functions and* c *be a nonnegative constant.*

(c_1) *If*

$$u(n) \le c + \sum_{s=0}^{n-1} f(s)\left[u(s) + \sum_{\sigma=0}^{s-1} k(s,\sigma)\,u(\sigma)\right], \tag{2.13}$$

for $n \in N_0$, *then*

$$u(n) \le c\left[1 + \sum_{s=0}^{n-1} f(s) \prod_{\sigma=0}^{s-1}\left[1 + f(\sigma) + B(\sigma)\right]\right], \tag{2.14}$$

for $n \in N_0$, *where*

$$B(n) = k(n+1,n)\sum_{\tau=0}^{n-1} \Delta_1 k(n,\tau), \tag{2.15}$$

for $n \in N_0$.

(c_2) *If*

$$u(n) \le a(n) + \sum_{s=0}^{n-1} f(s)\left[u(s) + \sum_{\sigma=0}^{s-1} k(s,\sigma)\,u(\sigma)\right], \tag{2.16}$$

for $n \in N_0$, *then*

$$u(n) \le a(n) + E(n)\left[1 + \sum_{s=0}^{n-1} f(s) \prod_{\sigma=0}^{s-1}\left[1 + f(\sigma) + B(\sigma)\right]\right], \tag{2.17}$$

for $n \in N_0$, *where*

$$E(n) = \sum_{s=0}^{n-1} f(s)\left[a(s) + \sum_{\sigma=0}^{s-1} k(s,\sigma)\,a(\sigma)\right], \tag{2.18}$$

for $n \in N_0$ *and* $B(n)$ *is defined by (2.15).*

Theorem 2.6. *Let* $u(x,y),\ f(x,y),\ a(x,y),\ k(x,y,s,t),\ \Delta_1 k(x,y,s,t),\ \Delta_2 k(x,y,s,t)$, $\Delta_1\Delta_2 k(x,y,s,t)$ *be nonnegative functions for* $0 \le s \le x$, $0 \le t \le y$, s, x, t, y *in* N_0 *and* c *be a nonnegative constant*

(d_1) *If*

$$u(x,y) \le c + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} f(s,t)\left[u(s,t) + \sum_{m=0}^{s-1}\sum_{n=0}^{t-1} k(s,t,m,n)\,u(m,n)\right], \tag{2.19}$$

for $x, y \in N_0$, *then*

$$u(x,y) \le c\left[1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} f(s,t) \prod_{m=0}^{s-1}\left[1 + \sum_{n=0}^{t-1}\left[f(m,n) + B(m,n)\right]\right]\right], \tag{2.20}$$

for $x, y \in N_0$, *where*

$$B(x,y) = k(x+1, y+1, x, y) + \sum_{\sigma=0}^{x-1} \Delta_1 k(x, y+1, \sigma, y) + \sum_{\tau=0}^{y-1} \Delta_2 k(x+1, y, x, \tau) + \sum_{\sigma=0}^{x-1} \sum_{\tau=0}^{y-1} \Delta_2 \Delta_1 k(x, y, \sigma, \tau), \tag{2.21}$$

for $x, y \in N_0$.

(d_2) *If*

$$u(x,y) \leq a(x,y) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} f(s,t) \left[u(s,t) + \sum_{m=0}^{s-1} \sum_{n=0}^{t-1} k(s,t,m,n)\, u(m,n) \right], \tag{2.22}$$

for $x, y \in N_0$, *then*

$$u(x,y) \leq a(x,y) + E(x,y) \left[1 + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} f(s,t) \times \prod_{m=0}^{s-1} \left[1 + \sum_{n=0}^{t-1} [f(m,n) + B(m,n)] \right] \right], \tag{2.23}$$

for $x, y \in N_0$, *where*

$$E(x,y) = \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} f(s,t) \left[a(s,t) + \sum_{m=0}^{s-1} \sum_{n=0}^{t-1} k(s,t,m,n)\, a(m,n) \right], \tag{2.24}$$

for $x, y \in N_0$ *and* $B(x,y)$ *is defined by (2.21).*

3. PROOFS OF THEOREMS 2.1, 2.3, 2.5 AND 2.6

Proof of Theorem 2.1. (a_1) Define a function $z(t)$ by the right hand side of (2.1). Then $z(0) = c$, $u(t) \leq z(t)$ and

$$\begin{aligned} z'(t) &= f(t) \left[u(t) + \int_0^t k(t,\sigma)\, u(\sigma)\, d\sigma \right] \\ &\leq f(t) \left[z(t) + \int_0^t k(t,\sigma)\, z(\sigma)\, d\sigma \right]. \end{aligned} \tag{3.1}$$

