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APPROXIMATION OF B-CONTINUOUS AND B-DIFFERENTIABLE FUNCTIONS BY GBS OPERATORS OF BERNSTEIN BIVARIATE POLYNOMIALS

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ABSTRACT. In this paper we give an approximation of B-continuous and B-differentiable functions by GBS operators of Bernstein bivariate polynomials.

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1. PRELIMINARIES

In this section, we recall some results which we will use in this article.

In the following, let X and Y be compact real intervals. A function $f : X \times Y \to \mathbb{R}$ is called a B-continuous (Bögel-continuous) function in $(x_0, y_0) \in X \times Y$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f\left((x,y),(x_0,y_0)\right) = 0.$$

Here

$$\Delta f\left((x,y),(x_0,y_0)\right) = f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0)$$

denotes a so-called mixed difference of f.

A function $f : X \times Y \to \mathbb{R}$ is called a B-differentiable (Bögel-differentiable) function in $(x_0, y_0) \in X \times Y$ if it exists and if the limit is finite:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f\left((x,y),(x_0,y_0)\right)}{(x-x_0)(y-y_0)}.$$

The limit is named the B-differential of f in the point (x_0, y_0) and is denoted by $D_B f(x_0, y_0)$.

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The definitions of B-continuity and B-differentiability were introduced by K. Bögel in the papers [5] and [6].

The function $f : X \times Y \to \mathbb{R}$ is B-bounded on $X \times Y$ if there exists $K > 0$ such that

$$|\Delta f\left((x, y), (s, t)\right)| \leq K$$

for any $(x, y), (s, t) \in X \times Y$.

We shall use the function sets $B(X \times Y) = \left\{f : X \times Y \to \mathbb{R} | f \text{ bounded on } X \times Y\right\}$ with the usual sup-norm $\|\cdot\|_\infty$, $B_b(X \times Y) = \left\{f : X \times Y \to \mathbb{R} | f\ B\text{-bounded on } X \times Y\right\}$ and we set $\|f\|_B = \sup\limits_{(x,y),(s,t)\in X\times Y} |\Delta f\left((x, y), (s, t)\right)|$, where

$$f \in B_b(X \times Y), \quad C_b(X \times Y) = \left\{f : X \times Y \to \mathbb{R} | f\ B - \text{continuous on } X \times Y\right\},$$
$$\text{and} \quad D_b(X \times Y) = \left\{f : X \times Y \to \mathbb{R} | f\ B - \text{differentiable on } X \times Y\right\}.$$

Let $f \in B_b(X \times Y)$. The function $\omega_{\text{mixed}}(f; \cdot, \cdot) : [0, \infty) \times [0, \infty) \to \mathbb{R}$, defined by

$$\omega_{\text{mixed}}(f; \delta_1, \delta_2) = \sup\left\{|\Delta f((x, y), (s, t))| : |x - s| \leq \delta_1, |y - t| \leq \delta_2\right\} \tag{1.1}$$

for any $(\delta_1, \delta_2) \in [0, \infty) \times [0, \infty)$ is called the mixed modulus of smoothness.

For related topics, see [1], [2], [3] and [10].

Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator. The operator $UL : C_b(X \times Y) \to B(X \times Y)$ defined for any function $f \in C_b(X \times Y)$ and any $(x, y) \in X \times Y$ by

$$(ULf)(x, y) = \left(L(f(\cdot, y) + f(x, *) - f(\cdot, *))\right)(x, y) \tag{1.2}$$

is called the GBS operator ("Generalized Boolean Sum" operator) associated to the operator L, where "$\cdot$" and "$*$" stand for the first and second variable.

Let the functions $e_{ij} : X \times Y \to \mathbb{R}$, $(e_{ij})(x, y) = x^i y^j$ for any $(x, y) \in X \times Y$, where $i, j \in \mathbb{N}$. The following theorem is proved in [1].

