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AN EXTENSION OF RESULTS OF A. MCD. MERCER AND I. GAVREA

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ABSTRACT. In this note we extend recent results of A. McD. Mercer and I. Gavrea on convex sequences to other classes of sequences.

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1. INTRODUCTION

The following result is valid [1, 2]. Let $\boldsymbol{a} = (a_0, a_1, \dots, a_n)$ be a real sequence. The inequality

$$\sum_{k=0}^{n} a_k u_k \geq 0 \tag{1.1}$$

holds for every convex sequence $\boldsymbol{u} = (u_0, u_1, \dots, u_n)$ if and only if the polynomial

$$P_{\boldsymbol{a}}(x) := \sum_{k=0}^{n} a_k x^k$$

has $x = 1$ as a double root and the coefficients c_k ($k = 0, 1, \dots, n-2$) of the polynomial

$$\frac{P_{\boldsymbol{a}}(x)}{(x-1)^2} = \sum_{k=0}^{n-2} c_k x^k$$

are non-negative. The sufficiency and necessity of this result are due, respectively, to A. McD. Mercer [2] and to I. Gavrea [1].

The purpose of this note is to extend the above result to other classes of sequences $\boldsymbol{u}$.

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2. BASIC LEMMA

A *convex cone* is a non-empty set $C \subset \mathbb{R}^{n+1}$ such that $\alpha C + \beta C \subset C$ for all non-negative scalars α and β. We say that a convex cone C is *generated by* a set $V \subset C$, and write $C = \operatorname{cone} V$, if every vector in C can be expressed as a non-negative linear combination of a finite number of vectors in V.

Let $\langle \cdot, \cdot \rangle$ stand for the standard inner product on $\mathbb{R}^{n+1}$. The *dual cone* of C is the cone defined by

$$\operatorname{dual} C := \{\boldsymbol{u} \in \mathbb{R}^{n+1} : \langle \boldsymbol{u}, \boldsymbol{v} \rangle \geq 0 \ \text{ for all } \boldsymbol{v} \in C\}.$$

It is well-known that

$$\operatorname{dual} \operatorname{dual} C = C \tag{2.1}$$

for any closed convex cone $C \subset \mathbb{R}^{n+1}$ (cf. [3, Theorem 14.1, p. 121]). The result below is a key fact in our considerations. It is a consequence of (2.1) for a finitely generated cone $C = \operatorname{cone}\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$.

Lemma 2.1 (Farkas lemma). *Let $\boldsymbol{v}, \boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ be vectors in $\mathbb{R}^{n+1}$. The following two statements are equivalent:*

(i): *The inequality $\langle \boldsymbol{u}, \boldsymbol{v} \rangle \geq 0$ holds for all $\boldsymbol{u} \in \mathbb{R}^{n+1}$ such that $\langle \boldsymbol{u}, \boldsymbol{v}_i \rangle \geq 0$, $i = 0, 1, \ldots, p$.*

(ii): *There exist non-negative scalars c_i, $i = 0, 1, \ldots, p$, such that*

$$\boldsymbol{v} = c_0 \boldsymbol{v}_0 + c_1 \boldsymbol{v}_1 + \cdots + c_p \boldsymbol{v}_p.$$

3. MAIN RESULT

Given a sequence $\boldsymbol{q} = (q_0, q_1, \ldots, q_r) \in \mathbb{R}^{r+1}$ with $0 \leq r \leq n$, we define

$$\boldsymbol{v}_i := (\underbrace{0, \ldots, 0}_{i \text{ times}}, q_0, q_1, \ldots, q_r, 0, \ldots, 0) = S^i \boldsymbol{v}_0 \in \mathbb{R}^{n+1} \ \text{ for } i = 0, 1, \ldots, n-r. \tag{3.1}$$

