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BOUNDS FOR ASYMPTOTE SINGULARITIES OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, bounds are obtained for the location of vertical asymptotes and other types of singularities of solutions to certain nonlinear differential equations. We consider several different families of nonlinear differential equations, but the main focus is on the second order initial value problem (IVP) of generalized superlinear Emden-Fowler type

$$y''(x) = p(x)[y(x)]^\eta, \quad y(x_0) = A, \quad y'(x_0) = B, \quad \eta > 1$$

A general method using bounded operators is developed to obtain some of the bounds derived in this paper. This method allows one to obtain lower bounds for the cases $A = 0$ and $A < 0$ under certain conditions, which are not handled by previously discussed bounds in the literature. We also make several small corrections to equations appearing in previous works. Enough numerical examples are given to compare the bounds, since no bound is uniformly better than the other bounds. In these comparisons, we also consider the bounds of Eliason [11] and Bobisud [5]. In addition, we indicate how to improve and generalize the bounds of these two authors.

Key words and phrases: Bounded operator, Comparison methods, Generalized Emden-Fowler equations, Nonlinear differential equations, Vertical asymptote.

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1. INTRODUCTION

Many papers have been written on oscillatory and/or nonoscillatory behavior of differential equations. The literature for this topic will not be cited here since it is too vast. However, most nonlinear differential equations do not have closed form solutions, so a numerical method must often be used, such as Runge-Kutta type methods. If a singularity is present in the solution, then such methods may give meaningless results. Hence, it would be useful to have easily computable (preferably closed form) bounds for the location of such singularities, since the interval of existence of the solutions must contain the interval on which the numerical method is applied. In this way, we can 'move forward' to the singularity starting at the initial value x_0.

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Hence, lower bounds for singularities to the right of x_0 are of especially important interest. It is the aim of this paper to supply a number of easily computable lower bounds. In some cases, we shall also obtain some upper bounds for the singularity. We focus on asymptote (vertical) singularities, but the methods used can work for other types of singularities as well.

In this paper, we shall present bounds for the location of certain types of singularities of certain nonlinear differential equations of order two or higher. We shall focus on those differential equations which have vertical asymptotes.

The common theme of this paper is maximization or minimization of certain operators combined with comparison techniques, in addition to standard integration techniques.

Definition 1.1. A solution $y = y(x)$ has a vertical asymptote at $x = c$ if $c > x_0$,

$$\lim_{x\to c^-} y(x) = +\infty, \quad \text{and} \quad \lim_{x\to c^-} y'(x) = +\infty\,.$$

Throughout this paper, we assume the existence of a singularity to the right of x_0. We shall be interested in the location of the first vertical asymptote to the right of x_0 in this paper. Conditions guaranteeing the existence of vertical asymptotes and other types of singularities/noncontinuation can be found in the works of Eliason ([9], [10], [11]), Bobisud [5], Hara et al. ([16], [17]), Burton [7], Burton and Grimmer [6], Petty and Johnson [27], Saito [28], Kwong [29], and Tsukamoto et al. [31]. Throughout this paper, we assume the existence of a singularity to the right of x_0 of some type. We mainly focus on the case of a vertical asymptote. For singularities to the left of x_0, the obvious modifications can be made. The emphasis is on obtaining easily computable bounds. Many of these bounds are obtained merely by finding the unique root of certain equations and are sometimes of closed form and computable by hand. We shall present enough numerical examples to compare the bounds discussed in this paper, since no single bound is always the best. A very general method is discussed to obtain lower bounds for the location of vertical asymptotes, which can be generalized to certain other kinds of singularities (such as a derivative blow-up). This general method handles some cases which are not handled by the bounds given in Eliason [11] and Bobisud [5]. It can also be extended to handle many families of n^{th} order nonlinear equations. Let us first consider methods for obtaining bounds for c for the generalized superlinear Emden-Fowler IVP:

$$y''(x) = p(x)[y(x)]^{\eta}, \quad y(x_0) = A, \quad y'(x_0) = B, \quad \eta > 1. \tag{1.1}$$

Several authors have discussed existence and uniqueness of solutions to (1.1). None of these results will be presented here. The interested reader should see the good survey paper by Erbe and Rao [12]. See also Taliaferro [29]. For results on oscillation and nonoscillation see Wong ([32], [33]). See also Biles [3], Dang et al. [8], Fowler [13], Habets [15], and Harris [18]. Only a few authors discuss locations of vertical asymptotes. Since this is the main point of interest of this paper, we briefly present the most germaine results of these authors here for the convenience of the reader.

First we present some results given in Eliason [11].

Theorem 1.1 (Eliason [11]). *Suppose $p(x)$ is continuous on $[x_0, c]$ and positive on $[x_0, c)$. Let $y(x)$ be a solution to (1.1). Suppose $A > 0$ and $B = 0$. If $y(x)$ is continuous on $[x_0, c)$, then upper and lower bounds for c satisfy*

$$A^{\frac{\eta-1}{2}} \int_{x_0}^{c} \sqrt{p_L(t)}dt \le z(\eta) \le A^{\frac{\eta-1}{2}} \int_{x_0}^{c} \sqrt{p_u(t)}dt\,, \tag{1.2}$$

where

$$p_L(t) = \inf_{0\le x\le t} p(x), \quad p_u(t) = \sup_{0\le x\le t} p(x) \tag{1.3}$$

and

$$z(\eta) = [2(\eta+1)]^{-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\eta-1}{2\eta+2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\eta-1}{2\eta+2}\right)}, \tag{1.4}$$

$\Gamma(\cdot)$ *denotes the gamma function. Let* $L_{E,1}$ *and* $U_{E,1}$ *denote the Eliason* [11] *lower and upper bounds for* c*, obtained from (1.2) above.*

Theorem 1.2 (Eliason [11])**.** *Suppose* $p(x)$ *is continuous on* $[x_0, c)$ *and positive on* (x_0, c)*. Suppose* $A > 0$ *and* $B = 0$*. If* $y(x)$ *is continuous on* $[x_0, x^*)$*, then the following hold:*

a) *If* $p(x)$ *is nondecreasing on* $[x_0, c]$*, then an upper bound for* c *is defined by*

$$A^{\frac{\eta-1}{2}} \int_{x_0}^{c} \sqrt{p_A(t)}dt \le z(\eta)\,, \tag{1.5}$$

where $p_A(x)$ *is given by (1.7).*

b) *If* $p(x)$ *is nonincreasing on* $[x_0, c]$*, then a lower bound for* c *satisfies*

$$A^{\frac{\eta-1}{2}} \int_{x_0}^{c} \sqrt{p_A(t)}dt \ge z(\eta)\,, \tag{1.6}$$

where $p_A(x)$ *is the average value of* $p(x)$ *on* $[x_0, x]$*, i.e.,*

$$p_A(x) = \begin{cases} (x-x_0)^{-1}\int_{x_0}^{x} p(t)dt, & \text{if } x > x_0, \\ p(x_0), & \text{if } x = x_0. \end{cases} \tag{1.7}$$

Let $L_{E,2}$ *and* $U_{E,2}$ *denote the lower and upper bounds of Eliason* [11] *obtained from (1.5) and (1.7) above. Note that the upper bounds of Theorems 1.1 and 1.2 are valid for* $B > 0$ *also. However, the lower bounds are not valid unless* $y'(x_0) = B = 0$*. We shall obtain later several new lower bounds for the case* $B > 0$*.*

Next, we present some results of Bobisud [5]. The lower bounds of Bobisud [5] are valid under more general conditions than the lower bounds of Eliason [11]. However, Bobisud [5] does not present any upper bounds for c. It should be mentioned that the lower bounds of Bobisud [5] are for the more general differential equation $y'' = p(x)f(y)$. However, they are the only lower bounds given in the literature for the case $B > 0$. (We shall also discuss the above more general differential equation later and discuss the case $A < 0$ for some choices of $f(y)$, a case not considered by Bobisud and Eliason.) We shall also discuss the case $A = 0$ when $p(x)$ may have a singularity at $x = x_0$.

Theorem 1.3 (Theorem 2 of Bobisud [5])**.** *Suppose* $p(x)$ *is continuous on* $[x_0, c]$ *and positive on* $[x_0, c)$*. Suppose* $y(x) \ge M > 0$ *for* $x_0 \le x < c$*; is the solution to*

$$y''(x) = p(x), \quad f(y(x)), \quad y(x_0) = A, \quad y'(x_0) = B. \tag{1.8}$$

Suppose $A \ge 0$*,* $B \ge 0$ *and* $A + B > 0$*. If* $f(y) > 0$ *and* $f'(y) \ge 0$*,* $M \le y < \infty$*, and if* $y(x)$ *has a vertical asymptote at* $x = c$*, then an implicit lower bound for* c *satisfies*

$$\int_{y_0}^{\infty} \frac{du}{f(u)} \le \int_{x_0}^{c} (c-w)p(w)dw + \frac{B}{f(A)}(c-x_0). \tag{1.9}$$

Let $L_{B,2}$ *denote the lower bound of Bobisud* [5] *obtained from (1.9).*

As a consequence of Theorem 1.3, we obtain the following corollary, which is a small correction of Theorem 2.2.8 of Erbe and Rao [12].

Corollary 1.4. *Suppose $p(x)$ is continuous on $[x_0, c]$. Suppose $y(x) \geq M > 0$ for $x_0 \leq x < c$. Then a lower bound for c in IVP (1.1) is implicitly given by*

$$\frac{A^{1-\eta}}{\eta - 1} \leq \int_{x_0}^{c} (c-w)p(w)dw + \frac{B}{A^{\eta}}(c - x_0). \tag{1.10}$$

Equivalently,

$$\frac{1}{\eta - 1} \leq A^{\eta-1} \int_{x_0}^{c} (c-w)p(w)dw + \frac{B}{A}(c - x_0)\,,$$

provided $A > 0$.

Theorem 1.5 (Theorem 3 of Bobisud [5])**.** *Let $y(x)$ be a solution to (1.8). Suppose $f(y)$ is continuous for $y \geq A$, with $f(y) > 0$ for $y \geq A$ if $A > 0$, and $f(y) > 0$ for $y > 0$ if $A = 0$. Suppose $p(x) > 0$ has a nonnegative derivative on $[A, \infty)$. If $A \geq 0$ and $B \geq 0$, then c satisfies*

$$\int_{A}^{\infty} \frac{dx}{\sqrt{\frac{B^2}{p(x_0)} + 2\int_{x_0}^{x} f(u)du}} \leq \int_{x_0}^{c} \sqrt{p(t)}dt. \tag{1.11}$$

Let $L_{B,3}$ denote the lower bound of c obtained from (1.11).

The results of Bobisud [5] and Eliason [11] require continuity of $p(x)$ at $x = x_0$. This limits the applicability of these results to (1.8) when $A = y(x_0) = 0$ since the initial condition $A = 0$ often will necessitate a singularity at $x = x_0$ in the function $p(x)$. One of the main contributions of this paper is to handle this singular case. In Eliason [11], the author remarks, in reference to (1.1), that 'due to the methods of our proof, we are not able to draw many conclusions for the case $y'(x_0) = B < 0$, nor for the boundary conditions $y(x_0) = A = 0$, $y'(x_0) = B > 0$.' In this paper, we shall present lower bounds for c even in some cases where $p(x)$ has a singularity at x_0. Moreover, we will show that the methods used can be extended to other differential equations of much more general form than Emden-Fowler type. The methods used are based on maximization and minimization of certain operators as well as classical integration techniques.

2. New Bounds for $A > 0$

Throughout this section, it is always assumed that $A > 0$. Consider the following IVP of Emden-Fowler type:

$$y''(x) = p(x)[y(x)]^{\eta}, \quad y(x_0) = A, \quad y'(x_0) = B, \quad \eta > 1. \tag{2.1}$$

To obtain bounds for the vertical asymptote c of (2.1), we first need a few lemmas. It will be helpful to consider the more general differential equation

$$y''(x) = (x - x_0)^{\theta} q(x) \cdot f(y(x)), \quad y(x_0) = A, \quad y'(x_0) = B, \tag{2.2}$$

where $q(x) > 0$ is continuous on $[x_0, \infty)$, $\eta > 1$, and θ is real. Thus, (2.2) allows for a singularity in the coefficient function at x_0 if $\theta < 0$. We will sometimes write (2.1) and (2.2) in the more respective compact forms $y'' = py^n$ and $y'' = (x - x_0)^{\theta} q(x) f(y)$. To prove some new results, we will first need some lemmas. Lemma 2.1 is a generalization and slight variation of Lemma 0.2 of Taliaferro [29].

Lemma 2.1 (Comparison lemma)**.** *Suppose $\phi_1(x)$ and $\phi_2(x)$ have the form*

$$\phi_1(x) = (x - x_0)^{\theta} q_1(x) \tag{2.3}$$

and

$$\phi_2(x) = (x - x_0)^{\theta} q_2(x), \tag{2.4}$$

where θ is a real number, and where $q_1(x)$ and $q_2(x)$ are continuous positive functions on $[x_0, \infty)$. Let $Y_1(x)$ and $Y_2(x)$ be respective solutions on some interval $I = [x_0, x_0 + \Delta), \Delta > 0$, of the equations

$$Y_1''(x) = (x - x_0)^\theta q_1(x) f(Y_1(x)) \tag{2.5}$$

and

$$Y_2''(x) = (x - x_0)^\theta q_2(x) f(Y_2(x))$$

where $f(y) \geq 0$ is continuous and nondecreasing. Suppose

$$Y_1(x_0) \leq Y_2(x_0) \quad \text{and} \quad Y_1'(x_0) \leq Y_2'(x_0)\,. \tag{2.6}$$

If $q_1(x) \leq q_2(x)$ on $[x_0, \infty)$, then $Y_1(x) \leq Y_2(x)$, for x in I.

