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BOUNDEDNESS OF SOLUTIONS OF A THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATION

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ABSTRACT. Sufficient conditions are established for the boundedness of all solutions of (1.1), and we also present some sufficient conditions, which ensure that the limits of first and second order derivatives of the solutions of (1.1) tend to zero as $t \to \infty$. Our results improve and include those results obtained by previous authors ([3], [5]).

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1. INTRODUCTION

We consider the third order non-linear and non-autonomous ordinary differential equation

$$\dddot{x} + f(x,\dot{x},\ddot{x})\ddot{x} + g(x,\dot{x}) + h(x,\dot{x},\ddot{x}) = p\left(t,x,\dot{x},\ddot{x}\right) \tag{1.1}$$

or its equivalent system

$$\begin{aligned} &\dot{x} = y, \dot{y} = z, \\ &\dot{z} = -f(x,y,z)z - g(x,y) - h(x,y,z) + p(t,x,y,z). \end{aligned} \tag{1.2}$$

It is assumed that f, g, h and p are continuous functions which depend only on the arguments displayed explicitly, and the dots denote differentiation with respect to t. The derivatives $\frac{\partial f(x,y,z)}{\partial x} \equiv f_x(x,y,z)$, $\frac{\partial f(x,y,z)}{\partial z} \equiv f_z(x,y,z)$, $\frac{\partial h(x,y,z)}{\partial x} \equiv h_x(x,y,z)$, $\frac{\partial h(x,y,z)}{\partial y} \equiv h_y(x,y,z)$, $\frac{\partial h(x,y,z)}{\partial z} \equiv h_z(x,y,z)$ and $\frac{\partial g(x,y)}{\partial x} \equiv g_x(x,y)$ exist and are continuous. Moreover, the existence and the uniqueness of the solutions of (1.1) will be assumed.

In recent years, the boundedness properties of solutions of certain non-linear differential equations of the third order have been investigated by a large number of mathematicians, and

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they have obtained many results for some special cases of the equation (1.1) with $h \equiv h(x)$, see ([1], [4], [5], [7] – [10]) and references therein. However, in the case $h \equiv h(x, \dot{x}, \ddot{x})$, the results about third order nonlinear differential equations are relatively scarce.

In ([5], [6]), Ezeilo discussed the ultimate boundedness and the existence of the periodic solutions of equations of the form

$$\dddot{x} + \psi(\dot{x})\ddot{x} + \varphi(x)\dot{x} + v(x, \dot{x}, \ddot{x}) = p(t).$$

Later, in a recent paper, Bereketoğlu and Györi [3] considered the differential equation described as follows

$$\dddot{x} + f(x, \dot{x})\ddot{x} + g(x, \dot{x}) + h(x, \dot{x}, \ddot{x}) = p(t, x, \dot{x}, \ddot{x}), \tag{1.3}$$

and the author established sufficient conditions under which all solutions of the non-autonomous differential equation (1.3) are bounded and the limits of first and second order derivatives of the solutions of (1.3) tend to zero as $t \to \infty$. In this paper, we shall be concerned with the boundedness results of the solutions of third-order non-linear differential equations of the form (1.1).

The motivation for the present work has come from the papers of Ezeilo ([5], [6]), Bereketoğlu and Györi [3] and the paper mentioned above. The results obtained herein are comparable in generality to the works of Bereketoğlu and Györi [3] and Ezeilo [5], and our results also include and improve the results in ([3], [5]). It should also be noted that the first result obtained here is proved without using the boundedness of $h(x, \dot{x}, \ddot{x})$.

2. MAIN RESULTS

The main results of this paper are the following.