Theorem 1.1. *Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator and $UL : C_b(X \times Y) \to B(X \times Y)$ the associated GBS operator. Then for any $f \in C_b(X \times Y)$, any $(x, y) \in (X \times Y)$ and any $\delta_1, \delta_2 > 0$, we have*

$$\begin{aligned} |f(x, y) - (ULf)(x, y)| &\leq |f(x, y)|\, |1 - (Le_{00})(x, y)| \\ &+ \Bigg[(Le_{00})(x, y) + \delta_1^{-1}\sqrt{\left(L(\cdot - x)^2\right)(x, y)} + \delta_2^{-1}\sqrt{\left(L(* - y)^2\right)(x, y)} \\ &\qquad + \delta_1^{-1}\delta_2^{-1}\sqrt{\left(L(\cdot - x)^2(* - y)^2\right)(x, y)}\Bigg]\omega_{mixed}(f; \delta_1, \delta_2). \end{aligned} \tag{1.3}$$

In the following, we need the following theorem for estimating the rate of the convergence of the B-differentiable functions (see [11]).

Theorem 1.2. *Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator and $UL : C_b(X \times Y) \to B(X \times Y)$ the associated GBS operator. Then for any $f \in D_b(X \times Y)$ with*

$D_B f \in B(X \times Y)$, *any* $(x,y) \in X \times Y$ *and any* $\delta_1, \delta_2 > 0$, *we have*

$$
\begin{aligned}
(1.4)\qquad & |f(x,y) - (ULf)(x,y)| \\
& \le |f(x,y)||1-(Le_{00})(x,y)|+3\|D_B f\|_\infty \sqrt{\left(L(\cdot - x)^2(* - y)^2\right)(x,y)} \\
& \quad + \Bigg[\sqrt{(L(\cdot - x)^2(* - y)^2)(x,y)} + \delta_1^{-1}\sqrt{(L(\cdot - x)^4(* - y)^2)(x,y)} \\
& \quad + \delta_2^{-1}\sqrt{(L(\cdot - x)^2(* - y)^4)(x,y)} \\
& \quad + \delta_1^{-1}\delta_2^{-1}\left(L(\cdot - x)^2(* - y)^2\right)(x,y)\Bigg]\omega_{mixed}(D_B f; \delta_1, \delta_2).
\end{aligned}
$$

2. MAIN RESULTS

Let the sets $\Delta_2 = \{(x,y) \in \mathbb{R} \times \mathbb{R} | x, y \ge 0, x + y \le 1\}$ and $\mathcal{F}(\Delta_2) = \{f | f : \Delta_2 \to \mathbb{R}\}$. For m a non zero natural number, let the operators $B_m : \mathcal{F}(\Delta_2) \to \mathcal{F}(\Delta_2)$, defined for any function $f \in \mathcal{F}(\Delta_2)$ by

$$
(B_m f)(x,y) = \sum_{\substack{k,j=0 \\ k+j\le m}} p_{m,k,j}(x,y) f\left(\frac{k}{m}, \frac{j}{m}\right) \tag{2.1}
$$

for any $(x,y) \in \Delta_2$, where

$$
p_{m,k,j}(x,y) = \frac{m!}{k!j!(m-k-j)!}\, x^k y^j (1-x-y)^{m-k-j}. \tag{2.2}
$$

The operators are named Bernstein bivariate polynomials (see [8]).

Lemma 2.1. *The operators* $(B_m)_{m\ge1}$ *are linear and positive on* $\mathcal{F}(\Delta_2)$.

Proof. The proof follows immediately. □

For m a non zero natural number, let the GBS operator of Bernstein bivariate polynomials UB_m (see [1]), $UB_m : C_b(\Delta_2) \to B(\Delta_2)$ defined for any function $f \in C_b(\Delta_2)$ and any $(x,y) \in \Delta_2$ by

$$
\begin{aligned}
(2.3)\qquad (UB_m f)(x,y) &= \left(B_m(f(x,*) + f(\cdot,y) - f(\cdot,*))\right)(x,y) \\
&= \sum_{\substack{k,j=0 \\ k+j\le m}} p_{m,k,j}(x,y)\left[f\left(x, \frac{j}{m}\right) + f\left(\frac{k}{m}, y\right) - f\left(\frac{k}{m}, \frac{j}{m}\right)\right].
\end{aligned}
$$