Here S is the *shift operator* from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$ defined by

$$S(z_0, z_1, \ldots, z_n) := (0, z_0, z_1, \ldots, z_{n-1}). \tag{3.2}$$

A sequence $\boldsymbol{u} = (u_0, u_1, \ldots, u_n) \in \mathbb{R}^{n+1}$ is said to be of $\boldsymbol{q}$*-class*, if

$$\langle \boldsymbol{u}, \boldsymbol{v}_i \rangle \geq 0 \ \text{ for all } i = 0, 1, \ldots, n-r. \tag{3.3}$$

In other words, the $\boldsymbol{q}$-class consists of all vectors of the cone

$$D := \operatorname{dual} \operatorname{cone}\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-r}\}. \tag{3.4}$$

Example 3.1.
(a). Set $r = 0$, $q_0 = 1$ and

$$\boldsymbol{v}_i = (\underbrace{0, \ldots, 0}_{i \text{ times}}, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1} \ \text{ for } i = 0, 1, \ldots, n.$$

Then (3.3) reduces to

$$u_i \geq 0 \ \text{ for } i = 0, 1, \ldots, n.$$

Thus D is the class of non-negative sequences.
(b). Put $r = 1$, $q_0 = -1$ and $q_1 = 1$, and denote

$$\boldsymbol{v}_i = (\underbrace{0, \ldots, 0}_{i \text{ times}}, -1, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1} \ \text{ for } i = 0, 1, \ldots, n-1.$$

Then (3.3) gives

$$u_i \leq u_{i+1} \ \text{ for } i = 0, 1, \ldots, n-1,$$

which means that D is the class of non-decreasing sequences.

(c). Consider $r = 2$, $q_0 = 1$, $q_1 = -2$, $q_2 = 1$ and

$$\boldsymbol{v}_i = (\underbrace{0, \dots, 0}_{i \text{ times}}, 1, -2, 1, 0, \dots, 0) \in \mathbb{R}^{n+1} \text{ for } i = 0, 1, \dots, n-2.$$

In this case, (3.3) is equivalent to

$$u_{i+1} \leq \frac{u_i + u_{i+2}}{2} \text{ for } i = 0, 1, \dots, n-2.$$

This says that $\boldsymbol{u}$ is a convex sequence. Therefore D is the class of convex sequences.

Theorem 3.1. *Let* $\boldsymbol{a} = (a_0, a_1, \dots, a_n) \in \mathbb{R}^{n+1}$ *and* $\boldsymbol{q} = (q_0, q_1, \dots, q_r) \in \mathbb{R}^{r+1}$ *be given with* $0 \leq r \leq n$*. Then the inequality*

$$\sum_{k=0}^{n} a_k u_k \geq 0 \tag{3.5}$$

holds for every sequence $\boldsymbol{u} = (u_0, u_1, \dots, u_n)$ *of* $\boldsymbol{q}$*-class if and only if the polynomial*

$$P_{\boldsymbol{a}}(x) := \sum_{k=0}^{n} a_k x^k$$

is divisible by the polynomial

$$P_{\boldsymbol{q}}(x) := \sum_{k=0}^{r} q_k x^k,$$

and the coefficients c_k *(*$k = 0, 1, \dots, n-r$*) of the polynomial*

$$\frac{P_{\boldsymbol{a}}(x)}{P_{\boldsymbol{q}}(x)} = \sum_{k=0}^{n-r} c_k x^k$$

are non-negative.

Proof. The map φ that assigns to each sequence $\boldsymbol{b} = (b_0, b_1, \dots, b_m)$ the polynomial

$$\varphi(\boldsymbol{b}) := P_{\boldsymbol{b}}(x) := \sum_{k=0}^{m} b_k x^k$$

is a one-to-one linear map from $\mathbb{R}^{m+1}$ to the space of polynomials of degree at most m. Also, $\psi := \varphi^{-1}$ is a one-to-one linear map. It is not difficult to check that