Proof. Since the proof is similar to that of Lemma 0.2 of Taliaferro [29], we merely sketch a few key steps that are different from the proof given in Taliaferro [29]. Proceeding as in Taliaferro [29] with some modifications, we obtain, for $x_0 \leq x < x_0 + \Delta$:

$$Y_i(x) = Y_i(x_0) + (x - x_0)Y_i'(x_0) + \int_{x_0}^{x} (x - t) \cdot (t - x_0)^\theta q_i(t) f(Y_i(t))dt, \quad i = 1, 2. \tag{2.7}$$

The above integral will exist (near $t = x_0$) since it is essentially an integrated form of the second derivative of Y_i, $i = 1, 2$. This is an important point especially for $\theta < 0$. Subtraction gives

$$Y_1(t) - Y_2(t) \leq \int_{x_0}^{x} (x - t)(t - x_0)^\theta [q_1(t) f(Y_1(t)) - q_2(t) f(Y_2(t))]dt,$$

which is nonpositive since the integrand is nonpositive. The lemma may also be proven by considering the difference function $D(t) = Y_1(t) - Y_2(t)$. For x in (x_0, c), there exists $d = d(x)$ in (x_0, x) such that $D(x) = \phi_1(d) f(Y_1(d)) - \phi_2(d) f(Y_2(d))$, which is nonpositive. However, equation (2.7) will be useful later. □

Remark 2.2. The famous Thomas-Fermi equation

$$y'' = x^{-1/2} y^{3/2} \tag{2.8}$$

has many applications in atomic physics and has the form (2.2) discussed in Lemma 2.1 as well as the Emden-Fowler equation

$$y'' = \pm x^\theta y^{1-\theta}.$$

See Hille ([19], [20]) for a discussion of (2.8). We shall consider differential equation (2.8) later in Sections 2 and 3. Before presenting the next few lemmas, we need to define some upper and lower coefficient functions. For IVP (2.2), define

$$q_L(x) = \inf_{x_0 \leq t \leq x} q(t), \quad q_u(x) = \sup_{x_0 \leq t \leq x} q(t). \tag{2.9}$$

Then $q_u(x)$ and $q_L(x)$ are nondecreasing and nonincreasing, respectively.

Lemma 2.3. *Consider IVP (2.2). Suppose $q(x) > 0$ is continuous on $[x_0, \infty)$ and $q(x)$ is differentiable on $[x_0, \infty)$. Let Z_q denote the zero set $Z_q = \{x \in [x_0, \infty) : \ q'(x) = 0\}$. Suppose that Z_q has no accumulation points. Then*

(a) *$q_L(x)$ and $q_u(x)$ are continuous on $[x_0, \infty)$.*
(b) *Let $q_L'(x) = \frac{d}{dx}(q_L(x))$, $q_u'(x) = \frac{d}{dx}(q_u(x))$. Then $q_L'(x)$ and $q_u'(x)$ are continuous on $[x_0, \infty) \setminus Z_q$, the complement of Z_q in $[x_0, \infty)$.*

(c) $q'_L(x)$ *and* $q'_u(x)$ *have finite left-handed limits at each point* $x \geq x_0$ *(but may not be continuous at* x *in* Z_q*), that is, for* $x \geq x_0$

$$\lim_{t \to x^-} q'_L(t) \qquad \textit{and} \qquad \lim_{t \to x^-} q'_u(t)$$

exist as real numbers.

Proof. We merely sketch a few key steps, since the result is intuitively clear. The conditions on the zero set Z_q guarantees that only a finite number of zeros can exist in $[x_0, \infty)$, by the Bolzano-Weierstrass Theorem. So q_L and q_u are piecewise continuous off Z_q in $[x_0, c)$. The same is true for q'_L and q'_u. Since there are only a finite number of continuous 'pieces', the results (a)–(c) now follow easily. □

Lemma 2.3 above will be needed in subsequent lemmas and theorems which use L'Hospital's Rule in a deleted left half neighborhood of x^*. From Lemma 2.3, it would follow that

$$\lim_{x \to (x_*)^-} q'_L(x) \qquad \text{and} \qquad \lim_{x \to (x^*)^-} q'_u(x) \tag{2.10}$$

exist, where x_* and x^* are any asymptotes of solutions to (2.2) with $q(x)$ replaced by $q_L(x)$ and $q_u(x)$, respectively. Throughout this paper, when we write $q'_L(x_*)$ and $q'_u(x^*)$, we shall mean the respective limits given in (c) above. Also, we assume throughout that Z_q has no accumulation points.

Remark 2.4. From Lemma 2.3, we can conclude that there exists an x_{00} in (x_0, x_*) or (x_0, x^*) such that on the respective interval, we have:

(a) $Y_u^{(3)}(x)$ and $Y_L^{(3)}(x)$ are continuous on the respective intervals (x_{00}, x_*) and (x_{00}, x^*), and

(b) q'_L and q'_u are continuous there.

Now let us give a major idea for comparison purposes throughout the rest of the paper. Many methods are based upon comparing the following three IVPs:

(1)

$$y(x) = (x - x_0)^\theta q(x) f(y(x)), \quad y(x_0) = A, \quad y'(x_0) = B. \tag{2.11}$$

Vertical asymptote at $x = c$ (actual IVP of interest in this paper)

(2)

$$Y_u(x) = (x - x_0)^\theta q_u(x) f(Y_u(x)), \quad Y_u(x_0) = A, \quad Y'_u(x_0) = B. \tag{2.12}$$

Vertical asymptote at $x = x^*$.

(3)

$$Y_L(x) = (x - x_0)^\theta q_L(x) f(Y_L(x)), \quad Y_L(x_0) = A, \quad Y'_L(x_0) = B. \tag{2.13}$$

Vertical asymptote at $x = x_*$.

By comparison, we have: $x^* \leq c \leq x_*$ and

$$Y_L(x) \leq y(x) \leq Y_u(x), \quad x_0 \leq x < x^*,$$

and $Y_L(x) \leq y(x)$, $x^* \leq x < c$. In some cases, it may be that only the solutions of (1) and (2) have asymptotes, in which case only a lower bound for c can be found. However, if (1) has an asymptote, then so does (2).

Lemma 2.5. *Let $Y_u(x)$ be a solution of (2.12) with $q(x) \geq 0$ continuously differentiable on $[x_0, \infty)$. Suppose*

$$Z = \lim_{w\to\infty} \frac{w \cdot f'(w)}{f(w)} > 1\,, \quad \textit{possibly infinite}.$$

Then

$$\lim_{x\to(x^*)^-} \frac{Y_u(x)}{{Y_u}'(x)} = 0\,.$$

Proof. Let

$$R = \limsup_{x\to(x^*)^-} \frac{Y_u(x)}{{Y_u}'(x)} \geq 0.$$

First, we establish that R is real. For $x > x_0$, there is a $d = d(x)$ in (x_0, x) such that

$$\frac{Y_u(x)}{{Y_u}'(x)} = \frac{A + {Y_u}'(d)(x - x_0)}{{Y_u}'(x)}\,,$$

from which it follows that $0 \leq R \leq x^* - x_0 < \infty$. Also,

$$\frac{{Y_u}'(x)}{Y_u(x)} = \frac{B + \int_{x_0}^{x} (t - x_0)^\theta q_u(t) f(Y_u(t))dt}{Y_u(x)}\,. \tag{2.14}$$

Let us consider two cases.

Case 1. $\theta \geq 0$. From (2.14), several consecutive applications of L'Hospital's Rule and Lemma 2.3 gives

$$\begin{aligned}
\lim_{x\to(x^*)^-} \frac{Y_u'(x)}{Y_u(x)} &= \lim_{x\to(x^*)^-} \frac{Y_u^{(3)}(x)}{Y_u''(x)} \\
&= \lim_{x\to(x^*)^-} \left[\frac{f'(Y_u(x)) \cdot {Y_u}'(x)}{f(Y_u(x))} + \frac{q_u'(x)}{q_u(x)} + \frac{\theta}{x - x_0}\right] \\
&\geq \lim_{x\to(x^*)^-} \left[Z\left(\frac{{Y_u}'(x)}{Y_u(x)}\right) + \left(\frac{q_u'(x)}{q_u(x)} + \frac{\theta}{x - x_0}\right)\right] \\
&\geq Z \lim_{x\to(x^*)^-} \left(\frac{{Y_u}'(x)}{Y_u(x)}\right)\,,
\end{aligned}$$

since the expression in parenthesis is nonnegative. Since $Z > 1$, this necessitates $\lim_{x\to(x^*)^-}\left(\frac{{Y_u}'(x)}{Y_u(x)}\right) = +\infty$, since $0 \leq R < \infty$. Thus, $\lim_{x\to(x^*)^-} \frac{Y_u(x)}{{Y_u}'(x)} = 0$.

Case 2. $\theta < 0$. From (2.14), we have

$$\frac{Y_u'(x)}{Y_u(x)} \geq \frac{B + (x^* - x_0)^\theta \int_{x_0}^{x} q_u(t) \cdot f(Y_u(t))dt}{Y_u(x)}\,.$$

Applying L'Hospital's Rule several times in succession in conjunction with Lemma 2.3 again and proceeding in much the same manner as done in Case 1 above, we obtain (we omit details)

$$\lim_{x\to(x^*)^-} \frac{Y_u'(x)}{Y_u(x)} \geq \lim_{x\to(x^*)^-} \left(Z \cdot \frac{Y_u'(x)}{Y_u(x)} + \frac{q_u'(x)}{q_u(x)}\right) \geq Z \cdot \lim_{x\to(x^*)^-} \frac{Y_u'(x)}{Y_u(x)}\,,$$

from which it follows again that $\lim_{x\to(x^*)^-} \frac{Y_u'(x)}{Y_u(x)} = +\infty$.

$\square$

Lemma 2.6. *Let $Y_u(t)$ be a solution of (2.12). Suppose*

$$\lim_{x\to(x^*)^-}\frac{Y_u(t)}{Y_u'(x)}=0 \text{ and } Z=\lim_{w\to\infty}\frac{w\cdot f'(w)}{f(w)}>1\,,\quad \text{possibly infinite}\,.$$

Suppose $q'(x)\geq 0$ is continuous on $[x_0,\infty)$. Then

$$\lim_{x\to(x^*)^-}\frac{Y_u(t)Y_u''(t)}{[Y_u'(t)]^2}=\frac{1+Z}{2}\,.$$

Proof. We apply L'Hospital's Rule and Lemma 2.3. After much cancellation and simplification, we finally obtain

$$\begin{aligned}\lim_{x\to(x^*)^-}\frac{Y_u(t)Y_u''(t)}{[Y_u'(t)]^2}&=\frac{1}{2}+\lim_{x\to(x^*)^-}\frac{Y_u(x)Y_u^{(3)}(x)}{Y_u'(x)Y_u''(x)}\\&=\frac{1}{2}+\lim_{x\to(x^*)^-}\left[\frac{Y_u(x)\cdot f'(Y_u(x))}{2f(Y_u(x))}+\frac{Y_u(x)}{2Y_u'(x)}\left(q_u'(x)+\frac{\theta q_u(x)}{x-x_0}\right)\right]\\&=\frac{1}{2}+\lim_{x\to(x^*)^-}\left[\frac{Y_u(x)f'(Y_u(x))}{2f(Y_u(x))}\right]=\frac{1+Z}{2}\,,\end{aligned}$$

upon application of Lemmas 2.3 and 2.5. □

Remark 2.7. The above lemmas remain true if $q(x)$ is merely differentiable on some left deleted neighborhood of x^* or x_* that is, on an interval $(x^*-\delta,x^*)$ or $(x_*-\delta,x_*)$ for some $\delta>0$. This will be an important observation needed later.

Remark 2.8. Lemma 2.6 holds in particular for the generalized Emden-Fowler choice $f(y)=y^n$, $n>1$, corresponding to IVP (1.1) with $Z=\eta$.

For comparison purposes, let $Y_L(x)$ and $Y_u(x)$ be solutions to

$$Y_L''(x)=p_L(x)[Y_L(x)]^\eta,\quad Y_L(x_0)=A,\quad Y_L'(x_0)=B \tag{2.15}$$

$$Y_u''(x)=p_u(x)[Y_u(x)]^\eta,\quad Y_u(x_0)=A,\quad Y_u'(x_0)=B \tag{2.16}$$

where

$$p_L(x)=\inf_{x_0\leq t\leq x}p(x)\quad\text{and}\quad p_u(x)=\sup_{x_0\leq t\leq x}p(x)\,.$$

Let $y(x)$ denote a solution to

$$y''(x)=p(x)[y(x)]^\eta,\quad y(x_0)=A,\quad y'(x_0)=B\,. \tag{2.17}$$

Lemma 2.9. *Consider IVP (2.12). Let $\epsilon=\frac{1-\eta}{2}$. Suppose $p(x)\geq 0$ is continuously differentiable $[x_0,\infty)$ and that $A>0$ and $B>0$. Then*

$$\lim_{x\to(x^*)^-}\epsilon[Y_u(x)]^{\epsilon-1}Y_u'(x)=\frac{\epsilon}{\sqrt{1-\epsilon}}\sqrt{p_u(x^*)}.$$

Proof. From Lemma 2.3, we have

$$\lim_{x\to(x^*)^-}\frac{(1-\epsilon)[Y_u'(x)]^2}{p_u(x)\cdot[Y_u(x)]^{\eta+1}}=1\,,$$

which implies that

$$\lim_{x\to(x^*)^-}\frac{\sqrt{1-\epsilon}Y_u'(x)}{\sqrt{p_u(x)}[Y_u(x)]^{1-\epsilon}}=1\,.$$

Rearranging terms, we finally conclude

$$\lim_{x\to(x^*)^-}[Y_u(x)]^{\epsilon-1}Y_u'(x)=\sqrt{\frac{p_u(x^*)}{1-\epsilon}}\,,$$

thereby proving the lemma. □

Lemma 2.10. *Let $Y_L(x)$ be a solution of (2.13) with $q(x) > 0$ continuously differentiable on $[x_0, \infty)$. Suppose $q_L(x)$ is continuously differentiable on $[x_0, \infty)$,*

$$Z = \lim_{w\to\infty} \frac{w \cdot f'(w)}{f(w)} > 1\,,$$

possibly infinite, that (2.11) has a vertical asymptote at $x = x_*$, and $\theta \le 0$. Then:

a) $\displaystyle\lim_{x\to(x_*)^-} \frac{Y_L(x)}{Y_L'(x)} = 0$*, provided*

$$Z > 1 - \theta + \sup_{x\ge x_0}\left[(x - x_0)\cdot\left(\frac{-q_L'(x)}{q_L(x)}\right)\right]. \tag{2.18}$$

b) $\displaystyle\lim_{x\to(x_*)^-} \frac{Y_L(t)Y_L''(t)}{[Y_L'(t)]^2} = \frac{1+Z}{2}.$

Proof. We prove part (a) only since the proof of (b) follows in a very similar way to the proof of Lemma 2.6. Proceeding as in the proof of Lemma 2.5, we obtain

$$\begin{aligned} R = \limsup_{x\to(x_*)^-} \frac{Y_L(x)}{Y_L'(x)} &\le \limsup_{x\to(x_*)^-} \frac{Y_L''(x)}{Y_L^{(3)}(x)} \\ &= \limsup_{x\to(x_*)^-} \frac{Y_L(x)}{ZY_L'(x) + Y_L(x)\cdot\left[\frac{\theta}{x-x_0} + \frac{q_L'(x)}{q_L(x)}\right]}\,, \end{aligned}$$

upon application of results in Taylor [30] on L'Hospital's Rule. Clearly $0 \le R < \infty$. We shall rule out $R > 0$, using (2.18). We have

$$\begin{aligned} R &\le \frac{R}{Z + RL}, \quad \text{where} \\ L &= \frac{\theta}{x_* - x_0} + \frac{q_L'(x_*)}{q_L(x_*)}. \end{aligned}$$

Suppose on the contrary that $R > 0$. Then $Z + RL \le 1$. By condition (2.18), we have

$$Z > 1 + (x_0 - x_*)\left(\frac{\theta}{x_* - x_0} + \frac{q_L'(x_*)}{q_L(x_*)}\right),$$

which gives $Z + RL > 1$, a contradiction. So $R = 0$ and $\lim_{x\to(x_*)^-} \frac{Y_L(x)}{Y_L'(x)} = 0$, as claimed. □

Part (a) can also be proven more easily by considering the divergence of the integral $\int_t^{x_*} \frac{y_L'(x)}{y_L(x)}dx$ as $t \to x_*^-$, but the L'Hospital's Rule argument used here will be needed in later sections.