Theorem 2.1. *Further to the basic assumptions on the functions f, g, h and p assume that the following conditions are satisfied (a, b, c, l, m and A- some positive constants):*

(i) $f(x, y, z) \geq a$ *and* $ab - c > 0$ *for all* x, y, z;

(ii) $\frac{g(x,y)}{y} \geq b$ *for all* $x, y \neq 0$;

(iii) $\frac{h(x,0,0)}{x} \geq c$ *for all* $x \neq 0$;

(iv) $0 < h_x(x, y, 0) \leq c$ *for all* x, y;

(v) $h_y(x, y, 0) \geq 0$ *for all* x, y;

(vi) $h_z(x, y, 0) \geq m$ *for all* x, y;

(vii) $yf_x(x, y, z) \leq 0, \quad yf_z(x, y, z) \geq 0$ *and* $g_x(x, y) \leq 0$ *for all* x, y, z;

(viii) $yzh_y(x, y, 0) + ayzh_z(x, y, z) \geq 0$ *for all* x, y, z;

(ix) $|p(t, x, y, z)| \leq e(t)$ *for all* $t \geq 0, x, y, z$, *where* $\int_0^t e(s)ds \leq A < \infty$.

Then given any finite numbers x_0, y_0, z_0 there is a finite constant $D = D(x_0, y_0, z_0)$ such that the unique solution $x(t)$ of (1.2) which is determined by the initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0 \tag{2.1}$$

satisfies

$$|x(t)| \leq D, \quad |y(t)| \leq D, \quad |z(t)| \leq D$$

for all $t \geq 0$.

Theorem 2.2. *Let all the conditions of Theorem 2.1 be satisfied; in addition, we assume that* $e(t)$ *is bounded for* $t \geq 0$ *, that is, there is a positive constant* M *such that* $|e(t)| \leq M$ *for all* $t \geq 0$. *Then every solution* $x(t)$ *of (1.2) determined by the initial conditions (2.1) satisfies*

$$\dot{x}(t) \to 0, \quad \ddot{x}(t) \to 0 \ \textit{as} \quad \to \infty.$$

Remark 2.3. Theorem 2.1 and Theorem 2.2 contain far less restrictive conditions than those established in Bereketoğlu and Györi [3, Theorem 1, Theorem 2]. Because the result established in [3] can be proved here without the assumptions $h_y(x, y, 0) \geq \frac{1}{4} > 0$ and $ab + \frac{al}{4} > a^2m + c$.

Remark 2.4. It should be noted that the function h satisfying conditions (iii)-(vi) essentially reduces to something like $h(x, y, z) = cx + h_0(x, z)$. For example, the function $h(x, y, z) := cx + z(x^2 + m)$ satisfies the above conditions.

The proofs of Theorem 2.1 and Theorem 2.2 depend on some certain fundamental properties of a continuously differentiable Lyapunov function $V = V(x, y, z)$ defined by:

$$\begin{aligned} V(x,y,z) &= a\int_0^x h(\xi,0,0)d\xi + h(x,0,0)y \\ &\quad + \int_0^y g(x,\eta)d\eta + a\int_0^y f(x,\eta,0)\eta d\eta + ayz + \frac{1}{2}z^2. \end{aligned} \tag{2.2}$$

Namely, this function and its time derivative satisfy some fundamental inequalities.

In the subsequent discussion we require the following lemmas.

Lemma 2.5. *Subject to the assumptions (i)-(vi) of Theorem 2.1,* $V(0, 0, 0) = 0$ *and there is a positive constant* K *depending only on* a, b *and* c *such that*

$$V(x,y,z) \geq K(x^2 + y^2 + z^2) \tag{2.3}$$

for all x, y, z.

Proof. It is clear that $V(0, 0, 0) = 0$. Since $h_x(x, y, z) \leq c$, $\frac{g(x,y)}{y} \geq b$ $(y \neq 0)$ and $f(x, y, z) \geq a$, the function $V(x, y, z)$ can be rearranged as follows (for $y \neq 0$):