Lemma 2.2. *The operators* $(B_m)_{m\ge1}$ *verify for any* $(x,y) \in \Delta_2$ *the following:*

$$
(B_m e_{00})(x,y) = 1; \tag{2.4}
$$

$$
\left(B_m(\cdot - x)^2\right)(x,y) = \frac{x(1-x)}{m}; \tag{2.5}
$$

$$
\left(B_m(* - y)^2\right)(x,y) = \frac{y(1-y)}{m}; \tag{2.6}
$$

$$
\left(B_m(\cdot - x)^2(* - y)^2\right)(x,y) = \frac{3(m-2)}{m^3}\, x^2y^2 - \frac{m-2}{m^3}\,(x^2y + xy^2) + \frac{m-1}{m^3}\, xy; \tag{2.7}
$$

$$(2.8)\quad \left(B_m(\cdot - x)^4(\ast - y)^2\right)(x,y)$$
$$= -\frac{5(3m^2-26m+24)}{m^5}\,x^4y^2 + \frac{6(3m^2-26m+24)}{m^5}\,x^3y^2 - \frac{6(m^2-7m+6)}{m^5}\,x^3y$$
$$-\frac{3m^2-41m+42}{m^5}\,x^2y^2 + \frac{3m^2-26m+24}{m^5}\,x^4y + \frac{3m^2-17m+14}{m^5}\,x^2y$$
$$-\frac{m-2}{m^5}\,xy^2 + \frac{m-1}{m^5}\,xy$$

and

$$(2.9)\quad \left(B_m(\cdot - x)^2(\ast - y)^4\right)(x,y)$$
$$= -\frac{5(m^2-26m+24)}{m^5}\,x^2y^4 + \frac{6(3m^2-26m+24)}{m^5}\,x^2y^3 - \frac{6(m^2-7m+6)}{m^5}\,xy^3$$
$$-\frac{3m^2-41m+42}{m^5}\,x^2y^2 + \frac{3m^2-26m+24}{m^5}\,xy^4 + \frac{3m^2-17m+14}{m^5}\,xy^2$$
$$-\frac{m-2}{m^5}\,x^2y + \frac{m-1}{m^5}\,xy$$

for any non zero natural number m.

Proof. Let $(x,y) \in \Delta_2$ and m be a non zero natural number. We have

$$(B_m e_{00})(x,y) = \sum_{\substack{k,j=0\\ k+j\le m}} \frac{m!}{k!j!(m-k-j)!}\,x^k y^j (1-x-y)^{m-k-j}$$
$$= (x+y+1-x-y)^m = 1,$$

so (2.4) holds,

$$(B_m e_{10})(x,y) = \sum_{\substack{k,j=0\\ k+j\le m}} \frac{m!}{k!j!(m-k-j)!}\,x^k y^j (1-x-y)^{m-k-j}\,\frac{k}{m}$$
$$= x \sum_{\substack{k=1,j=0\\ k+j\le m}} \frac{(m-1)!}{(k-1)!j!(m-k-j)!}\,x^{k-1} y^j (1-x-y)^{m-k-j}$$
$$= x,$$

it results that

$$(2.10)\qquad (B_m e_{10})(x,y) = x$$

and similarly

$$(2.11)\qquad (B_m e_{01})(x,y) = y.$$

In the same way, using the formulas

$$k^2 = k(k-1) + k,$$
$$k^3 = k(k-1)(k-2) + 3k(k-1) + k,$$
$$k^4 = k(k-1)(k-2)(k-3) + 6k(k-1)(k-2) + 7k(k-1) + k,$$

we obtain

$$(2.12)\qquad (B_m e_{20})(x,y) = \frac{m-1}{m}\,x^2 + \frac{1}{m}\,x,$$

$$(2.13)\qquad (B_m e_{30})(x,y) = \frac{(m-1)(m-2)}{m^2}\,x^3 + \frac{3(m-1)}{m^2}\,x^2 + \frac{1}{m^2}\,x,$$

$$(2.14)\quad (B_m e_{40})(x,y) = \frac{(m-1)(m-2)(m-3)}{m^3}\,x^4 + \frac{6(m-1)(m-2)}{m^3}\,x^3 + \frac{7(m-1)}{m^3}\,x^2 + \frac{1}{m^3}\,x$$

and similarly the relations $(B_m e_{02})(x,y)$, $(B_m e_{03})(x,y)$, $(B_m e_{04})(x,y)$.