Remark 2.11. For the generalized Emden-Fowler IVP (1.1), condition (2.18) reduces to $(\theta = 0$ and $Z = \eta)$

$$\eta > \sup_{x\ge x_0}\left(\frac{-p_L'(x)}{p_L(x)}\cdot(x - x_0)\right) + 1$$

which holds automatically if $p(x)$ is nondecreasing in x, in particular. It will also hold for certain choices of nonincreasing $p(x)$ provided that $p(x)$ does not decrease 'too fast'.

Lemma 2.12. *Consider IVP (2.13). Suppose $p(x) > 0$ is continuously differentiable on $[x_0, \infty)$, $p_L(x)$ is continuously differentiable, $A > 0$ and $B > 0$. Suppose IVP (2.13) has a vertical asymptote at $x = x_*$. Then, if $\theta \leq 0$, we have*

$$\lim_{x \to (x_*)^-} \epsilon (Y_L(x))^{\epsilon - 1} Y_L'(x) = \frac{\epsilon}{\sqrt{1-\epsilon}} \sqrt{q_L(x_*)}\,,$$

provided

$$\eta > \sup_{x \geq x_0} \left(\frac{-q_L'(x)}{q_L(x)} \cdot (x - x_0) \right) + 1 - \theta. \tag{2.19}$$

Proof. Follow the proof of Lemma 2.9, using Lemma 2.10, part (b), instead of Lemma 2.3. □

Note that Lemmas 2.10 and 2.12 remain true if $\sup_{x \geq x_0}$ is replaced by $\sup_{x \geq L}$, where L is any lower bound for c.

Remark 2.13. In Lemmas 2.10, 2.12, the sets of conditions under which $q_L'(x)$ and $p_L'(x)$ will be continuously differentiable include (but are not exhausted by) the cases below:

(1) $q_L(x)$, $p_L(x)$ are nonincreasing or nondecreasing on $[x_0, \infty)$.
(2) $q_L(x)$, $p_L(x)$ are 'bath-tub' shaped, that is, there is an $x' > x_0$ with:

$$q_L(x) = \begin{cases} q_1(x), & \text{if} \quad x_0 \leq x \leq x', \\ q_2(x), & \text{if} \quad x > x', \end{cases}$$

where $q_1(\cdot)$, $q_2(\cdot)$ continuously differentiable functions are such that $q_1'(x') = q_2'(x')$, $q_1(x)$ is nonincreasing on $[x_0, x')$ and $q_2(x)$ is nondecreasing on $[x', \infty)$.
(3) $q_L(x)$ and $p_L(x)$ are unimodal.

Lemmas of type 2.5 and 2.6 will be indispensable throughout this paper.

We are now in a position to state and prove several main results. Throughout Sections 2–4 below, we assume the existence of a vertical asymptote at $x = c > x_0$.

Theorem 2.14. *Let $y(x)$ be a solution to IVP (1.1). Suppose $A > 0$. Suppose $p(x) \geq 0$ is continuously differentiable on $[x_0, c]$. Let $\epsilon = \frac{1-\eta}{2}$ and let $Z_p = \{x \geq x_0 : p'(x) = 0\}$. Suppose Z_p has no accumulation points. Then:*

a) *Let $p_u(x) = \sup_{x_0 \leq t \leq x} p(t)$ and*

$$g_1(x) = \min \left(\epsilon A^{\epsilon - 1} B,\ \frac{\epsilon}{\sqrt{1-\epsilon}} \sqrt{p_u(x)} \right),\ x > x_0. \tag{2.20}$$

Then a lower bound L_1 for c is the unique root (value of x) satisfying

$$(x_0 - x) g_1(x) = A^\epsilon. \tag{2.21}$$

b) *Let $p_L(x) = \inf_{x_0 \leq t \leq x} p(t)$. Suppose $p(x) > 0$ on $[x_0, \infty)$. Suppose $p_L'(x)$ is continuous on $[x_0, \infty)$. Let*

$$h_1(x) = \max \left(\epsilon A^{\epsilon - 1} B,\ \frac{\epsilon}{\sqrt{1-\epsilon}} \sqrt{p_L(x)} \right),\ x > x_0. \tag{2.22}$$

Suppose

(H1) $$\liminf_{x\to\infty}\left((x-x_0)\cdot\sqrt{p_L(x)}\right) > \frac{A^\epsilon\sqrt{1-\epsilon}}{(-\epsilon)},\ \textit{possibly infinite},$$

$$\textit{or}\ \limsup_{x\to\infty}\left((x-x_0)\sqrt{p_L(x)}\right) < \frac{A^\epsilon\sqrt{1-\epsilon}}{(-\epsilon)}\,.$$

(H2) $$\sup_{x\geq x_0}(x_0-x)h_1(x) > A^\epsilon,\ \textit{and}$$

(H3) $$1+\sup_{x\geq x_0}(x-x_0)\left(\frac{-p_L'(x)}{p_L(x)}\right) < \eta\ \textit{all hold}\,.$$

Then an upper bound U_1 for c is the largest root of

(2.23) $$(x_0-x)h_1(x) = A^\epsilon\,.$$

Proof. Of (a): Let $u = u(x) = [Y_u(x)]^\epsilon$. By the Mean Value Theorem, there exists $d = d(x)$ in (x_0, x) such that

(2.24) $$u(x) = u(x_0) + u'(d)\cdot(x-x_0)\,.$$

Differentiation of $u(x)$ produces

(2.25) $$u' = \epsilon(Y_u)^{\epsilon-1}Y_u'$$

and

(2.26) $$u'' = \epsilon(Y_u)^{\epsilon-1}Y_u'' + \epsilon(\epsilon-1)Y_u^{\epsilon-2}(Y_u')^2\,.$$

To bound $u'(d)$ in (2.24), we obtain from (2.26) that $u''(x) = 0$ for values of x (if any) satisfying

(2.27) $$Y_u'(x) = \sqrt{\frac{p_u(x)[Y_u(x)]^\eta}{1-\epsilon}}\,.$$

Substitution of this into (2.25) results in

$$|u'(x)| = \frac{|\epsilon|}{\sqrt{1-\epsilon}}[Y_u(x)]^{\epsilon-1+\frac{\eta}{2}}\sqrt{p_u(x)}\,.$$

By choice of ϵ, for any x satisfying (2.27), we have

(2.28) $$|u'(x)| = \frac{|\epsilon|}{\sqrt{1-\epsilon}}\sqrt{p_u(x)}\,.$$

By Lemma 2.9, (2.28) holds also as $x\to(x^*)^-$. From all this, we may infer that

$$|u'(d(x))| \leq -g_1(x^*)\ \text{and}\ u'(d(x)) \geq g_1(x^*),\ x_0 < x < x^*\,,$$

which implies that

$$x \geq x_0 - \frac{A^\epsilon}{g_1(x^*)},\quad x_0 < x < x^*\,.$$

Since $u(x_0) = A^\epsilon$ and $u'(x_0) = \epsilon A^{\epsilon-1}B$, we obtain from (2.24) that

$$u'(d(x^*)) = \frac{A^\epsilon}{x_0-x^*}$$

and

$$x_0 - \frac{A^\epsilon}{g_1(x^*)} \leq x^*\,,$$

upon letting $x \to (x^*)^-$. Thus, $(x_0-x^*)g_1(x^*) \geq A^\epsilon$ holds. But we also have $(x_0-L_1)g_1(L_1) = A^\epsilon$. Since $(x_0-x)g_1(x)$ is strictly increasing in x, we must have $x^* \geq L_1$. Since $c \geq x^*$, we have $c \geq L_1$. This completes the proof of part (a).

The proof of part (b) is analogous except we use h_1 instead of g_1 and p_L instead of p_u in the above arguments, along with Lemma 2.10. For this reason, we only give the details for the parts of the proof that are different from the proof of part (a).

To prove (b), we proceed as in the proof of (a). Hypothesis (H1) guarantees that the zero set of the equation $(x_0-x)h_1(x) = A^\epsilon$ is bounded above by a real number $\tilde{x} > x_0$. Hypothesis (H2) guarantees that this zero set is nonempty. Hypothesis (H3) allows us to apply Lemma 2.12 to obtain $|u'(d(x))| \geq \frac{|\epsilon|}{\sqrt{1-\epsilon}}\sqrt{p_L(x)}$ and

$$u'(d(x)) < h_1(x), \quad x_0 \leq x < x^*.$$

Thus, $(x_0-x_*)h_1(x_*) \leq A^\epsilon$ holds. Now suppose on the contrary that $x_* > U_1$. By the Intermediate Value Theorem, there would exist a root x' in $[x_*, \tilde{x})$ of the equation $(x_0-x)h_1(x) = A^\epsilon$. So $x' > U_1$ and $(x_0-x')h_1(x') = A^\epsilon$. So x' is a root of $(x_0-x)h_1(x) = A^\epsilon$, a contradiction to U_1 being the largest such root. We can conclude that $x_* \leq U_1$. Since $c \leq x_*$, we must have $c \leq U_1$ and U_1 is an upper bound of c. This completes the proof. □

Remark 2.15. If any upper bound for c is available (call it U), then it should be computed first. Then a search for a lower bound of c can be confined to a search on the compact interval $[x_0, U]$, since $x_0 \leq L \leq U$ must hold. However, if (H3) does not hold, it may be the case that condition (H3') holds:

$$\text{(H3')} \qquad \eta > 1 + \sup_{x \geq L}(x-x_0)\left(\frac{-p_L'(x)}{p_L(x)}\right),$$

where L is any lower bound of c. In this situation, we would want to compute $L = L_1$ first instead. In any case, (H3) can also be replaced by the requirement $\lim_{x\to(x_*)^-}\frac{Y_L(x)}{Y_L'(x)} = 0$.

Remark 2.16. The above theorem makes use of the operator $u = Y_u^\epsilon$. In this paper, we shall also consider operators of the form: $u = e^{\alpha Y_u}$, $\alpha < 0$, $u = (x-x_0)^{-1}Y_u$. The author has also considered the operators $u = Y_u^{\epsilon_1}(Y_u')^{\epsilon_2}$, $\epsilon_1 < 0$, $\epsilon_2 < 0$, and $u = (Y_u + aY_u' + b)^\epsilon$, but these did not consistently provide better lower bounds.

Remark 2.17. Note that the existence of a lower bound L_1 does not depend on the initial values $A > 0$ and $B > 0$. However, the existence of an upper bound U_1 may depend on the values of A and B, if $p_L(x)$ is not constant ($p(x)$ is nondecreasing), for example. In fact, we shall see that more stringent conditions guaranteeing the existence of an upper bound of x^* are usually more necessary than those guaranteeing the existence of a lower bound of x^*, for the remaining problems considered in this paper.

Remark 2.18. In his concluding comments section, Eliason [11] mentions that the methods used in his paper cannot be used for the following cases:

$$(1) \quad B < 0 \qquad\qquad (2) \quad A = 0 \quad \text{and} \quad B > 0.$$

He also mentions that these cases certainly are of interest. A check of the literature revealed no subsequent work providing bounds for these two cases. When $\theta < 0$. Bobisud [5] provides lower bounds when $\theta = 0$ for a more general $f(y)$, however. Theorem 2.14 clearly provides bounds in Case 1. In Section 3, we shall offer bounds for Case 2. Moreover, the methods of this paper can also provide bounds for the following cases:

$$(3) \quad A < 0 \quad \text{and} \quad B > 0 \qquad\qquad (4) \quad A = 0,\ B = 0,\ y''(x_0) > 0.$$

We elect to discuss (3) and (4) in a future work. However, for an example of Case (3), see Example 4.8.

Remark 2.19. We can relax the requirement that $p(x) > 0$. We merely need $p(x)$ to be eventually positive, at least in the case of providing lower bounds for c. We would use $p_u^+(x)$ instead of $p_u(x)$ in part (a) of Theorem 2.14 above, where

$$p_u^+(x) = \max(0, p_u(x))\,.$$

The condition $p(x) > 0$ on $[x_0, \infty)$ is necessary to have any chance to obtain upper bounds for c, however.

Next, we consider a few other bounds for c in the case of $A > 0$. One is a modification of a bound given in Eliason [11]. The others are based upon a numerical integration, after a transformation, of the differential equation (2.17).

The following theorem is a generalization of a theorem given in Hille [20] and Eliason [11].

Theorem 2.20. *Consider IVP (1.1). Suppose $p'(x) \le 0$, $x \ge x_0$. Suppose $\int_{x_0}^x \sqrt{p(t)}dt \to \infty$ as $x \to \infty$. Then:*

a) *there exists c with $x_0 < c < \infty$ and $\lim\limits_{x\to c^-} y(x) = +\infty$*

b) *If, in addition, $B^2 \ge \frac{2p(x_0)}{\eta+1}A^{\eta+1}$, then*

$$\sqrt{\frac{2}{1+\eta}}\int_{x_0}^c \sqrt{p(t)}dt \le \frac{2}{\eta-1}A^{\epsilon}, \tag{2.29}$$

where $\epsilon = \frac{1-\eta}{2}$. Let U_H denote the upper bound for c given by (2.29).

c) *If $B^2 \ge \frac{2p(x_0)}{\eta+1}A^{\eta+1}$, then there exists a constant $M > 0$ independent of x such that*

$$y(x) \le M(c-x)^{\frac{2}{1-\eta}}, \quad x_0 \le x < c\,.$$

Proof. The proof is similar to that given in Hille [20], except $p(t) = t^{-3/2}$ there. Multiplication of (2.17) by $2y'$ gives

$$2y'(t)y''(t) = 2y'(t)p(t)y(t)^{\eta}.$$

Integration by parts from x_0 to x and using $p'(t) \le 0$ gives

$$y'(x)^2 \ge \frac{2p(x)}{\eta+1}y(x)^{\eta+1} + \left[B^2 - \frac{2p(x_0)}{\eta+1}A^{\eta+1}\right] \tag{2.30}$$

$$= \frac{2p(x)}{\eta+1}y(x)^{\eta+1} + c_0, \text{ say}\,. \tag{2.31}$$

Let U_H denote the upper bound of c defined by (2.29). Multiplying by $y(x)^{-1-\eta}$ and taking square roots, we get

$$y'(x)\cdot y(x)^{\frac{-1-\eta}{2}} \ge \sqrt{\frac{2}{1+\eta}p(x) + c_0 y(x)^{-1-\eta}}, \tag{2.32}$$

where (2.32) is valid on $x_0 \le d < c$ for some $d \ge x_0$. Integration of (2.32) from $x = d$ to $x = w$ gives

$$\frac{2}{1-\eta}\left[y(w)^{\frac{1-\eta}{2}} - y(d)^{\frac{1-\eta}{2}}\right] > \int_d^w \sqrt{\frac{2}{1+\eta}p(x) + c_0 y(x)^{-1-\eta}}dx\,. \tag{2.33}$$

The left-hand side of (2.33) remains bounded as $w \to \infty$. The integral in (2.33) diverges to ∞ as $w \to \infty$, since $\int_{x_0}^x \sqrt{p(t)}dt \to \infty$ as $x \to \infty$. Thus, there exists c with $x_0 < c < \infty$ such that $\lim\limits_{x\to c^-} y(x) = +\infty$. This proves part (a).