$$\begin{aligned} V(x,y,z) &\geq a\int_0^x h(\xi,0,0)d\xi + h(x,0,0)y + \frac{b}{2}y^2 + \frac{1}{2}a^2y^2 + ayz + \frac{1}{2}z^2 \\ &= \frac{1}{2b}\left[by + h(x,0,0)\right]^2 + \frac{1}{2}\left[ay + z\right]^2 \\ &\qquad + \frac{1}{2by^2}\left\{4\int_0^x h(\xi,0,0)\left[\int_0^y (ab - h_\xi(\xi,0,0))\eta d\eta\right]d\xi\right\} \\ &\geq \frac{1}{2b}\left[by + h(x,0,0)\right]^2 + \frac{1}{2}\left[ay + z\right]^2 \\ &\qquad + \frac{1}{2by^2}\left\{4\int_0^x h(\xi,0,0)\left[\int_0^y (ab - c)\eta d\eta\right]d\xi\right\}, \quad (\text{for } y \neq 0). \end{aligned} \tag{2.4}$$

Now, it is obvious from (2.4) that the function $V(x, y, z)$ defined in (2.2) is a positive definite function which has infinite inferior limit and infinitesimal upper limit. Hence, there is a positive constant K such that

$$V(x,y,z) \geq K(x^2 + y^2 + z^2).$$

The proof of this lemma is now complete. □

Lemma 2.6. *Under the assumptions of Theorem 2.1, there are positive constants D_1 and D_2 depending only on a and m such that, if $(x(t), y(t), z(t))$ is any solution of (1.2), then*

$$\dot{V} = \frac{d}{dt}V(x(t), y(t), z(t)) \leq -D_1(y^2 + z^2) + D_2(|y| + |z|)e(t). \tag{2.5}$$

Proof. An easy calculation from (2.2) and (1.2) yields that

$$\begin{aligned}\dot{V} = {} & y^2 h_x(x,0,0) + y\int_0^y g_x(x,\eta)d\eta + ay\int_0^y f_x(x,\eta,0)\eta d\eta \\ & + az^2 - f(x,y,z)z^2 - ayg(x,y) - W_1 - W_2 - W_3 + (ay+z)p(t,x,y,z),\end{aligned} \tag{2.6}$$

where

$$\begin{aligned} W_1 &= af(x,y,z)yz - af(x,y,0)yz, \\ W_2 &= -h(x,0,0)z + h(x,y,z)z, \\ W_3 &= -ayh(x,0,0) + ayh(x,y,z). \end{aligned}$$

By (vii), we get

$$y\int_0^y g_x(x,\eta)d\eta \leq 0, \qquad y\int_0^y f_x(x,\eta,0)\eta d\eta \leq 0.$$

It also follows from (vii), for $z \neq 0$, that

$$W_1 = ayz^2\left[\frac{f(x,y,z) - f(x,y,0)}{z}\right] = yz^2 f_z(x,y,\theta_1 z) \geq 0, \quad 0 \leq \theta_1 \leq 1,$$

but $W_1 = 0$ when $z = 0$. Hence

$$W_1 \geq 0 \text{ for all } x, y, z.$$

Similarly, it is clear that

$$\begin{aligned} W_2 &= yzh_y(x,\theta_2 y,0) + z^2 h_z(x,y,\theta_3 z) \quad ,0 \leq \theta_2 \leq 1, \quad 0 \leq \theta_3 \leq 1 \\ W_3 &= ay^2 h_y(x,\theta_4 y,0) + ayzh_z(x,y,\theta_5 z), \quad 0 \leq \theta_4 \leq 1, \quad 0 \leq \theta_5 \leq 1. \end{aligned}$$

Then, combining the estimates for W_1, W_2, W_3 with (2.6) we obtain

$$\begin{aligned}\dot{V} \leq {} & y^2 h_x(x,0,0) - yzh_y(x,\theta_2 y,0) - z^2 h_z(x,y,\theta_3 z) - ay^2 h_y(x,\theta_4 y,0) \\ & - ayzh_z(x,y,\theta_5 z) + az^2 - f(x,y,z)z^2 - ayg(x,y) + (ay+z)p(t,x,y,z).\end{aligned}$$

The assumption (viii) shows that

$$\begin{aligned}\dot{V} \leq {} & -ay^2 h_y(x,\theta_4 y,0) + y^2 h_x(x,0,0) - z^2 h_z(x,y,\theta_3 z) \\ & + az^2 - f(x,y,z)z^2 - ayg(x,y) + (ay+z)p(t,x,y,z).\end{aligned} \tag{2.7}$$