$$\psi(x^k P_{\boldsymbol{b}}(x)) = S^k \psi(P_{\boldsymbol{b}}(x)).$$

Therefore, for any polynomial

$$P_{\boldsymbol{c}}(x) := c_0 + c_1 x + \dots + c_{n-r} x^{n-r},$$

we have

$$\psi(P_{\boldsymbol{c}}(x) P_{\boldsymbol{q}}(x)) = c_0 S^0 \boldsymbol{v}_0 + c_1 S^1 \boldsymbol{v}_0 + \dots + c_{n-r} S^{n-r} \boldsymbol{v}_0 = c_0 \boldsymbol{v}_0 + c_1 \boldsymbol{v}_1 + \dots + c_{n-r} \boldsymbol{v}_{n-r},$$

where $\boldsymbol{v}_i$ are given by (3.1). In other words,

$$P_{\boldsymbol{c}}(x) P_{\boldsymbol{q}}(x) = \varphi(c_0 \boldsymbol{v}_0 + c_1 \boldsymbol{v}_1 + \dots + c_{n-r} \boldsymbol{v}_{n-r}) \text{ for any } \boldsymbol{c} = (c_0, c_1, \dots, c_{n-r}). \tag{3.6}$$

We are now in a position to show that the following statements are mutually equivalent:

(i): Inequality (3.5) holds for every $\boldsymbol{u}$ of $\boldsymbol{q}$-class.

(ii): $\langle \boldsymbol{a}, \boldsymbol{u} \rangle \geq 0$ for every $\boldsymbol{u} \in \operatorname{dual cone} \{\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_{n-r}\}$.

(iii): There exist non-negative scalars $c_0, c_1, \dots, c_{n-r}$ such that $\boldsymbol{a} = c_0\boldsymbol{v}_0 + c_1\boldsymbol{v}_1 + \cdots + c_{n-r}\boldsymbol{v}_{n-r}$.

(iv): There exist non-negative scalars $c_0, c_1, \dots, c_{n-r}$ such that $P_{\boldsymbol{a}}(x) = (c_0 + c_1x + \cdots + c_{n-r}x^{n-r})P_{\boldsymbol{q}}(x)$.

In fact, (ii) is an easy reformulation of (i) (see (3.4)). That (ii) and (iii) are equivalent is a direct consequence of Farkas lemma (see Lemma 2.1). We now show the validity of the implication (iii) ⇒ (iv). By (iii) and (3.6), we have

$$P_{\boldsymbol{a}}(x) = \varphi(\boldsymbol{a}) = \varphi(c_0\boldsymbol{v}_0 + c_1\boldsymbol{v}_1 + \cdots + c_{n-r}\boldsymbol{v}_{n-r}) = P_{\boldsymbol{c}}(x)P_{\boldsymbol{q}}(x)$$

for certain scalars $c_k \geq 0$, $k = 0, 1, \dots, n-r$. Thus (iv) is proved.

To prove the implication (iv) ⇒ (iii) assume (iv) holds, that is $P_{\boldsymbol{a}}(x) = P_{\boldsymbol{c}}(x)P_{\boldsymbol{q}}(x)$ with $c_k \geq 0$, $k = 0, 1, \dots, n-r$. Then by (3.6),

$$\begin{aligned}\boldsymbol{a} &= \psi(P_{\boldsymbol{a}}(x)) = \psi(P_{\boldsymbol{c}}(x)P_{\boldsymbol{q}}(x)) \\ &= \psi\varphi(c_0\boldsymbol{v}_0 + c_1\boldsymbol{v}_1 + \cdots + c_{n-r}\boldsymbol{v}_{n-r}) \\ &= c_0\boldsymbol{v}_0 + c_1\boldsymbol{v}_1 + \cdots + c_{n-r}\boldsymbol{v}_{n-r}.\end{aligned}$$

This completes the proof of Theorem 3.1. □

4. APPLICATIONS FOR CONVEX SEQUENCES OF ORDER r

In this section we study special types of sequences related to difference calculus and generalized convex sequences.