To prove part (b), we proceed in an analogous manner, starting with (2.29) above. Since $c_0 \geq 0$ by assumption, from (2.33), we obtain

$$\frac{2}{\eta-1} y(x)^{\frac{1-\eta}{2}} > \sqrt{\frac{2}{1+\eta}} \int_x^c \sqrt{p(t)}dt \,.$$

Letting $x = x_0$ proves part (b). By the Mean Value Theorem, for integrals, there is an x' in (x, c) with

$$\sqrt{\frac{2}{1+\eta}} \int_x^c \sqrt{p(t)}dt = \sqrt{\frac{2}{1+\eta}} \sqrt{p(x')}(c-x) \,.$$

Since $p'(t) \leq 0$, we deduce

$$\frac{2}{\eta-1} y(x)^{\frac{1-\eta}{2}} \geq \sqrt{\frac{2}{1+\eta}} \sqrt{p(c)} \cdot (c-x) \,.$$

Consequently, we obtain

$$y(x) \leq \left[\left(\frac{\eta-1}{2} \right) \sqrt{\frac{2}{1+\eta}} \right]^{\frac{2}{1-\eta}} [p(c)]^{\frac{1}{1-\eta}} (c-x)^{\frac{2}{1-\eta}} \,. \tag{2.34}$$

This proves part (c). □

Example 2.1. Consider the IVP

$$y'' = \frac{1}{(x+1)^2} y^3, \quad y(0) = 1, \quad y'(0) = 1 \,.$$

Then a vertical asymptote exists to the solution $y(x)$, by Theorem 2.20 above. Theorem 1 of Bobisud [5] is not applicable here, since $\frac{|p'(x)|}{[p(x)]^{3/2}} \equiv 2$ does not satisfy $\lim\limits_{x\to\infty} \frac{|p'(x)|}{[p(x)]^{3/2}} = 0$.

Remark 2.21. Theorem 2.20 requires $p'(x) \leq 0$. It is interesting to note that if $p'(x) \geq 0$ and $p(x)$ is absolutely monotone on $[x_0, \infty)$ (as defined by Boas [4]), then certain derivative inequalities found in Boas [4] and Pečarič [26], together with Bernstein's Theorem, can be used to prove existence of vertical asymptotes using Lemmas 2.5 and 2.6 above. We omit details here, since the emphasis is on bounds for c.

Remark 2.22. In Theorem 2.20, the condition $B^2 \geq \frac{2p(x_0)}{\eta+1} A^{\eta+1}$ is a generalization of a condition given by Hille ([19], [20]) and Eliason which guarantees the existence of a vertical asymptote for the Thomas-Fermi equation

$$\begin{aligned} &y'' = x^{-1/2} y^{3/2} \\ &y(x_0) = A, \quad y'(x_0) = B, \quad (x_0 > 0) \,. \end{aligned}$$

Application of Theorem 2.20 in this case leads to the following upper bound on c:

$$c < \left(x_0^{3/4} + \frac{3}{2}\sqrt{5} A^{-1/4} \right)^{\frac{4}{3}}$$

which is Equation (1.9) of Eliason [11] and Equation (4.4) of Hille [20]. We shall have a little more to say about the Thomas-Fermi equation in Section 3 later.

Theorem 2.23. *Consider the IVP (1.1). Suppose $p(x) > 0$ is continuous on $[x_0, \infty)$.*

a) *Let u^* denote any initial upper bound for c. Let*

$$w_1 = B^2 \quad \textit{and} \quad w_2 = \frac{2p_u(u^*)A^{\eta+1}}{\eta+1} \,.$$

Let $D_1 = w_1 - w_2$ *and* $D_2 = \frac{2\eta+2}{\eta-1}$ *and suppose* $D_1 \leq 0$*. Then a lower bound* L_2 *for* c *is given by*

$$L_2 = x_0 + \frac{2A}{\eta-1}\left[w_2 + D_1\left(\frac{1}{2}\right)^{D_2}\right]^{-\frac{1}{2}}, \tag{2.35}$$

provided the expression in brackets is positive.

b) *Suppose that either* u^* *exists or that*

$$D_3 = \inf_{x_0 \leq x < \infty} p(x) \geq 0\,.$$

Define D_4 *as follows:*

$$D_4 = \begin{cases} D_3, & \textit{if} \quad D_3 > 0 \\ u^*, & \textit{if} \quad D_3 = 0\,. \end{cases}$$

Let $w_3 = \frac{2p_L(D_4)A^{\eta+1}}{\eta+1}$*. If* $w_1 \leq w_3$*, then an upper bound* U_2 *for* c *is given by*

$$U_2 = x_0 + \frac{A}{\eta-1}\left(w_1^{-1/2} + w_3^{-1/2}\right). \tag{2.36}$$

Proof. Starting with

$$y''y' = p(x)y^\eta y' \tag{2.37}$$

and integration of both sides of (2.37) eventually results in

$$c - x_0 = \int_A^\infty \left(B^2 - \frac{2p(t')}{\eta+1}A^{\eta+1} + \frac{2p(t')}{\eta+1}y(t)^{\eta+1}\right)^{-\frac{1}{2}} dt\,,$$

for some $t' \in (x_0, c)$, by the Generalized Mean Value Theorem for integrals. By the change of variable $z = 1 - \left(\frac{A}{y(t)}\right)^{\frac{\eta-1}{2}}$, we obtain

$$c - x_0 = \frac{2A}{\eta-1}\int_0^1 \left[w_2 + \left(w_1 - \frac{2p(t')}{\eta+1}A^{\eta+1}\right)z^{D_2}\right]^{-\frac{1}{2}} dz\,. \tag{2.38}$$

Since $p(t') \leq p_u(U_1)$, the result now follows by the convexity of the integrand of (2.38) and the Midpoint Rule approximation to the integral of (2.38). This completes the proof of part (a).

The proof of (b) is very similar except the Trapezoidal Rule approximation is used at the end of the proof instead. Hence, the proof of (b) is omitted. □

Remark 2.24. We may take $u^* = U_1$, where U_1 is given in Theorem 2.14 above. Also, the lower bound L would exist, in particular, if $p(x)$ is nonincreasing in x, and the upper bound U would exist, in particular, if $p(x)$ is nondecreasing in x, provided that $y(x_0) = A$ and $y'(x_0) = B$ satisfied the other conditions of the theorem. The bounds of this theorem are of closed form and are offered as more easily computable alternatives to other bounds already discussed. We shall numerically compare many of the bounds discussed in this paper in subsequent examples. We may replace $p_u(u^*)$ in part (a) by any constant P, if there exists $P > 0$ such that $0 < p(x) \leq P$, for $x \geq x_0$. Similarly, we may replace $P_L(D_4)$ by any constant Q such that $Q > 0$ and $Q \leq P(x)$, for $x \geq x_0$. For example, if $x_0 = 0$, and $p(x) = \frac{2x+2}{x+2}$, we may use $P = 1$ and $Q = 2$.

Next, we show that the methods of Eliason [11] can be modified to produce a lower bound for c in the case $A > 0$ and $B > 0$, after applying comparison results discussed earlier.

Theorem 2.25. *Suppose $A > 0$ and $B > 0$ in IVP (1.1). Under the conditions stated in Theorem B, a lower bound L_3 for c is the unique root (value of x) of:*

$$A^{\frac{\eta-1}{2}} \int_{x_0}^{x} \sqrt{P_M(t)}dt = z(\eta)\,, \tag{2.39}$$

where $z(\eta)$ is given by (1.4) and

$$P_M(x) = \sup_{x_0 \le t \le x} \left[p(t) \cdot \left(1 + \frac{B(t-x_0)}{A} \right)^{\eta} \right] . \tag{2.40}$$

Proof. Let w be the operator given by $w = w(x) = Y_u(x) - B(x - x_0)$. Then

$$w' = Y_u' - A, \quad w'' = Y_u'', \quad w(x_0) = A, \quad w'(x_0) = 0\,.$$

IVP (1.1) becomes

$$w''(x) = \left(p(x) \cdot \left[\frac{w(x) + B(x-x_0)}{w(x)} \right]^{\eta} \right) \cdot [w(x)]^{\eta}, \quad w(x_0) = A, \quad w'(x_0) = 0\,. \tag{2.41}$$

By comparison with the IVP

$$W''(x) = (p_M(x)) \cdot [W(x)]^{\eta}, \quad W(x_0) = A, \quad W'(x_0) = 0\,, \tag{2.42}$$

applying Lemma 2.1, we see that if IVP (1.1) has an asymptote at $x = c$ and IVP (2.42) has asymptote at $x = x^*$, then $c \ge x^*$. But x^* is at least as large as the unique root of (2.39) above, by Theorem 1.1. This completes the proof. $\square$

Next, we state that Theorem 1.5 (Theorem 3 of Bobisud [5]) can be extended to $p(x)$ that are not nondecreasing after applying Lemma 2.1. We omit the straightforward proof.

Theorem 2.26. *Consider IVP (1.1). Under the conditions of Theorem 1.5, omitting the nonnegative derivative of $p(x)$ requirement, a lower bound $L_{B,4}$ for c is the unique root of the equation*

$$\int_A^{\infty} \frac{dx}{\sqrt{\frac{B^2}{p(x_0)} + 2\int_{x_0}^{x} f(u)du}} = \int_{x_0}^{x} \sqrt{P_u(t)}dt\,. \tag{2.43}$$

Next, we state a famous inequality which will be used to obtain one more lower bound for c, when $p(x)$ is nonincreasing in x. Since there are many versions of this inequality, we state a form most convenient for our use here. It is the Grüss inequality, a special case of Chebyshev-type inequalities. For a discussion on these inequalities, see Barza and Persson [1], Beesack and Pečarić [2], and Mitrinović, Pečarić and Fink [22].

Grüss Inequality. Let $F(x)$ and $G(x)$ be continuous on $[a, b]$. Suppose F is nonincreasing and G is nondecreasing on $[a, b]$. Then

$$(b-a) \int_a^b F(x)G(x)dx \le \int_a^b F(x)dx \cdot \int_a^b G(x)dx\,. \tag{2.44}$$

Theorem 2.27. *Consider IVP (1.1) with $A > 0$, $B \ge 0$. Suppose $p(x) \ge 0$ is continuous on $[A, \infty)$. If $p(x)$ is nonincreasing on $[A, \infty)$, then a lower bound L_G for c is the unique root of the equation*

$$\frac{1}{\eta - 1} A^{1-\eta} = (x - x_0)^{-1} A_1(x) A_2(x)\,, \tag{2.45}$$

where

$$A_1(x) = \int_{x_0}^{x} I(t)dt, \quad A_2(x) = \int_{x_0}^{x} \frac{1}{\eta+1} \left[\left(1 + \frac{B(t-x_0)}{A}\right)^{\eta+1} \frac{A}{B} - 1 \right] dt,$$

$$I(t) = (t-x_0)^{-1} \int_{x_0}^{t} p(u)du\,. \tag{2.46}$$

Proof. As done in the proof of Theorem 2.25, the substitution $w = w(x) = Y_u(x) - B(x-x_0)$ into (1.1) leads to the equivalent IVP

$$w''(x) = \left(p(x) \cdot \left[\frac{w(x) + B(x-x_0)}{w(x)} \right]^{\eta} \right) \cdot (w(x))^{\eta}\,,$$

$$w(x_0) = A, \quad w'(x_0) = 0\,. \tag{2.47}$$

So

$$w''(t) \le \left(p(t) \cdot \left[1 + \frac{B(t-x_0)}{A} \right]^{\eta} \right) \cdot (w(t))^{\eta}\,. \tag{2.48}$$

Integration of (2.48) from x_0 to x, using Grüss' Inequality, gives

$$w'(x) \le I(x) \left(\int_{x_0}^{x} \left[1 + \frac{B(t-x_0)}{A} \right]^{\eta} dt \right) \cdot [w(x)]^{\eta}\,, \tag{2.49}$$

using the obvious inequality $\int_{x_0}^{x} w(t)^{\eta} dt \le (x-x_0)(w(x))^{\eta}$. Integration of (2.49) one more time yields, after division by $w(x)^{\eta}$, $\frac{1}{\eta-1} A^{1-\eta} \le (x-x_0)^{-1} A_1(x) A_2(x)$. The result follows upon letting $x \to (x^*)$ and using $L_G \le x^* \le c$.

Next, we shall numerically compare the new bounds given in this paper to those of Eliason [11] and Bobisud [5]. The bounds of Bobisud [5] are more general from the standpoint of being valid for the more general differential equation $y'' = p(x)f(y)$. However, $p(x)$ is not allowed to have a singularity in his Theorems 2 and 3 (Theorems 1.3 and 1.5 above), whereas we shall, in Section 3, allow for the possibility of a singularity in $p(x)$ at $x = x_0$. Thus, the new bounds complement, and in some cases, improve on the bounds of the above two authors as we shall see in subsequent examples. □

Next, we present a few numerical examples to compare the lower bounds and upper bounds (if they exist) of c.

Example 2.2. Table 2.1 below gives a numerical comparison of various lower and upper bounds for c for the IVP

$$y''(x) = (3e^x + e^{2x})[y(x)]^3, \quad y(x_0) = A > 0, \quad y'(x_0) = B > 0$$

where $y = (3-e^x)^{-1}$ for various choices of x_0. The actual value of c is $c = Ln3 \approx 1.0986123$. Here, $p(x) = 3e^x + e^{2x}$, $\eta = 3$, $\epsilon = -1$.