Also under the assumptions of the theorem we have

$$\begin{gathered} -ay^2 h_y(x,\theta_4 y,0) \leq 0 \quad \text{for all } x, y; \\ y^2 h_x(x,0,0) \leq cy^2 \quad \text{for all } x, y; \\ -z^2 h_z(x,y,\theta_3 z) \leq -mz^2 \quad \text{for all } x, y, z; \\ -f(x,y,z)z^2 \leq -az^2 \quad \text{for all } x, y, z; \\ -ayg(x,y) \leq -aby^2 \quad \text{for all } x, y; \end{gathered}$$

$$\begin{aligned}(ay+z)p(t,x,y,z) &\leq |ay+z|\,|p(t,x,y,z)| \\ &\leq (a\,|y|+|z|)e(t) \\ &\leq \max\{a,1\}\,(|y|+|z|)e(t).\end{aligned}$$

Now, let $D_1 = \min\{ab-c, m\}$ and $D_2 = \max\{a,1\}$.

From the estimates just stated above and (2.7) we obtain

$$\begin{aligned}\dot{V} &\leq -(ab-c)y^2 - mz^2 + \max\{a,1\}\,(|y|+|z|)e(t) \\ &\leq -D_1(y^2+z^2) + D_2(|y|+|z|)e(t).\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.7. *Let f be a non-negative function defined on $[0,\infty)$ such that f is integrable on $[0,\infty)$ and uniformly continuous on $[0,\infty)$. Then*

$$\lim_{t\to\infty} f(t) = 0.$$

Proof. See ([2]). □

3. PROOF OF THEOREMS

Proof of Theorem 2.1. Consider the Lyapunov function $V(x,y,z)$ defined by (2.2). By Lemma 2.5, it is obvious that

$$\begin{aligned}&V(x,y,z) = 0, \quad \text{at } x^2+y^2+z^2 = 0, \\ &V(x,y,z) > 0, \quad \text{if } x^2+y^2+z^2 \neq 0, \\ &V(x,y,z) \to \infty, \quad \text{as } x^2+y^2+z^2 \to \infty.\end{aligned}$$

Next suppose $(x(t), y(t), z(t))$ is any solution of (1.2) which satisfies the initial conditions

$$x(0) = x_0, y(0) = y_0, z(0) = z_0.$$

Set

$$V(t) \equiv V(x(t), y(t), z(t)).$$

Then just as in Lemma 2.6,

$$\dot{V} \leq -D_1(y^2+z^2) + D_2(|y|+|z|)e(t),$$

so that

$$\dot{V} \leq D_2(|y|+|z|)e(t).$$

It follows from the obvious inequalities

$$|y| < 1+y^2, \qquad |z| < 1+z^2$$

and

$$y^2+z^2 \leq \frac{1}{K}V(x,y,z)$$

that

$$\begin{aligned}\dot{V}(t) &\leq D_2(2+y^2+z^2)e(t) \\ &\leq \frac{D_2}{K}e(t)V(t) + 2D_2e(t).\end{aligned}$$

Integrating both sides of this inequality between 0 and t $(t \geq 0)$ and using Gronwall-Reid-Bellman inequality, we obtain

$$V(t) \leq \frac{1}{\chi(t)}\left(V(0) + 2D_2\int_0^t \chi(s)e(s)ds\right),$$

where

$$\chi(t) = \exp\left(-\frac{D_2}{K}\int_0^t e(s)ds\right).$$

Since $\chi(t) \leq 1,$ and using (ix) we have

$$V(t) \leq (V(0) + 2D_2A)\exp\left(\frac{D_2}{K}A\right) \quad \text{for } t \geq 0.$$

As $V(0) = V(x_0, y_0, z_0)$, this completes the proof. □

Proof of Theorem 2.2. The proof of this theorem is similar to that of Bereketoğlu and Györi [3, Theorem 2] and hence it is omitted. □

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