Define a function $v(t)$ by

$$v(t) = z(t) + \int_0^t k(t,\sigma)\, z(\sigma)\, d\sigma. \tag{3.2}$$

Then $v(0) = z(0) = c$, $z(t) \leq v(t)$, $z'(t) \leq f(t)v(t)$ and $v(t)$ is nondecreasing in t, $t \in \mathbb{R}_+$, we have

$$\begin{aligned} v'(t) &= z'(t) + k(t,t)z(t) + \int_0^t D_1 k(t,\sigma) z(\sigma) d\sigma \\ &\leq f(t)v(t) + k(t,t)v(t) + \int_0^t D_1 k(t,\sigma) v(\sigma) d\sigma \\ &\leq \left[f(t) + k(t,t) + \int_0^t D_1 k(t,\sigma) d\sigma \right] v(t) \\ &= [f(t) + A(t)] v(t), \end{aligned}$$

implying

$$v(t) \leq c \exp\left(\int_0^s [f(\sigma) + A(\sigma)] d\sigma \right), \tag{3.3}$$

where $A(t)$ is defined by (2.3). Using (3.3) in (3.1) and integrating the resulting inequality from 0 to t, $t \in \mathbb{R}_+$, we get

$$z(t) \leq c \left[1 + \int_0^t f(s) \exp\left(\int_0^s [f(\sigma) + A(\sigma)] d\sigma \right) ds \right]. \tag{3.4}$$

The desired inequality in (2.2) follows by using (3.4) in $u(t) \leq z(t)$.

(a_2) Define a function $z(t)$ by

$$z(t) = \int_0^t f(s) \left[u(s) + \int_0^s k(s,\sigma) u(\sigma) d\sigma \right] ds. \tag{3.5}$$

Then from (2.4), $u(t) \leq a(t) + z(t)$ and using this in (3.5), we get

$$\begin{aligned} z(t) &\leq \int_0^t f(s) \left[a(s) + z(s) + \int_0^s k(s,\sigma)(a(\sigma) + z(\sigma)) d\sigma \right] ds \\ &= e(t) + \int_0^t f(s) \left[z(s) + \int_0^s k(s,\sigma) z(\sigma) d\sigma \right] ds, \end{aligned} \tag{3.6}$$

where $e(t)$ is defined by (2.6). Clearly $e(t)$ is nonnegative, continuous and nondecreasing in t, $t \in \mathbb{R}_+$. First, we assume that $e(t) > 0$ for $t \in \mathbb{R}_+$. From (3.6) it is easy to observe that

$$\frac{z(t)}{e(t)} \leq 1 + \int_0^t f(s) \left[\frac{z(s)}{e(s)} + \int_0^s k(s,\sigma) \frac{z(\sigma)}{e(\sigma)} d\sigma \right] ds.$$

Now, an application of the inequality in (a_1) we have

$$\frac{z(t)}{e(t)} \leq \left[1 + \int_0^t f(s) \exp\left(\int_0^s [f(\sigma) + A(\sigma)] d\sigma \right) ds \right]. \tag{3.7}$$

The desired inequality in (2.5) follows from (3.7) and the fact that $u(t) \leq a(t) + z(t)$. If $e(t) \geq 0$, we carry out the above procedure with $e(t) + \varepsilon$ instead of $e(t)$, where $\varepsilon > 0$ is an arbitrary small constant, and then subsequently pass to the limit as $\varepsilon \to 0$ to obtain (2.5). □

Remark 3.1. By replacing the function $u(\sigma)$ in the inner integral on the right hand side of (2.1) by $u^p(\sigma)$, $0 \leq p < 1$, and by following the proof of (a_1) with suitable modifications, we get the corrected version of Theorem 2.7 given in [2].