This example illustrates many points which seem to hold in many other examples considered by the author, but not presented here. These are:

1) If $p(x)$ is nondecreasing in x and $\eta > 2$ the Theorem 3 lower bound of Bobisud [5], $L_{B,3} = L_{B,4}$ is the best bound, unless x_0 is near the asymptote, in which case the new bound, L_1, is best. The Theorem 2 bound of Bobisud [5], $L_{B,3}$, is more generally applicable, but not as good as L_1, in this case.
2) No bound is the best in all cases (all choices of x_0). This has been observed in many other IVPs for a wide range of $p(x)$ behavior and value of $\eta > 1$. It does not seem possible to easily compare all bounds in this paper analytically for this reason. Hence,

x_0	L_1	L_2	$L_{B,2}$	$L_{B,3}$	L_3	U_1	U_2	$U_{E,1}$	$U_{E,2}$
-2.0	0.011	-1.990	0.317	0.882	0.864	19.17	11.69	6.16	1.80
-1.0	0.419	-0.992	0.373	0.954	0.887	6.15	4.25	3.38	1.63
0.0	0.819	0.333	0.573	1.034	0.942	2.00	1.71	1.86	1.41
0.5	0.989	0.890	0.769	1.067	0.996	1.32	1.26	1.41	1.28
1.0	1.094	1.093	1.038	1.087	1.078	1.10	1.17	1.13	1.13

Table 2.1: $L_{B,3} = L_{B,4}$ since $p(x)$ is nondecreasing.

we shall give enough numerical examples to compare the various asymptote bounds and also to illustrate the application of theorems obtained in this paper.

3) Among the upper bounds, the upper bound of Eliason [11], $U_{E,2}$ and U_H are usually the best. However, there were many IVPs for which the computer algebra package MAPLE would not compute $P_A(x)$ given by (1.7), which is needed to compute $U_{E,2}$ mainly because $P_A(x)$ is often not of closed form. The above example was chosen so that $P_A(x)$ would be of closed form. Note that the upper bound U_1 is of closed form and is easily hand computable. These two new upper bounds are also more accurate than $U_{E,2}$, if x_0 is not too far from c.
4) An iterative version of the Runge-Kutta (4,4) (RK) numerical method was applied to this problem to obtain a sequence of 'pseudo'-lower and upper bounds. The current lower bound value of L_1 was taken as the new value of x_0 at each iteration. Thus, the RK method was successively applied on intervals of the form $I_K = [L_1^{(K)}, L_1^{(K+1)}]$, where $L^{(K)}$ = iteration K value of the L_1 lower bound. Thus, $L_1^{(K+1)}$ is the L_1 lower bound for the IVP

$$y'' = (3e^x + e^{2x})y^3\,,$$
$$y(L_1^{(K)}) = \widehat{y}(L_1^{(K)}), \quad y'(L_1^{(K)}) = \widehat{y}'(L_1^{(K)})\,,$$

where $\widehat{y}(\cdot)$ and $\widehat{y}'(\cdot)$ are the RK approximations of y and y', respectively. The RK method was applied to each I_K interval, moving forward from below to a final lower bound approximation to x^*. Table 2.2 below gives the values of $L_1^{(K)}$ and $U_1^{(K)}$, the K^{th} iteration value (approximation of $U_1(L_1^{(K)})$. After 20 iterations, however, these

K	$L_1^{(K)}$	$U_1^{(K)}$
1	0.818507	2.000000
2	1.068355	1.141719
3	1.098107	1.09909
4	1.098545	1.098545
5	1.098545	1.098545
6	1.098545	1.098545

Table 2.2:

values start deteriorating, because of truncation and round-off error. Also, the 'pseudo'-upper bounds are no longer real upper bounds for $K \geq 4$. Here, the 'final' value of

$L_1 = 1.098545$ is in error by only 6.73×10^{-5}, using 5000 equally-spaced values in interval I_K at each iteration.

Example 2.3. Consider the IVPs below. The Runge-Kutta method was used to approximate c in IVPs (a) and (b), since the exact solution was unknown. ($U_{E,2}$ not applicable or computable not considered). The results are given in Table 2.3

(a) $y''(x) = (x+1)^{-1}[y(x)]^3$, $y(0) = 2$, $y'(0) = 1$ $(c \approx 0.960)$.
(b) $y''(x) = (x+1)^{-1}[y(x)]^4$, $y(0) = 1$, $y'(0) = \frac{1}{2}$ $(c \approx 1.33)$.
(c) $y''(x) = \left[\frac{12x+4}{9(x+1)^4}\right][y(x)]^7, \quad y(0) = 1, \quad y'(0) = \frac{2}{3}$. The exact solution is $y(x) = \left(\frac{x+1}{1-x}\right)^{1/3}$, with $c = 1.00$.
(d) $y''(x) = [\exp(x^2-2x+4)]\cdot(y(x))^3, \quad y(0) = \frac{1}{10}, \quad y'(0) = \frac{1}{100}$, $p(x)$ is nonmonotonic (bathtub-shaped), c is unknown.
(e) $y''(x) = \left[\frac{3\sqrt{x+8}-1}{4(x+8)^{3/2}}\right](y(x))^3, \quad y(0) = 5.828, \quad y'(0) = 6.005$. The exact solution is $y(x) = (3-\sqrt{x+8})^{-1}$, with $c = 1$, $p(x)$ is decreasing.

IVP	L_1	L_2	$L_{B,2}$	$L_{B,3}$	$L_{B,4}$	L_G	L_3	U_1	U_2	$U_{E,1}$	U_H
(a)	0.71	0.73	0.41	NA	0.78	0.30	0.93	2.00	1.61	1.14	NA
(b)	1.05	1.07	0.47	NA	1.01	0.97	1.10	1.75	1.54	1.96	NA
(c)	0.50	0.82	0.23	NA	0.60	0.88	0.65	2.59	2.43	3.57	1.67
(d)	0.50	NA	0.24	NA	0.98	NA	1.15	NA	NA	3.94	NA
(e)	0.97	NA	0.36	NA	0.79	0.76	0.39	1.03	NC	1.332	1.008

Table 2.3: NA = not applicable, NC = not computable

Again, we see mixed results, although the new lower bounds L_1, $L_{B,4}$ do well. The new upper bounds U_1 and U_2 sometimes do better than $U_{E,1}$, although they are not always computable, whereas the Eliason bound always is, although it is harder to compute. Note that L_G is the best lower bound for IVP (c) and that each of the new lower bounds (L_1, L_2, L_3, $L_{B,4}$) is best or nearly the best in at least one of the IVPs above. This is the reason why we discuss many lower bounds in this paper. Among the new lower bounds, L_1 and L_2 are the easiest to compute. However, L_1 can be found in more general situations. When it exists, U_H is a better bound than $U_{E,1}$ in most cases. However, $U_{E,1}$ is more generally applicable.

We shall next present some examples where $p(x)$ is not nondecreasing in x, to illustrate Theorem 2.14, part (b).

Example 2.4. Consider the following IVPs.

(a) $$y''(x) = [(x^2-22x+131)e^{-x/5}]\cdot(y(x))^{6/5}, \quad y(0) = 1, \quad y'(0) = 11.$$

The exact solution is $y(x) = \frac{e^x}{(1-x)^{10}}$. Here, $c = 1$, $\eta = 6/5$, $\epsilon = \frac{-1}{10}$, $p(x)$ is decreasing in x. We obtain: $L_1 = 0.909$, $L_{B,2} = 0.206$, $L_{B,4} = 0.264$, ($L_{B,3}$ is not applicable), $L_2 = 0.907$, $L_3 = 0.482$, $L_G = 0.439$. The Eliason upper bound is $U_{E,1} = 1.074$. Since $\eta = \frac{6}{5}$ is close to 1, the new bounds are better than $L_{B,2}$. Note that U_1 and U_2 are not applicable here. Also, $U_H = 1.0047$. So U_H does better than $U_{E,1}$ here. However, $U_{E,1}$ is more generally applicable.

(b) The IVP of Thomas-Fermi type (see Hille [19], [20] for a discussion of this equation)
$$y''(x) = x^{-1/2}(y(x))^{3/2}, \quad y(1) = 2, \quad y'(1) = 1.$$

It is easily verified that conditions (H1)–(H3) of Theorem 2.14 hold. We obtain ($L_{B,3}$ is not applicable) $L_1 = 4.761$, $L_{B,2} = 2.465$, $L_G = 2.274$, $L_3 = 4.445$, $L_{B,4} = 3.065$, $L_2 = 4.763$, $U_1 = 9.000$, $U_{E,1} = 6.763$, $U_2 = 8.256$, U_H is not applicable here. The exact value of c is unknown. We have: $4.763 < c < 6.763$ by the foregoing, however. The RK method discussed earlier converged to $c \approx 6.164$. Extensive curve-fitting of Padé approximants done by the author found asymptote estimates ranging from $c \approx 5.964$ to $c \approx 6.063$. These findings are obviously consistent with all the above bounds. Here, the new lower bounds L_1 and L_2 are best and the Eliason upper bound $U_{E,1}$ is best. (MAPLE would not compute $U_{E,2}$.)

The new lower bounds often considerably improve on the bounds of Bobisud [5] when either $p(x)$ is nonincreasing in x or $\eta \leq 2$. Many numerical examples considered besides the ones presented in this paper seem to confirm this observation. For $p(x)$ nondecreasing and $\eta > 2$, the Bobisud bound $L_{B,3} = L_{B,4}$ seems to be best. The new upper bounds improve on the Eliason upper bounds especially in the cases with $A \gg 0$ and $B \gg 0$.

We shall soon discuss the case $A = 0$ when $p(x) = (x - x_0)^\theta q(x)$ has at least one singularity, first at $x = x_0$ (when $\theta < 0$). We shall obtain bounds for c for IVPs in which it is not possible to apply the previous lower bounds of Bobisud [5] or Eliason [11], in Section 3. Let us consider a few examples with nonmonotonic $p(x)$.

Example 2.5. Consider the IVP

$$y''(x) = \left[\frac{4}{(2x^2 - x + 1)^3}\right][y(x)]^3, \quad y(0) = 1, \quad y'(0) = 0\,.$$

The exact solution is $y(x) = \frac{2x^2-x+1}{1-x}$, with $c = 1.00$. Here $p(x)$ is not monotonic, is unimodal with maximum value $p(0.25) \approx 5.971$. Since the methods of Eliason [11] require $B = 0$, it should be the case that the Eliason bounds are best here. This is in fact the case for upper bounds, but not lower. The following bounds were obtained:
$L_1 = 0.579$, $L_2 = 0.598$, $L_{B,2} = 0.438$, $L_{B,3}$ is not applicable, $L_{B,4} = 0.904$, $L_3 = 0.776$, $L_{E,1} = 0.776$.
Only the Eliason upper bounds are applicable: $U_{E,1} = 1.488$. We see that the new modified Bobisud bound $L_{B,4} = 0.904$ is the best lower bound here.

Example 2.6. Consider the IVP

$$y''(x) = \left[\frac{-x^4 + 4x^3 + 6x^2 - 4x + 7}{4(x^2+1)^4}\right][y(x)]^5, \quad y(-1) = +1, \quad y'(-1) = -\frac{1}{4}\,.$$

The exact solution is $y = \sqrt{\frac{x^2+1}{1-x}}$ with $c = 1$, and $p(x)$ is unimodal here with maximum $p(-0.0840) \approx 1.7928$. Only the new bound L_1 is applicable here since $A = y'(-1) < 0$, and both the bounds of Bobisud [5] and Eliason [11] require $B \geq 0$. We obtain $(\epsilon = -2.00)$

$$L_1 = -0.299.$$

L_1 can generally be used (under conditions of Theorem 2.14) if $y'(x_0) = B < 0$, provided $y(x)$ remains positive. In this case, we modify $g_1(x)$ of Theorem 2.14 to get

$$g_1(x) = \min\left(\epsilon A^{\epsilon-1}|B|,\ \frac{\epsilon}{\sqrt{1-\epsilon}}\sqrt{p_L(x)}\right)\,.$$

3. THE CASE $A = 0$ AND $B > 0$

In this section, we present methods for finding bounds for c in the case $A = 0$ and $B > 0$. Not only do we present bounds for the continuous coefficient $(p(x))$ case, we also present bounds for the case of a singularity at $x = x_0$ of a certain form. As remarked by Eliason [11, p. 483], 'bounds for the case $A = 0$ are certainly of interest'. In this section, we will in fact obtain bounds for the case $A = 0$ for a more general IVP which allows for a singularity at $x = x_0$.

We consider the IVP

$$y''(x) = p(x)[y(x)]^\eta, \quad y(x_0) = A = 0, \quad y'(x_0) = B > 0, \tag{3.1}$$

where

$$p(x) = (x - x_0)^\theta q(x), \quad q(x) > 0, \quad \theta \leq 0\,,$$

with vertical asymptote at $x = c$.

If $\theta = 0$, then we may use the new bound $L_{B,4}$ given earlier.

When $\theta \leq 0$ and $\theta + \eta \geq 1$, then Theorem 3.1 below demonstrates that a modification (after a transformation of $Y_u(x)$) of the method of Eliason [11] provides a lower bound for c when $A = 0$.

Theorem 3.1. *Suppose $\theta + \eta \geq 1$ in IVP (3.1). Suppose $q(x) > 0$ on $[x_0, \infty)$ is continuously differentiable. Consider the auxiliary IVP:*

$$Z''(x) = \left[(x - x_0)^{\theta+\eta-1} q_u(x)\right] \cdot [Z(x)]^\eta, \quad Z(x_0) = B, \quad Z'(x_0) = 0\,. \tag{3.2}$$

Then any lower bound for a vertical asymptote of (3.2) is also a lower bound for c.

Proof. Let $u(x) = \frac{Y_u(x)}{x - x_0}$. Then IVP (2.12) with $f(y) = y^\eta$ becomes

$$\begin{gathered}(x - x_0)u''(x) + 2u'(x) = [(x - x_0)^\theta q(x)(x - x_0)^\eta u(x)^\eta], \\ u(x_0) = B, \quad u'(x_0) = 0,\end{gathered} \tag{3.3}$$

where we have applied L'Hospital's Rule to obtain

$$\begin{aligned} u'(x_0) &= \lim_{x \to x_0} \frac{(x - x_0)Y_u'(x) - Y_u(x)}{(x - x_0)^2} \\ &= \lim_{x \to x_0} \frac{Y_u''(x)}{2} - \lim_{x \to x_0} \frac{Y_u'(x)}{2(x - x_0)} \\ &= \frac{Y_u''(x_0)}{2} - \lim_{x \to x_0} \frac{Y_u''(x)}{2} = 0\,. \end{aligned}$$

A solution to (3.3) must be bounded above by a solution to

$$Z''(x) = \left[(x - x_0)^{\theta+\eta-1} q_u(x)\right] \cdot [Z(x)]^\eta, \quad Z(x_0) = B, \quad Z'(x_0) = 0 \tag{3.4}$$

upon application of comparison techniques to IVPs (3.3) and (3.4), and using $u'(x) \geq 0$, $q(x) \leq q_u(x)$.