We have

$$(B_m e_{11})(x,y) = \frac{m-1}{m}\,y \sum_{\substack{k=0,j=1\\k+j\le m}} \frac{(m-1)!}{k!(j-1)!(m-k-j)!}\,x^k y^{j-1}(1-x-y)^{m-k-j}\,\frac{k}{m-1} = \frac{m-1}{m}\,y(B_{m-1}e_{10})(x,y),$$

$$(B_m e_{21})(x,y) = \left(\frac{m-1}{m}\right)^2 y \sum_{\substack{k=0,j=1\\k+j\le m}} \frac{(m-1)!}{k!(j-1)!(m-k-j)!}\,x^k y^{j-1}(1-x-y)^{m-k-j}\left(\frac{k}{m-1}\right)^2 = \left(\frac{m-1}{m}\right)^2 y(B_{m-1}e_{20})(x,y),$$

and in the same way, we write $(B_m e_{31})(x,y)$, $(B_m e_{41})(x,y)$, $(B_m e_{32})(x,y)$, $(B_m e_{42})(x,y)$. Taking (2.12) - (2.14) into account, we obtain

$$(2.15)\quad (B_m e_{11})(x,y) = \frac{m-1}{m}\,xy,$$

$$(2.16)\quad (B_m e_{21})(x,y) = \frac{(m-1)(m-2)}{m^2}\,x^2 y + \frac{m-1}{m^2}\,xy,$$

$$(2.17)\quad (B_m e_{31})(x,y) = \frac{(m-1)(m-2)(m-3)}{m^3}\,x^3 y + \frac{3(m-1)(m-2)}{m^3}\,x^2 y + \frac{m-1}{m^3}\,xy,$$

$$(2.18)\quad (B_m e_{41})(x,y) = \frac{(m-1)(m-2)(m-3)(m-4)}{m^4}\,x^4 y + \frac{6(m-1)(m-2)(m-3)}{m^4}\,x^3 y + \frac{7(m-1)(m-2)}{m^4}\,x^2 y + \frac{m-1}{m^4}\,xy,$$

$$(2.19)\quad (B_m e_{22})(x,y) = \frac{(m-1)(m-2)(m-3)}{m^3}\,x^2 y^2 + \frac{(m-1)(m-2)}{m^3}\,(x^2 y + xy^2) + \frac{m-1}{m^3}\,xy,$$

$$(2.20)\quad (B_m e_{32})(x,y) = \frac{(m-1)(m-2)(m-3)(m-4)}{m^4}\,x^3 y^2 + \frac{(m-1)(m-2)(m-3)}{m^4}\,x^3 y + \frac{3(m-1)(m-2)(m-3)}{m^4}\,x^2 y^2 + \frac{3(m-1)(m-2)}{m^4}\,x^2 y + \frac{(m-1)(m-2)}{m^4}\,xy^2 + \frac{m-1}{m^4}\,xy,$$

$$(B_m e_{42})(x,y) = \frac{(m-1)(m-2)(m-3)(m-4)(m-5)}{m^5}\, x^4y^2 + \frac{(m-1)(m-2)(m-3)(m-4)}{m^5}\, x^4y + \frac{6(m-1)(m-2)(m-3)(m-4)}{m^5}\, x^3y^2 + \frac{6(m-1)(m-2)(m-3)}{m^5}\, x^3y + \frac{7(m-1)(m-2)(m-3)}{m^5}\, x^2y^2 + \frac{7(m-1)(m-2)}{m^5}\, x^2y + \frac{(m-1)(m-2)}{m^5}\, xy^2 + \frac{m-1}{m^5}\, xy \tag{2.21}$$

and similarly the relations $(B_m e_{12})(x,y)$, $(B_m e_{13})(x,y)$, $(B_m e_{14})(x,y)$, $(B_m e_{23})(x,y)$, $(B_m e_{24})(x,y)$.

Now, we have

$$(B_m(\cdot - x)^2)(x,y) = (B_m e_{20})(x,y) - 2x(B_m e_{10})(x,y) + x^2(B_m e_{02})(x,y),$$

$$(B_m(\cdot - x)^2(* - y)^2)(x,y) = (B_m e_{22})(x,y) - 2y(B_m e_{21})(x,y) + y^2(B_m e_{20})(x,y) - 2x(B_m e_{12})(x,y) + 4xy(B_m e_{11})(x,y) - 2xy^2(B_m e_{10})(x,y) + x^2(B_m e_{02})(x,y) - 2x^2y(B_m e_{01})(x,y) + x^2y^2(B_m e_{00})(x,y),$$