Proof of Theorem 2.3. (b_1) Let $c > 0$ and define a function $z(x,y)$ by the right hand side of (2.7). Then $z(0,y) = z(x,0) = c$, $u(x,y) \le z(x,y)$, and

$$D_1D_2z(x,y) = f(x,y)\left[u(x,y) + \int_0^x\int_0^y k(x,y,\sigma,\xi)\,u(\sigma,\xi)\,d\xi d\sigma\right] \le f(x,y)\left[z(x,y) + \int_0^x\int_0^y k(x,y,\sigma,\xi)\,z(\sigma,\xi)\,d\xi d\sigma\right]. \tag{3.8}$$

Define a function $v(x,y)$ by

$$v(x,y) = z(x,y) + \int_0^x\int_0^y k(x,y,\sigma,\xi)\,z(\sigma,\xi)\,d\xi d\sigma. \tag{3.9}$$

Then, $v(0,y) = z(0,y) = c$, $v(x,0) = z(x,0) = c$, $z(x,y) \le v(x,y)$, $D_1D_2z(x,y) \le f(x,y)\,v(x,y)$, $v(x,y)$ is nondecreasing for $x, y \in \mathbb{R}_+$ and

$$\begin{aligned} D_1D_2v(x,y) &= D_1D_2z(x,y) + k(x,y,x,y)\,z(x,y) + \int_0^x D_1k(x,y,\sigma,y)\,z(\sigma,y)\,d\sigma \\ &\quad + \int_0^y D_2k(x,y,x,\xi)\,z(x,\xi)\,d\xi + \int_0^x\int_0^y D_1D_2k(x,y,\sigma,\xi)\,z(\sigma,\xi)\,d\xi d\sigma \\ &\le f(x,y)\,v(x,y) + k(x,y,x,y)\,v(x,y) + \int_0^x D_1k(x,y,\sigma,y)\,v(\sigma,y)\,d\sigma \\ &\quad + \int_0^y D_2k(x,y,x,\xi)\,v(x,\xi)\,d\xi + \int_0^x\int_0^y D_1D_2k(x,y,\sigma,\xi)\,v(\sigma,\xi)\,d\xi d\sigma \\ &\le [f(x,y) + A(x,y)]\,v(x,y), \end{aligned} \tag{3.10}$$

where $A(x,y)$ is defined by (2.9). Now, by following the proof of Theorem 4.2.1 given in [4], inequality (3.10) implies

$$v(x,y) \le c\exp\left(\int_0^x\int_0^y [f(\sigma,\xi) + A(\sigma,\xi)]\,d\xi d\sigma\right). \tag{3.11}$$

Using (3.11) in (3.8) and integrating the resulting inequality first from 0 to y and then from 0 to x for $x, y \in \mathbb{R}_+$, we get

$$z(x,y) \le c\left[1 + \int_0^x\int_0^y f(s,t)\exp\left(\int_0^s\int_0^t [f(\sigma,\xi) + A(\sigma,\xi)]\,d\xi d\sigma\right)dtds\right]. \tag{3.12}$$

Using (3.12) in $u(x,y) \le z(x,y)$, we get the required inequality in (2.8). If $c \ge 0$, we carry out the above procedure with $c + \varepsilon$ instead of c, where $\varepsilon > 0$ is an arbitrary small constant, and then subsequently pass to the limit as $\varepsilon \to 0$ to obtain (2.8).