Let L_5 denote the lower bound $L_{E,1}$ applied to IVP (3.4) instead.

Next, we offer a simple lower bound in the case $\theta + \eta > -1$, which is more generally applicable than $L_{B,4}$ or L_5. □

Theorem 3.2. *Consider IVP (3.1) with $A = 0$, $B > 0$. Suppose $q(x) > 0$ is continuous on $[x_0, \infty)$. Suppose $\theta + \eta > -1$. Then a lower bound L_6 for c is the unique root of the equation*

$$\int_{x_0}^{x} (t - x_0)^{\theta+\eta} q_u(t)dt = \frac{B^{1-\eta}}{\eta - 1}\,. \tag{3.5}$$

Proof. IVP (3.1) will be transformed to obtain an auxiliary IVP for comparison purposes.

$$Y_u''(x) = (x-x_0)^{\theta} q_u(x)[Y_u(x)]^{\eta} \tag{3.6}$$
$$= (x-x_0)^{\theta+\eta} q_u(x) \left[\frac{Y_u(x)}{x-x_0}\right]^{\eta}$$
$$\leq (x-x_0)^{\theta+\eta} q_u(x)[Y_u'(x)]^{\eta}.$$

Letting $V = Y_u'$, we obtain the auxiliary IVP

$$V_u'(x) = \left[(x-x_0)^{\theta+\eta} q_u(x)\right] \cdot (V(x))^{\eta}, \quad V(x_0) = B\,. \tag{3.7}$$

Clearly, the location of the vertical asymptote of (3.7) will be to the right of any vertical asymptote of (3.6), by comparison lemmas given earlier. Integration of (3.7) produces

$$\int_B^{\infty} V^{-\eta} dV = \int_{x_0}^{x} (t-x_0)^{\theta+\eta} q_u(t)dt$$

or

$$\frac{B^{1-\eta}}{\eta - 1} = \int_{x_0}^{x} (t-x_0)^{\theta+\eta} q_u(t)dt\,.$$

The proof is complete. □

For our final method, we present a method which uses a variation of the method used to derive L_1 in Section 2. Since $A = 0$, however, we cannot use the $u(x) = [Y_u(x)]^{\epsilon}$ transformation as done there. Instead, we consider the operator $(x-x_0)^{-1}Y_u$, i.e.,

$$w(x) = \left(\frac{Y_u(x)}{x-x_0}\right)^{\epsilon}\,. \tag{3.8}$$

First, we need some lemmas.

Lemma 3.3. *Let* $u(x) = \frac{Y_u(x)}{x-x_0}$, $x_0 < x < x^*$. *Suppose* $q(x) > 0$ *is continuously differentiable on* $[x_0, \infty)$. *Then*

$$\lim_{x\to(x^*)^-} \frac{u(x)u''(x)}{[u'(x)]^2} = \frac{1+\eta}{2}\,.$$

Proof. Applying L'Hospital's Rule, we obtain $(Y_u = Y_u(x), Y_u'' = Y_u''(x))$

$$\lim_{x\to(x^*)^-} \frac{u(x)u''(x)}{[u'(x)]^2} = \lim_{x\to(x^*)^-} \left[\frac{(x-x_0)^2 Y_u Y_u'' - 2(x-x_0)Y_u Y_u' + 2Y_u^2}{(x-x_0)^2 (Y_u')^2 - 2(x-x_0)Y_u Y_u' + Y_u^2}\right]$$
$$= \lim_{x\to(x^*)^-} \frac{Y_u(x)Y_u''(x)}{[Y_u'(x)]^2} = \frac{1+\eta}{2},$$

upon application of Lemmas 2.5 and 2.6 with $f(y) = y^{\eta}$. □

Lemma 3.4. *Suppose* $q(x) > 0$ *is continuously differentiable on* $[x_0, \infty)$. *Let* $T(x) = (x - x_0)[u(x)]^{\epsilon-1}u'(x)$, $\epsilon = \frac{1-\eta}{2}$. *Then*

$$\limsup_{x\to(x^*)^-} T(x) \leq \frac{(x^*-x_0)^{\frac{\theta+\eta+1}{2}} \sqrt{q_u(x^*)}}{\sqrt{1-\epsilon}} = M\,.$$

Proof.

$$\begin{aligned}\limsup_{x\to(x^*)^-}\frac{u'}{u^{1-\epsilon}} &\le \limsup_{x\to(x^*)^-}\frac{u''}{(1-\epsilon)u'u^{-\epsilon}}\\ &\le \limsup_{x\to(x^*)^-}\left[\frac{u''}{(1-\epsilon)u^{-\epsilon}\sqrt{\frac{uu''}{1-\epsilon}}}\right]\cdot\lim_{x\to(x^*)^-}\left(\frac{\sqrt{\frac{uu''}{1-\epsilon}}}{u'}\right).\end{aligned}$$

By Lemma 3.3, the latter limit equals 1. So

$$\limsup_{x\to(x^*)^-}T(x)\le\limsup_{x\to(x^*)^-}\sqrt{\frac{(x-x_0)^{\theta+\eta}q_u(x)\cdot u(x)^\eta}{(x-x_0)(1-\epsilon)u(x)^{1-2\epsilon}}}\,,$$

using $u''\le(x-x_0)^{-1}Y_u''$, as $x\to(x^*)^-$. (In fact, the above argument shows that all 'lim sup's above can be replaced by usual limits.) Thus

$$\lim_{x\to(x^*)^-}T(x)\le\frac{(x^*-x_0)^{\frac{\theta+\eta+1}{2}}\sqrt{q_u(x^*)}}{\sqrt{1-\epsilon}}=M\,.$$

□

Lemma 3.5. *Let $K(x)$ be continuously differentiable and strictly at one sign on $[x_0,\infty)$. Suppose $\limsup_{x\to(x^*)^-}|K(x)|<\infty$. Then $|K(x)|\le K_0$ on $[x_0,x^*)$, where $K_0=\max(K_1,K_2,K_3)$,* $K_1=|K(x_0)|$, $K_2=\sup_{t\in Z_k}|K(t)|<\infty$, $K_3=\limsup_{x\to(x^*)^-}|K(x)|$, $Z_k=\{t\in[x_0,x^*):K'(t)=0\}$.

Proof. Let $A_n=[x_0,x^*-\frac{1}{n}]$, $B_n=[x^*-\frac{1}{n},x^*)$, $n=1,2,3,\dots$ Clearly, since A_n is compact,

$$\sup_{x\in A_n}|K(x)|\le\max\left(K_1,K_2,\left|K\left(x^*-\frac{1}{n}\right)\right|\right)$$

and

$$\left|K\left(x^*-\frac{1}{n}\right)\right|\le\sup_{x\in B_n}|K(x)|\,.$$

Let us show $\sup_{x\in B_n}|K(x)|\le K_3+\delta$ for arbitrarily small $\delta>0$. By definition of limit superior, $\exists$ integer N such that

$$\sup_{x\in B_N}|K(x)|\le K_3+\delta\,.$$

Thus,

$$\begin{aligned}\sup_{x\in[x_0,x^*)}|K(x)| &\le\max\left(\sup_{x\in A_n}|K(x)|,\quad\sup_{x\in B_n}|K(x)|\right)\\ &\le\max(K_1,K_2,K_3+\delta),\end{aligned}$$

implying that

$$\sup_{x\in[x_0,x^*)}K(x)\le K_0+\delta\,.$$

Since $\delta>0$ is arbitrary, the result follows. □

Armed with the knowledge of the above lemmas, we are now ready to prove the following theorem.

Theorem 3.6. *Consider IVP (3.1). Suppose $q(x) > 0$ is continuously differentiable on $[x_0, \infty)$. Let*

$$G_7(x) = \min\left(\frac{\epsilon}{2}B^{\epsilon+\eta-1}q(x_0), \quad \frac{\epsilon}{\sqrt{1-\epsilon}}(x-x_0)^{\frac{\theta+\eta-1}{2}}\sqrt{q_u(x)}\right), \tag{3.9}$$

where $\epsilon = \frac{1-\eta}{2}$. Suppose that either $\theta+\eta \geq 1$ or $\theta+\eta \geq 0$ and $w(x) = \left(\frac{Y_u(x)}{x-x_0}\right)^{\epsilon}$ has a strictly monotonic derivative $w'(x)$. Then a lower bound L_7 for c is the unique root (value of x) satisfying the equation

$$(x_0 - x)G_7(x) = B^{\epsilon}. \tag{3.10}$$

Proof. It is useful to recall the method of proof of Theorem 2.14 earlier. We use a device and method here that is similar, except we use a different operator. Let w be the operator given by

$$w = w(x) = \left(\frac{Y_u(x)}{x-x_0}\right)^{\epsilon} = u(x)^{\epsilon},$$

where $u(x) = \frac{Y_u(x)}{x-x_0}$, $x \neq x_0$, and $u(x_0) = B$. Then:

$$\begin{aligned} w' &= \epsilon u^{\epsilon-1}u', \\ w'' &= \epsilon u^{\epsilon-1}u'' + (u')^2\epsilon(\epsilon-1)u^{\epsilon-2}, \end{aligned}$$

which equals zero at a value of $x > x_0$ (if any) satisfying

$$u'(x) = \sqrt{\frac{u(x)u''(x)}{1-\epsilon}}. \tag{3.11}$$

Now the Mean Value Theorem implies

$$w(x) = w(x_0) + w'(d(x)) \cdot (x-x_0), \quad x_0 < d(x) < x, \quad x_0 < x < x^*.$$

Proceeding as in the proof of Theorem 2.14, a direct computation shows that $w'(x)$, for such values of x, satisfies:

$$|w'(x)| = \frac{-\epsilon}{\sqrt{1-\epsilon}}u(x)^{\epsilon-1}\sqrt{\frac{u(x)\cdot(x-x_0)^{\theta}q_u(x)Y_u(x)^{\eta} - 2u'(x))}{x-x_0}}, \tag{3.12}$$

assuming the right-hand side radical exists for the moment. Clearly $u'(x) \geq 0$. From (3.12), we have

$$\begin{aligned} |w'(x)| &\leq \frac{-\epsilon}{\sqrt{1-\epsilon}}u(x)^{\epsilon-1}\sqrt{\frac{u(x)\cdot((x-x_0)^{\theta}q_u(x)\cdot(x-x_0)^{\eta}u(x)^{\eta})}{x-x_0}} \\ &\leq \frac{-\epsilon}{\sqrt{1-\epsilon}}u(x)^{\epsilon+\frac{\eta}{2}-\frac{1}{2}}(x-x_0)^{\frac{\theta+\eta-1}{2}}\sqrt{q_u(x)}. \end{aligned} \tag{3.13}$$

If the square root in (3.12) does not exist, then we can still bound $|u'(x)|$. In either case, $|w'(x)|$ will be either maximized at $x = x_0$, at a value of x satisfying (3.11) or as $x \to (x^*)^-$ (where we shall apply the lemmas given above). Once we bound $|w'(x)|$ on $[x_0, x^*)$, this will allow us to get a lower bound for x^*, hence for c. From all the foregoing, we have, for $x_0 \leq x < x^*$:

$$\begin{aligned} &(x-x_0)\sup_{x_0\leq t\leq x}|w'(t)| \\ &\quad\leq \max\left((x-x_0)\cdot|w'(x_0)|,\ \frac{-\epsilon}{\sqrt{1-\epsilon}}(x-x_0)^{\frac{\theta+\eta+1}{2}}\sqrt{q_u(x)},\ \frac{(-\epsilon)}{\sqrt{1-\epsilon}}(x-x_0)M^*\right), \end{aligned}$$

upon application of Lemma 3.5, where

$$M^* = \lim_{x\to(x^*)^-} (u(x))^{\epsilon-1} u'(x)\,.$$

By direct computation, $|w'(x_0)| \le |\frac{\epsilon}{2} B^{\epsilon+\eta-1} q(x_0)|$, with equality holding if $\theta+\eta=0$. By Lemma 3.4,

$$\frac{-\epsilon}{\sqrt{1-\epsilon}} (x^*-x_0)^{\frac{\theta+\eta+1}{2}} \sqrt{q_u(x^*)} = (-\epsilon)(x^*-x_0)M^*\,.$$

Thus

$$\begin{aligned}(3.14)\quad & (x^*-x_0) \sup_{x_0\le t<x^*} |w'(t)| \\ & \le \max\left((x^*-x_0)\left(-\frac{\epsilon}{2}\right) B^{\epsilon+\eta-1} q(x_0),\ \frac{-\epsilon}{\sqrt{1-\epsilon}} (x^*-x_0)^{\frac{\theta+\eta+1}{2}} \sqrt{q_u(x^*)} \right)\,.\end{aligned}$$

But $w(x_0) = B^\epsilon$. So

$$w(x) = B^\epsilon + w'(d(x))(x-x_0)\,. \tag{3.15}$$

Since $\lim\limits_{x\to(x^*)^-} w(x) = 0$, letting $x \to (x^*)^-$ in (3.15), we obtain

$$x^* - x_0 = \frac{-B^\epsilon}{w'(d(x^*))}\,.$$

Since $|w'(x)| \le -G_7(x)$, $x > x_0$, we have $(x^*-x_0)(-G_7(x^*)) \ge B^\epsilon$. But $(L_7-x_0)(-G_7(L_7)) = B^\epsilon$. Since $(x-x_0)(-G_7(x))$ is increasing in x, we have $L_7 \le x^*$. Since $x^* \le c$, then $L_7 \le c$. This completes the proof. □

Remark 3.7. When $\theta = 0$ ($p(x)$ is continuous), we may also consider the Bobisud bound $L_{B,3}$ when $p(x)$ is nondecreasing. In this case, $L_{B,3} = L_{B,4}$. When $p(x)$ is not nondecreasing, we shall use the new modified bound $L_{B,4}$ since $L_{B,3}$ is not applicable in this case.

Remark 3.8. The author has also obtained bounds in the case $y(x_0) = A < 0$ and $B > 0$ for the equation $y'' = (x-x_0)^\theta q(x)|y|^\eta \operatorname{sgn}(y)$ using a two stage procedure. These results will be discussed in a forthcoming paper.

Next, we present some numerical examples to compare the bounds $L_{B,4}$, L_5, L_6 and L_7 in the case $A = 0$, $B > 0$. First, we consider an example where $\theta = 0$. In this case, all four bounds exist.