We introduce the *difference operator* on sequences $\boldsymbol{z} = (z_0, z_1, \dots, z_m)$ by

$$\Delta\boldsymbol{z} := (z_1 - z_0, z_2 - z_1, \dots, z_m - z_{m-1}).$$

Notice that $\Delta = \Delta_m$ acts from $\mathbb{R}^{m+1}$ to $\mathbb{R}^m$. We define

$$\Delta^0\boldsymbol{z} := \boldsymbol{z} \ \text{ and } \ \Delta^r\boldsymbol{z} := \Delta_{m-r+1}\cdots\Delta_{m-1}\Delta_m\boldsymbol{z} \ \text{ for } r = 1, 2, \dots, m.$$

A sequence $\boldsymbol{u} \in R^{n+1}$ is said to be *convex of order* r (in short, r*-convex*), if

$$\Delta^r\boldsymbol{u} \geq 0.$$

The last inequality is meant in the componentwise sense in $\mathbb{R}^{n+1-r}$, that is

$$\langle\Delta^r\boldsymbol{u}, \boldsymbol{e}_i\rangle \geq 0 \ \text{ for } i = 0, 1, \dots, n-r, \tag{4.1}$$

where

$$\boldsymbol{e}_i := (\underbrace{0, \dots, 0}_{i \text{ times}}, 1, 0, \dots, 0) \in \mathbb{R}^{n+1-r}.$$

In order to relate the r-convex sequences to the $\boldsymbol{q}$-class of Section 3, observe that (4.1) amounts to

$$\langle\boldsymbol{u}, (\Delta^r)^T\boldsymbol{e}_i\rangle \geq 0 \ \text{ for } i = 0, 1, \dots, n-r,$$

where $(\cdot)^T$ denotes the transpose. By a standard induction argument, we get

$$(\Delta^r)^T\boldsymbol{e}_i = S^i\boldsymbol{v}_0 \ \text{ for } i = 0, 1, \dots, n-r,$$

where S is the shift operator from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$ given by (3.2), and

$$\boldsymbol{v}_0 := (\boldsymbol{q}, 0, \dots, 0) \in \mathbb{R}^{n+1} \quad \text{and} \quad \boldsymbol{q} := (q_0, q_1, \dots, q_r) \ \text{ with } \ q_j := \binom{r}{j}(-1)^{r-j}. \tag{4.2}$$

As in (3.1), we set

$$\boldsymbol{v}_i := S^i\boldsymbol{v}_0 \ \text{ for } i = 0, 1, \dots, n-r.$$

In summary, the r-convex sequences form the $\boldsymbol{q}$-class for $\boldsymbol{q}$ given by (4.2). For example, the class of r-convex sequences for $r = 0$ (resp. $r = 1$, $r = 2$) is the class of non-negative (resp. non-decreasing, convex) sequences in $\mathbb{R}^{n+1}$ (cf. Example 3.1).

By virtue of (4.2) we get

$$P_{\boldsymbol{q}}(x) = \sum_{k=0}^{r} q_k x^k = (x-1)^r.$$

Therefore we obtain from Theorem 3.1

Corollary 4.1. *Let* $\boldsymbol{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$ *be given with* $0 \le r \le n$*. Then the inequality*

$$\sum_{k=0}^{n} a_k u_k \ge 0 \tag{4.3}$$

holds for every r*-convex sequence* $\boldsymbol{u} = (u_0, u_1, \ldots, u_n)$ *if and only if the polynomial*

$$P_{\boldsymbol{a}}(x) = \sum_{k=0}^{n} a_k x^k$$

has $x = 1$ *as a root of multiplicity at least* r*, and the coefficients* c_k $(k = 0, 1, \ldots, n-r)$ *of the polynomial*

$$\frac{P_{\boldsymbol{a}}(x)}{(x-1)^r} = \sum_{k=0}^{n-r} c_k x^k$$

are non-negative.

Corollary 4.1 extends the mentioned results of A. McD. Mercer and I. Gavrea from $r = 2$ to an arbitrary $0 \le r \le n$.

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