(b_2) The proof can be completed by closely looking at the proofs of (a_2) and (b_1) given above. Here we omit the details. □

Proof of Theorem 2.5. (c_1) Define a function $z(n)$ by the right hand side of (2.13), then $z(0) = c$, $u(n) \le z(n)$ and

$$\Delta z(n) = f(n)\left[u(n) + \sum_{\sigma=0}^{n-1} k(n,\sigma)\,u(\sigma)\right] \le f(n)\left[z(n) + \sum_{\sigma=0}^{n-1} k(n,\sigma)\,z(\sigma)\right]. \tag{3.13}$$

Define a function $v(n)$ by

$$v(n) = z(n) + \sum_{\sigma=0}^{n-1} k(n,\sigma) z(\sigma). \tag{3.14}$$

Then $v(0) = z(0) = c$, $z(n) \leq v(n)$, $\Delta z(n) \leq f(n) v(n)$ and $v(n)$ is nondecreasing in n, $n \in N_0$, we have

$$\begin{aligned} \Delta v(n) &= \Delta z(n) + \sum_{\sigma=0}^{n} k(n+1,\sigma) z(\sigma) - \sum_{\sigma=0}^{n-1} k(n,\sigma) z(\sigma) \\ &= \Delta z(n) + k(n+1,n) z(n) + \sum_{\sigma=0}^{n-1} \Delta_1 k(n,\sigma) z(\sigma) \\ &\leq [f(n) + B(n)] v(n), \end{aligned} \tag{3.15}$$

where $B(n)$ is defined by (2.15). The inequality (3.15) implies

$$v(n) \leq c \prod_{\sigma=0}^{n-1} [1 + f(\sigma) + B(\sigma)]. \tag{3.16}$$

Using (3.16) in (3.11) we get

$$\Delta z(n) \leq c f(n) \prod_{\sigma=0}^{n-1} [1 + f(\sigma) + B(\sigma)]. \tag{3.17}$$

The inequality (3.17) implies the estimate

$$z(n) \leq c \left[1 + \sum_{s=0}^{n-1} f(s) \prod_{\sigma=0}^{s-1} [1 + f(\sigma) + B(\sigma)] \right]. \tag{3.18}$$

Using (3.18) in $u(n) \leq z(n)$ we get the desired inequality in (2.14).
(c_2) The proof of can be completed by closely looking at the proofs of (a_2) and (c_2) given above. □

Proof of Theorem 2.6. (d_1) and (d_2) can be completed by following the proofs of the inequalities given above and closely looking at the proofs of the similar results given in [5]. □

4. APPLICATIONS

In this section, we present some applications of the inequality (b_1) in Theorem 2.3 to study certain properties of solutions of the nonlinear hyperbolic partial integrodifferential equation

$$u_{xy}(x,y) = F\left(x, y, u(x,y), \int_0^x \int_0^y h(x,y,\sigma,\xi,u(\sigma,\xi))\, d\xi d\sigma \right), \tag{4.1}$$

with the given initial boundary conditions

$$u(x,0) = \alpha_1(x),\ u(0,y) = \alpha_2(y),\ \alpha_1(0) = \alpha_2(0) = 0, \tag{4.2}$$

where $u \in C(\mathbb{R}_+^2, \mathbb{R})$, $h \in C(G_2 \times \mathbb{R}, \mathbb{R})$, $F \in C(\mathbb{R}_+^2 \times \mathbb{R}^2, \mathbb{R})$.

The following theorem deals with the estimate on the solution of (4.1) – (4.2).

Theorem 4.1. *Suppose that*

$$|h(x,y,s,t,u(s,t))| \leq k(x,y,s,t) |u(s,t)|, \tag{4.3}$$

$$|F(x,y,u,v)| \leq f(x,y) [|u| + |v|], \tag{4.4}$$

$$|\alpha_1(x) + \alpha_2(y)| \leq c, \tag{4.5}$$

where k, f and c are as defined in Theorem 2.3. If $u(x,y)$, $x, y \in \mathbb{R}_+$ is any solution of (4.1) – (4.2), then

$$|u(x,y)| \le c\left[1+\int_0^x\int_0^y f(s,t)\exp\left(\int_0^s\int_0^t [f(\sigma,\xi)+A(\sigma,\xi)]\,d\xi d\sigma\right)dtds\right], \tag{4.6}$$

for $x, y \in \mathbb{R}_+$, where $A(x,y)$ is defined by (2.9).