Example 3.1. Consider the IVP

$$y''(x) = \left[\frac{20-30x+12x^2}{(x^4-x+1)^3}\right] \cdot (y(x))^3, \quad y(0)=0, \quad y'(0)=1\,.$$

Here, $p(x)$ is unimodal on $[0,\infty)$ with

$$\sup_{x\in[0,1]} \sqrt{p(x)} = \sqrt{p(0.526)} \approx 45.19\,.$$

Hence,

$$p_u(x) = \begin{cases} p(x), & 0 \le x \le 0.526 \\ 45.19, & x > 0.526 \end{cases}$$

The exact solution is $y(x) = \frac{x^5-x^2+x}{1-x}$, with $c = 1$. Here, $\theta = 0$, $\eta = 3$, $\epsilon = -1$, $B = 1$. We obtain the lower bounds:

$$L_{B,4} = 0.815, \quad L_5 = 0.593, \quad L_6 = 0.614, \text{ and } L_7 = 0.462\,.$$

Here, the modified bound $L_{B,4}$ is the best. Note that the original bounds, $L_{B,2}$ and $L_{B,3}$ of Bobisud [5] are not applicable here, nor are the bounds of Eliason [11]. However, in this paper, two of the four bounds proposed in this paper are modified versions of these 'authors' bounds $(L_{B,4}, L_5)$ and two are derived totally from scratch (L_6 and L_7).

Example 3.2. Consider the IVP

$$y''(x) = x^{-3}\left(\frac{12-4x^2+x^3}{e^{2x}}\right)[y(x)]^3, \quad y(0)=0, \quad y'(0)=\frac{1}{2}\,.$$

Here, $\theta = -3$, $\eta = 3$, $q(x) = \frac{12-4x^2+x^3}{e^{2x}}$ is decreasing. We obtain: $L_6 = 0.204$, $L_7 = 0.667$. The exact solution is $y(x) = \frac{xe^x}{2-x}$ with $c = 2$. Note that $L_{B,4}$ is not applicable here. Neither are the bounds of Eliason [11] and Bobisud [5].

Example 3.3. Consider the IVP

$$y''(x) = 4x^{-3}[y(x)]^3, \quad y(0)=0, \quad y'(0)=\frac{1}{2}\,.$$

The exact solution is $y(x) = \frac{x}{2-x}$ with $c = 2$. We obtain: $L_6 = 0.500$ and $L_7 = 2.000$. We see that the bound L_7 is exact here. It can be shown that the L_7 (and L_1) bounds are sharp bounds, in general.

Example 3.4. Consider the IVP

$$y''(x) = x^{-3}((6-2e^x+xe^x+3x)e^x)[y(x)]^3, \quad y(0)=0, \quad y'(0)=\frac{1}{2}\,.$$

The exact solution is $y(x) = \frac{x}{3-e^x}$ with $c = \ln 3 \approx 1.0987$. The only lower bounds applicable are the new bounds $L_6 = 0.375$ and $L_7 = 0.712$.

Next, we apply the methods of this section to the Thomas-Fermi equation with $A = 0$, $x_0 = 0$.

$$y''(x) = x^{-1/2}[y(x)]^{3/2}, \quad y(x_0)=0, \quad y'(x_0)=B>0\,. \tag{3.16}$$

The bounds of Bobisud [5] are not applicable here, since $x_0 = 0$. Neither are the bounds given in Equations (5.2)–(5.3) of Eliason [11]. Incidentally, there is a small error in (5.2). In his paper, x_0 is denoted by a. The term $[y(a)]^{1/2}$ should be replaced by $[y(a)]^{1/4}$ throughout in (5.2). Also, '$\lambda = 0$' should be replaced by '$\lambda < 0$' right before Equation (5.10).

Example 3.5. Consider the Thomas-Fermi equation

$$y''(x) = x^{-1/2}(y(x))^{3/2}, \quad y(0)=0, \quad y'(0)=B>0\,.$$

We obtain

$$L_5 = \frac{z\left(\frac{3}{2}\right)}{B^{1/4}} \approx \frac{5.064}{B^{1/4}}, \quad L_6 = \frac{2}{B^{1/4}}\,.$$

Clearly L_5 is a better lower bound than L_6. However,

$$L_7 = \frac{B^{-1/4}}{\max\left(\frac{1}{8}B^{1/4}, \frac{\sqrt{5}}{10}\right)}$$

and from Table 3.1 below, we see that L_7 is only much inferior to L_5 for $B \geq 100.0$. For $B \leq 10.24$, $L_5 = 1.13L_7$. For $B > 10.24$, $\frac{L_5}{L_7} > 1.13$.

B	0.001	0.01	0.10	0.33	0.50	1.00	2.00
L_5	28.48	16.01	9.01	6.66	6.02	5.06	4.26
L_7	25.15	14.14	7.95	5.89	5.32	4.47	3.76
B	3.00	5.00	10.00	100.00	1000.0	10^5	10^8
L_5	3.85	3.39	2.85	1.60	0.901	0.285	0.0506
L_7	3.40	2.99	2.51	0.80	0.253	0.0253	0.0008

Table 3.1: L_5 and L_7 values.

4. SOME OTHER FAMILIES

In this section, we turn our attention to some other families of differential equations. First, consider the IVP:

$$y''(x) = p(x)f(y(x)), \quad y(x_0) = A, \quad y'(x_0) = B > 0 \tag{4.1}$$

or just $y'' = p(x)f(y)$.

First, let us consider the case $f(y) = e^{\beta y}$ for some $\beta > 0$. Consider the IVP

$$y''(x) = p(x)e^{\beta y(x)}, \quad y(x_0) = A, \quad y'(x_0) = B > 0\,. \tag{4.2}$$

First, we state the following lemma. The proofs are omitted since they are similar to proofs of earlier lemmas.

Lemma 4.1. *Consider IVP (4.2). Let $p(x)$ be continuously differentiable on $[x_0, \infty)$. Let $Y_u(x)$ and $Y_L(x)$ be given by (2.12) and (2.13), resp., with $\theta = 0$ $(p(x) \equiv q(x))$. Let $\alpha = -\beta/2$. Then*

(a) $\lim_{x \to (x^*)^-} -\alpha e^{\alpha Y_u(x)} \cdot Y_u'(x) = \sqrt{-\alpha}\sqrt{p_u(x^*)}$.

(b) $\lim_{x \to (x_*)^-} -\alpha e^{\alpha Y_L(x)} \cdot Y_L'(x) = \sqrt{-\alpha}\sqrt{p_L(x^*)}$.

(c) *Let $w_1(x) = e^{\alpha Y_u(x)}$. Then* $\lim_{x \to (x^*)^-} |w_1'(x)| = \sqrt{-\alpha}\sqrt{p_u(x^*)}$.

(d) *Let $w_2(x) = e^{\alpha Y_L(x)}$. Then* $\lim_{x \to (x_*)^-} |w_2'(x)| = \sqrt{-\alpha}\sqrt{p_L(x^*)}$.

Theorem 4.2. *Consider IVP (4.2). Let $\alpha = \frac{-\beta}{2}$. Let*

$$g_8(x) = \min\left(\alpha e^{\alpha A}B,\ -\sqrt{-\alpha}\sqrt{p_u(x)}\right)$$
$$h_8(x) = \max\left(\alpha e^{\alpha A}B,\ -\sqrt{-\alpha}\sqrt{p_L(x)}\right)\,.$$

Then

a) *a lower bound L_8 for c is the unique root of*

$$(x_0 - x)g_8(x) = e^{\alpha A}\,. \tag{4.3}$$

b) *Suppose that conditions (H4)–(H5) below hold. Then an upper-bound U_8 for c is the largest root of*

$$(x_0 - x)h_8(x) = e^{\alpha A} \tag{4.4}$$

(H4)

$$\liminf_{x\to\infty}\left[(x-x_0)\cdot\sqrt{p_L(x)}\right] > \frac{e^{\alpha A}}{\sqrt{-\alpha}}$$

or

$$\limsup_{x\to\infty}\left[(x-x_0)\sqrt{p_L(x)}\right] < \frac{e^{\alpha A}}{\sqrt{-\alpha}}\,.$$

(H5)

$$\sup_{x\geq x_0}(x_0-x)h_8(x) > e^{\alpha A}$$

Proof. The proof is similar to the proof of Theorem 2.14, so we merely sketch key steps and any new ideas needed. For part (a), let $u(x) = u = e^{\alpha Y_u(x)}$. Then

$$u''(x) = \alpha e^{\alpha Y_u} Y_u'' + (Y_u')^2 \alpha^2 e^{\alpha y} = 0$$

when

$$Y_u' = \sqrt{\frac{p_u(x)e^{\beta Y_u}}{-\alpha}}\,. \tag{4.5}$$

We can obtain the bound, for any value of x satisfying (4.5),

$$|w'(x)| \leq \sqrt{-\alpha}e^{\left(\alpha+\frac{\beta}{2}\right)Y_u}\sqrt{p_u(x)} = \sqrt{-\alpha}\sqrt{p_u(x)}\,. \tag{4.6}$$

Lemma 4.1, parts (a) and (c) show that (4.6) holds as $x \to (x^*)^-$ also. Now apply the Mean Value Theorem as done in the proof of Theorem 2.14.

The proof of part (b) is similar to the proof of part (b) of Theorem 2.14. We do not need a third condition "(H6)" to parallel condition (H3) of Theorem 2.14, since $Z = \infty$ in Lemma 2.10, so that the condition

$$Z > \sup_{x\geq x_0}\left(\frac{-p_L'(x)}{p_L(x)}\right) + 1$$

is automatically satisfied, as can be seen by considering

$$\lim_{x\to(x_*)^-}\frac{Y_L''(x)}{(Y_L'(x))^2} = -\alpha = \frac{\beta}{2}\,.$$

□

It is noteworthy to mention that the operator $u = e^{\alpha Y_u(x)}$ was also considered for the generalized Emden-Fowler IVP (1.1). However, the bounds by this operator were found inferior to those given earlier for that IVP.

Next, we consider a few numerical examples to compare L_8 to some other lower bounds discussed earlier. No upper bounds have previously been given in the literature for IVP (2.3) when A is allowed to be nonpositive. However, we also present U_8 when it can be shown to be a valid a priori upper bound in some examples below.

Example 4.1. Consider the IVP

$$y''(x) = e^{y(x)}, \quad y(0) = 2, \quad y'(0) = 1\,.$$

We obtain the Bobisud bound of $L_{B,3} = 0.705$, which is exact since $p(x) \equiv 1$ is constant. Theorem 4.2 gives $L_8 = 0.520$ and $U_8 = 1.414$, both of which are of closed form and computed by hand. Clearly, $L_8 < c = L_{B,3} = 0.705 < U_8$.

Example 4.2. Consider the IVP

$$y''(x) = \left[\frac{10x-5}{(x+2)^4}\right] e^{2y(x)}, \quad y(x_0) = A, \quad y'(x_0) = B,$$

where the exact solution is

$$y(x) = Ln\left(\frac{x+2}{3-x}\right) \quad \text{with} \quad c = 3.00\,.$$

Only the new lower bound L_8 is applicable for $A < 0$. Only the new bound U_8 is available as an upper bound. We obtain the following bounds ('NA' $=$ not applicable) for various x_0:

x_0	0.00	1.50	2.00	2.50	2.75
A	-0.405	0.847	1.386	2.197	2.944
B	0.833	0.952	1.250	2.222	2.988
$L_{B,3}$	NA	1.901	2.295	2.662	2.836
$L_{B,4}$	NA	2.808	2.943	2.984	2.996
L_8	1.200	2.550	2.800	2.950	2.988
U_8	NA	4.217	3.337	3.063	3.114

The new bounds $L_{B,4}$ and L_8 perform best. Only L_8 exists for $A < 0$. For $x_0 \geq 1.50$, $L_{B,4} < B < U_8$ holds. $(c = 3)$

Note that L_8 is easier to compute than either $L_{B,3}$ or $L_{B,4}$, and is a better bound than $L_{B,3}$.

Example 4.3. Consider the IVP

$$y''(x) = (x^2+1)e^{3y}, \quad y(0) = \frac{1}{100}, \quad y'(0) = \frac{1}{10}\,.$$

The exact solution is unknown to the author. We obtain $L_{B,3} = L_{B,4} = 0.689$ and $L_8 = 0.668$. Here the Bobisud bound is better. However, only U_8 is available as an upper bound with $U_8 = 8.165$. So we may conclude $0.689 < c < 8.165$ holds.

Next, we consider the generalized Emden-Fowler IVP below with y' present:

$$\begin{aligned} y''(x) &= a(x)y'(x) + p(x) \cdot [y(x)]^\eta \\ y(x_0) &= A, \quad y'(x_0) = B, \quad A > 0, \quad B > 0, \quad \eta > 1\,. \end{aligned} \tag{4.7}$$

Suppose $y(x)$ has a vertical asymptote at $x = c$. Note that (4.7) generalizes the IVP of generalized Emden-Fowler type (where $a(x) \equiv 0$ and $y'(x)$ is missing) considered earlier. Hara, et al. ([16], [17]) discuss noncontinuability of such equations. The only type of noncontinuability we consider here is the case where $y(x)$ has a vertical asymptote at $x = c$. First, we need the following lemmas which we state without proofs, the proofs being similar to proofs of previous lemmas, part (a) following from L'Hospital's Rule, and (b) following from (a).

Lemma 4.3. *Consider IVP (4.7). Let $a(x)$, $b(x)$ be continuously differentiable on $[x_0, \infty)$, with $a(x) \geq 0$, $b(x) \geq 0$ on $[x_0, \infty)$,*

$$a_u(x) = \sup_{x_0 \leq t \leq x} a(t), \quad p_u(x) = \sup_{x_0 \leq t \leq x} p(t)\,.$$

Let $Y_u(x)$ be the solution to the auxiliary IVP

$$Y_u''(x) = a_u(x)Y_u'(x) + p_u(x) \cdot [Y_u(x)]^\eta,$$
$$Y_u(x_0) = A, \quad Y_u'(x_0) = B, \quad A > 0, \quad B > 0.$$

Suppose $Y_u(x)$ has a vertical asymptote at $x = x^*$. If a_u and b_u are continuously differentiable near x^*, then

a) $\lim_{x\to(x^*)^-} \frac{Y_u(x)}{Y_u'(x)} = 0$ *and*

b) $\lim_{x\to(x^*)^-} \frac{Y_u(x)Y_u''(x)}{[Y_u'(x)]^2} = \frac{1+\eta}{2}$.