Proof. The solution $u(x,y)$ of (4.1) – (4.2) can be written as

$$\begin{aligned} u(x,y) &= \alpha_1(x)+\alpha_2(y) \\ &\quad + \int_0^x\int_0^y F\left(s,t,u(s,t),\int_0^s\int_0^t h(s,t,\sigma,\xi,u(\sigma,\xi))\,d\xi d\sigma\right)dtds. \end{aligned} \tag{4.7}$$

Using (4.3) – (4.5) in (4.7) we have

$$\begin{aligned} |u(x,y)| \le c + \int_0^x\int_0^y f(s,t)\Bigg[&|u(s,t)| \\ &+\left(\int_0^s\int_0^t k(s,t,\sigma,\xi)\,|u(\sigma,\xi)|\,d\xi d\sigma\right)\Bigg]dtds. \end{aligned} \tag{4.8}$$

Now, an application of the inequality (b_1) in Theorem 2.3 yields the desired estimate in (4.6). □

Our next result deals with the uniqueness of the solutions of (4.1) – (4.2).

Theorem 4.2. *Suppose that the functions h, F in (4.1) satisfy the conditions*

$$|h(x,y,s,t,u_1)-h(x,y,s,t,u_2)| \le k(x,y,s,t)\,|u_1-u_2|, \tag{4.9}$$

$$|F(x,y,u_1,u_2)-F(x,y,v_1,v_2)| \le f(x,y)\,[|u_1-v_1|+|u_2-v_2|], \tag{4.10}$$

where k and f are as in Theorem 2.3. Then the problem (4.1) – (4.2) has at most one solution on $\mathbb{R}_+^2$.

Proof. Let $u_1(x,y)$ and $u_2(x,y)$ be two solutions of (4.1) – (4.2) on $\mathbb{R}_+^2$, then we have

$$\begin{aligned} &u_1(x,y)-u_2(x,y) \\ &= \int_0^x\int_0^y\Bigg[F\left(s,t,u_1(s,t),\int_0^s\int_0^t h(s,t,\sigma,\xi,u_1(\sigma,\xi))\,d\xi d\sigma\right) \\ &\qquad - F\left(s,t,u_2(s,t),\int_0^s\int_0^t h(s,t,\sigma,\xi,u_2(\sigma,\xi))\,d\xi d\sigma\right)\Bigg]dtds. \end{aligned} \tag{4.11}$$

From (4.9), (4.10) and (4.11) we have

$$\begin{aligned} |u_1(x,y)-u_2(x,y)| \le \int_0^x\int_0^y f(s,t)\Bigg[&|u_1(s,t)-u_2(s,t)| \\ &+\int_0^s\int_0^t k(s,t,\sigma,\xi)\,|u_1(\sigma,\xi)-u_2(\sigma,\xi)|\,d\xi d\sigma\Bigg]dtds. \end{aligned} \tag{4.12}$$

As an application of the inequality (b_1) in Theorem 2.3 with $c=0$ yields $|u_1(x,y)-u_2(x,y)| \le 0$. Therefore, $u_1(x,y)=u_2(x,y)$, i.e., there is at most one solution of (4.1) – (4.2) on $\mathbb{R}_+^2$. □

We note that the inequality (d_1) in Theorem 2.6 can be used to obtain the bound and uniqueness of solutions of the following partial sum-difference equation

$$\Delta_2\Delta_1 z(x,y) = H\left(x, y, z(x,y), \sum_{m=0}^{x-1}\sum_{n=0}^{y-1} g(x,y,m,n,z(m,n))\right), \tag{4.13}$$

with the given conditions

$$z(x,0) = \beta_1(x), \quad z(0,y) = \beta_2(y), \quad \beta_1(0) = \beta_2(0) = 0, \tag{4.14}$$

under some suitable conditions on the functions involved in (4.13) – (4.14). For various other applications of the inequalities similar to that given above, see [4, 5].

In concluding, we note that in another paper [1], Oguntuase has given the upper bounds on certain integral inequalities involving functions of several variables. However, the results given in [1] are also not correct. In fact, in the proof of Theorem 2.1, the equality in (2.3) and in the proof of Theorem 3.1 on page 5, the equality on line 10 (from above) are not correct. For a number of inequalities involving functions of many independent variables and their applications, see [4, 5].

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