Theorem 4.4. *Consider IVP (4.7). Under the conditions of the previous lemma, a lower bound L_9 for c is the unique root of*

$$(x_0 - x)g_9(x) = B^\epsilon\,, \tag{4.8}$$

where

$$g_9(x) = \min(\epsilon A^{\epsilon-1}B,\ R(x))\,, \tag{4.9}$$

$\epsilon = \frac{1-n}{2}$ *and*

$$R(x) = \frac{\epsilon}{2(1-\epsilon)}a_u(x)A^\epsilon + \frac{\epsilon}{2\sqrt{1-\epsilon}}\sqrt{(a_u(x))^2A^{2\epsilon} + 4(1-\epsilon)p_u(x)}\,. \tag{4.10}$$

Sketch of proof. The proof is very similar to the proofs of Theorems 2.14 and 3.6 given earlier. Let $u(x) = u = [Y_u(x)]^\epsilon$, as done earlier. Then $u''(x) = 0$ when

$$Y_u'(x) = \sqrt{\frac{Y_u(x)\cdot(a_u(x)Y_u'(x) + p_u(x)(Y_u(x))^\eta}{1-\epsilon}}\,. \tag{4.11}$$

At any such x, we have

$$(1-\epsilon)(Y_u'(x))^2 - a_u(x)Y_u(x)Y_u'(x) - p_u(x)\cdot Y_u(x))^{\eta+1} = 0\,.$$

Solving for $Y_u'(x)$, we obtain

$$Y_u'(x) = \frac{a_u(x)Y_u(x) + \sqrt{(a_u(x))^2Y_u(x)^2 + 4(1-\epsilon)p_u(x)(Y_u(x))^{\eta+1}}}{2(1-\epsilon)}\,,$$

the plus sign being retained since $Y_u'(x) \geq 0$.

For values of x satisfying (4.11), we have

$$\begin{aligned}|u'(x)| &\leq |\epsilon|(Y_u(x))^{\epsilon-1}\left[\frac{a_u(x)Y_u(x)}{2(1-\epsilon)} + \frac{\sqrt{(a_u(x))^2Y_u(x)^2 + 4(1-\epsilon)p_u(x)Y_u(x)^{\eta+1}}}{2(1-\epsilon)}\right]\\ &= \frac{|\epsilon|}{2(1-\epsilon)}a_u(x)Y_u(x)^\epsilon + \frac{|\epsilon|}{2\sqrt{1-\epsilon}}\sqrt{(a_u(x))^2(Y_u(x))^{2\epsilon} + 4(1-\epsilon)p_u(x)}\,,\end{aligned}$$

using $(Y_u(x))^{2\epsilon+\eta-1} \equiv 1$. So, for values of x satisfying (4.11), if any, we have

$$|u'(x)| \leq \frac{|\epsilon|}{2(1-\epsilon)}a_u(x)A^\epsilon + \frac{\epsilon}{2\sqrt{1-\epsilon}}\sqrt{(a_u(x))^2A^{2\epsilon} + 4(1-\epsilon)p_u(x)}\,.$$

The rest of the proof proceeds as in the last part of the proof of Theorem 2.14, using Lemma 4.3 instead, and is left to the reader.

A theorem could be presented for upper bounds as well, but we omit it here. ☐

Example 4.4. Consider the IVP

$$y''(x) = (3e^{-x})y'(x) + p(x)[y(x)]^2, \quad y(2) = 2, \quad y'(2) = \frac{17}{4}\,.$$

where

$$p(x) = \frac{(15x^2 + 110x + 135) + (18x^3 + 48x^2 - 174x - 396)e^{-x}}{4(x+2)^{5/2}}\,,$$

which is unimodal with maximum $p(2.2) \approx 2.84$. This example was constructed with the exact solution

$$y(x) = \frac{\sqrt{x+2}}{(3-x)^2}, \text{ and } c = 3.00\,.$$

None of the bounds of Eliason [11] or Bobisud [5] are applicable here. Among the new bounds, only L_9 is applicable. We obtain $L_9 = 2.576$. Clearly, $L_9 = 2.576 < c = 3.000$ holds.

Finally, we indicate how we might obtain bounds for c for other types of differential equations not previously considered. Theorems could be presented here; however, to save space, we merely indicate general strategies and operators likely to be useful for obtaining bounds. We do this via several examples to conclude this paper. We also indicate possibilities for further research. The scope of the applicability of the methods given in this paper appear large indeed.

Example 4.5. Consider the IVP

$$y''(x) = \frac{3y^2 + y^4}{1+x^2}, \quad y(0) = 1, \quad y'(0) = 1\,. \tag{4.12}$$

The exact value of c is unknown. We obtain $L_{B,2} = 0.323$, $L_{B,4} = 0.712$. The calculation of these two bounds requires numerical integration. We now demonstrate a slight variation of the L_1 bound which will enable a closed form hand computation of a lower bound for c. Clearly, we may rewrite (4.12) as: $(y = y(x))$

$$y''(x) = \left[\left(\frac{3y^2 + y^4}{y^4}\right)\left(\frac{1}{1+x^2}\right)\right] y^4.$$

We may treat our 'coefficient function' as the expression in brackets which is clearly bounded above by '$P_u(x)$'$\equiv 4$ and bounded below by '$P_L(x)$'$\equiv 1$. For the variation of L_1 (similarly for U_1), consider the auxiliary (majorant) IVP:

$$Y''(x) = 4[Y(x)]^4, \quad Y(0) = 1, \quad Y'(0) = 1\,,$$

we obtain $L_1 = \sqrt{\frac{5}{18}} = 0.527$, which is better than $L_{B,2}$ and not much worse than $L_{B,4}$. We also obtain $U_1 = 1.054$. So we may conclude that $0.712 < c < 1.054$. This example demonstrates that we may easily obtain lower bounds for the IVP

$$y''(x) = \sum_{i=1}^{k} p_i(x) g_i(y(x)), \quad y(x_0) = A > 0, \quad y'(x_0) = B > 0,$$

where $g_i(\cdot)$ are given positive functions, and $p_i(x)$ are given 'coefficients'.

Example 4.6. Consider the IVP

$$y''(x) = e^{xy}, \quad y(0) = \frac{1}{2}, \quad y'(0) = \frac{1}{3}\,.$$

This is not of any of the forms considered earlier. Only the new lower bound L_1 below will handle this IVP. We use the operator

$$u = e^{\delta x y(x)} \quad \delta < 0\,.$$

Then $u''(x) = 0$ when

$$xy'(x) + y(x) = \sqrt{\frac{xy''(x) + 2y'(x)}{-\delta}} \tag{4.13}$$

with a bound on $|u'(x)|$, for any x satisfying (4.13), of

$$\begin{aligned}|u'(x)| &\le \frac{|\delta|}{\sqrt{-\delta}}e^{\delta xy}\sqrt{xe^{xy}+2y'}\\ &\le \frac{|\delta|}{\sqrt{-\delta}}e^{\delta xy}\sqrt{xe^{xy}+2xe^{xy}+\frac{2}{3}} = \frac{|\delta|}{\sqrt{-\delta}}e^{\delta xy}\sqrt{3xe^{xy}+\frac{2}{3}},\\ |u'(x)| &\le \sqrt{-\delta}e^{(\delta+\frac{1}{2})xy}\left(\sqrt{3x+\frac{2}{3}e^{-xy}}\right).\end{aligned}$$

Let $\delta = -\frac{1}{2}$. Then

$$|u'(x)| \le \sqrt{\frac{1}{2}}\left(\sqrt{3x+\frac{2}{3}}\right)$$

for all x satisfying (4.13). As done in the proof of Theorem 4.2, let

$$g_1(x) = \min\left(-e^{-\frac{1}{2}x_0 A}, \quad \frac{-\sqrt{2}}{2}\left(\sqrt{3x+\frac{2}{3}}\right)\right) \quad (x_0 = 0).$$

Then L_1, a lower bound for c, satisfies $(x_0 - x)g_1(x) = 1$ or

$$x\left(\frac{\sqrt{2}}{2}\left(\sqrt{3x+\frac{2}{3}}\right)\right) = e^{-0} = 1,$$

which gives $x = L_1 \approx 0.805$.

Example 4.7. Consider the IVP

$$y''(x) = (y'(x))^2 \cdot \left(\frac{1+2y(x)}{1+y(x)^2}\right), \quad y(0) = 0, \quad y'(0) = 1.$$

The exact solution is $y(x) = -\tan(\ln(1-x))$, with $c = 1 - \epsilon^{-\pi/2} \approx 0.7921$ Only the methods of this paper will provide a lower bound for the asymptote singularity location. Since $y(0) = 0$, we use the operator $u(x) = [y'(x)]^{\epsilon}$, $\epsilon < 0$. Note that no comparison results are needed here. Then by direct but messy computation, $u''(x) = 0$ when

$$y''(x) = \sqrt{\frac{y'(x)y^{(3)}(x)}{1-\epsilon}}, \tag{4.14}$$

where

$$y^{(3)}(x) = \frac{[y'(x)]^3[4+6y(x)+6y(x)^2]}{[1+y(x)^2]^2}.$$

L'Hospital's Rule establishes

$$\lim_{x\to c^-}\frac{y'(x)y^{(3)}(x)}{(y''(x))^2} = \frac{3}{2} < \frac{\sqrt{1-\epsilon}}{|\epsilon|} = 2.$$

Also, at any $x \ge 0$ satisfying (4.14), we have

$$|u'(x)| \le \frac{|\epsilon|}{\sqrt{1-\epsilon}}(y'(x))^{\epsilon+1}\sqrt{\frac{4+6y(x)+6y(x)^2}{(1+y(x)^2)^2}}.$$

Letting $\epsilon = -1$ and using the fact that the maximum value of the radicand expression occurs when $y = y(x) \approx 0.4299$, we obtain, at any x satisfying (4.14),

$$|u'(x)| \leq \max\left(1, \sqrt{\frac{2}{3}}(2.34026)\right) = 1.9108\,.$$

A lower bound L for c is therefore given by

$$L = (1.9108)^{-1} = 0.5233\,.$$

Clearly, $L \leq c = 0.7921$ holds.

Example 4.8. We now consider an example where $\lim\limits_{x\to c^-} y(x) = -\infty$ with $A = y(x_0) > 0$. Consider the IVP

$$y''(x) = -2y(x) - 2(y(x))^3,\ y\left(-\frac{\pi}{4}\right) = 1,\ y'\left(-\frac{\pi}{4}\right) = -2\,.$$

The exact solution is $y = -\tan x$ with $c = \frac{\pi}{2} \approx 1.571$. The transformation $Y(x) = -y(x)$ produces the IVP

$$Y''(x) = 2Y(x) + 2(Y(x))^3,\ Y\left(-\frac{\pi}{4}\right) = -1,\ Y'\left(-\frac{\pi}{4}\right) = 2\,.$$

We may use the same operator $u(x) = e^{\epsilon Y(x)}$, $\epsilon < 0$, used in Theorem 4.2 earlier. Proceeding as in the proof of Theorem 4.2 (we omit details), a lower bound for c is the unique root of

$$\left(x + \frac{\pi}{4}\right) g(x) = e^{-\epsilon}\,, \tag{4.15}$$

where

$$g(x) = \max\left(-2\epsilon e^{-\epsilon},\ \sqrt{-\epsilon}\, M(\epsilon)\right)\,,$$
$$M(\epsilon) = \sup_{t\geq -1} |2t + 2t^3|^{1/2} e^{\epsilon t}\,.$$

Unlike earlier, there is no clearcut choice for $\epsilon < 0$. Let $L(\epsilon)$ denote the lower bound which is the root of (4.15), given $\epsilon < 0$. We thus determined $\epsilon_0 < 0$ satisfying

$$L(\epsilon_0) = \sup_{\epsilon<0} L(\epsilon)\,.$$

This found $\epsilon_0 = -0.353$ with a best lower bound of $L(\epsilon_0) = 0.0562$. Clearly, $L(\epsilon_0) \leq c$ holds. However, it is probable that other operators can be found which will produce better lower bounds. The presence of an inflection point in the solution may contribute to the poor bounds obtained here. Table 4.1 below gives the values of $L(\epsilon)$ for various values of $\epsilon < 0$, including $\epsilon = \epsilon_0$, to see the dependence of the lower bound $L(\epsilon)$ on the 'operator parameter' ϵ. In a future paper, we shall discuss more general formulas for lower bounds of c, including the

ϵ	-2.00	-1.75	-1.50	-1.25	-1.00
$L(\epsilon)$	-0.535	-0.500	-0.452	-0.385	-0.286
ϵ	-0.75	-0.50	-0.40	-0.36	-0.353
$L(\epsilon)$	-0.208	-0.085	$+0.005$	$+0.048$	$+0.056$
ϵ	-0.35	-0.34	-0.30	-0.20	-0.10
$L(\epsilon)$	$+0.048$	$+0.018$	-0.101	-0.367	-0.595

Table 4.1:

case $A > 0$, $\lim_{x\to c} y(x) = -\infty$ in the Emden-Fowler case.

Many more examples of IVPs which are handled only by the methods of this paper could be given. Some will be given in forthcoming papers of third and higher order IVPs and BVPs.

5. Concluding Remarks

In this paper, we have presented several methods, including a bounded operator approach, for finding bounds for the vertical asymptote c of a solution to a given IVP. In some instances, the new bounds are the only bounds available. In other cases, new bounds improve on bounds of previous authors in some cases. Although the new bounds are sometimes for a less general IVP than considered by Bobisud [5], they handle some cases where the coefficient function $p(x)$ has a left endpoint singularity and some cases where $y(x_0) = A \leq 0$ (the case $A < 0$ to be discussed in a forthcoming paper in the generalized Emden-Fowler case).

5.1. Possibilities for Future Research.

(1) Can an upper bound for c be found in the case $A = 0$ and $\theta < 0$ in Theorems 3.1, 3.2 and 3.6?
(2) The Runge-Kutta (4,4) method does not seem too efficient when numerically approximating the solution to an IVP near a vertical asymptote. Can a modification of Runge-Kutta (4,4) (or other RK) be used to improve efficiency in this case?
(3) Can the interval analysis methods given in Moore [24] be used in conjunction with lower bounds for c to get improved bounds for c (both upper and lower)?

Other operators of use but not discussed in this paper are $u = (y + a(x - x_0))^\epsilon$, $u = (y + by' + c)^\epsilon$, $u = e^{\delta y + y'}$, $u = (y')^{\delta_1}(y)^{\delta_2}$, and $u = [y(x)]^{\epsilon y(x)}$, if $y(x) \geq 1$. In each case, we try to bound $|u'(x)|$ at a value of x where $u''(x) = 0$, if any, and examine what happens as $x \to x^*$, x_* or c. Lower bounds can almost always be found by a judicious choice of the operator 'parameters', which are ϵ, δ, δ_1, δ_2 above. The parameters are chosen to eliminate, as much as possible, having to know the $y(x)$ value at a certain x, so that we may (a priori) bound $|u'(x)|$, at those x values where $u''(x) = 0$ and at $x = x_0$, x^*, x_*, possibly.

The methods in this paper can be extended to handle:

(1) 3rd and higher order generalized Emden-Fowler IVPs, (details to come in a forthcoming paper)
(2) problems with derivative blow-up and other IVPs which have noncontinuable solutions
(3) boundary value problems
(4) IVPs with horizontal asymptotes present in their solutions.

However, there are some extra complications in the above problems. The author will report on further research on these topics in the future.

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