$$\begin{aligned}(B_m(\cdot - x)^4(* - y)^2)(x,y) &= (B_m e_{40})(x,y) - 2y(B_m e_{41})(x,y) + y^2(B_m e_{40})(x,y)\\ &- 4x(B_m e_{32})(x,y) + 8xy(B_m e_{31})(x,y) - 4xy^2(B_m e_{30})(x,y)\\ &+ 6x^2(B_m e_{22})(x,y) - 12x^2y(B_m e_{21})(x,y) + 6x^2y^2(B_m e_{20})(x,y)\\ &- 4x^3(B_m e_{12})(x,y) + 8x^3y(B_m e_{11})(x,y) - 4x^3y^2(B_m e_{10})(x,y)\\ &+ x^4(B_m e_{02})(x,y) - 2x^4y(B_m e_{01})(x,y) + x^4y^2(B_m e_{00})(x,y)\end{aligned}$$

and taking (2.9) – (2.21) into account, we obtain (2.5), (2.7) and (2.8). Similarly we obtain (2.9). □

Lemma 2.3. *The operators $(B_m)_{m\geq 1}$ verify for any $(x,y)\in\Delta_2$ the following inequalities:*

$$(B_m(\cdot - x)^2)(x,y) \leq \frac{1}{4m}, \tag{2.22}$$

$$(B_m(* - y)^2)(x,y) \leq \frac{1}{4m}, \tag{2.23}$$

for any non zero natural number m,

$$\left(B_m(\cdot - x)^2(* - y)^2\right)(x,y) \leq \frac{9}{4m^2}, \tag{2.24}$$

for any natural number m, $m \geq 2$,

$$\left(B_m(\cdot - x)^4(* - y)^2\right)(x,y) \leq \frac{9}{m^3}, \tag{2.25}$$

$$\left(B_m(\cdot - x)^2(* - y)^4\right)(x,y) \leq \frac{9}{m^3}, \tag{2.26}$$

for any natural number m, $m \geq 8$.

Proof. Because $x(1-x) \leq \frac{1}{4}$ for any $x \in [0,1]$, (2.22) and (2.23) results.

From (2.7), we have

$$\begin{aligned}\left(B_m(\cdot - x)^2(* - y)^2\right)(x,y) &= \frac{2(m-2)}{m^3}\, x^2y^2 + \frac{m-2}{m^3}\, x(1-x)y(1-y) + \frac{1}{m^3}\, xy \\ &\leq \frac{2(m-2)}{m^3} + \frac{m-2}{16m^3} + \frac{1}{m^3} \\ &= \frac{33m-50}{16m^3}\,,\end{aligned}$$

from where (2.24) results.

From (2.8), we have

$$\begin{aligned}&\left(B_m(\cdot - x)^4(* - y)^2\right)(x,y) \\ &\quad = \frac{6(3m^2-26m+24)}{m^5}\, x^3y^2(1-x) + \frac{3m^2-26m+24}{m^5}\, x^4y(y+1) \\ &\qquad - \frac{6(m^2-7m+6)}{m^5}\, x^3y + \frac{3m^2-17m+14}{m^5}\, x^2y(1-y) \\ &\qquad\qquad + \frac{24m-28}{m^5}\, x^2y^2 + \frac{m-2}{m^5}\, xy(1-y) + xy.\end{aligned}$$

But

$$\begin{aligned}\frac{3m^2-26m+24}{m^5}\, x^4y(y+1) &\leq 2\frac{3m^2-26m+24}{m^5}\, x^2y \\ &= \frac{6m^2-42m+36}{m^5}\, x^2y - \frac{10m-12}{m^5}\, x^2y \\ &\leq \frac{6m^2-42m+36}{m^5}\, x^2y - \frac{10m-12}{m^5}\, x^3y^2\end{aligned}$$

and then, from the inequalities above, we obtain

$$\begin{aligned}&\left(B_m(\cdot - x)^4(* - y)^2\right)(x,y) \\ &\quad \leq \frac{6(3m^2-26m+24)}{m^5}\, x^3y^2(1-x) + \frac{6m^2-42m+36}{m^5}\, x^2y(1-y) \\ &\qquad + \frac{3m^2-17m+14}{m^5}\, x^2y(1-y) + \frac{10m-12}{m^5}\, x^2y^2(1-y) \\ &\qquad\qquad + \frac{14m-16}{m^5}\, x^2y^2 + \frac{m-2}{m^5}\, xy(1-y) + xy.\end{aligned} \tag{2.27}$$

Because $x(1-x) \leq \frac{1}{4}$, $y(1-y) \leq \frac{1}{4}$, $xy \leq 1$ for any $x, y \in [0,1]$, from (2.27) we have

$$\begin{aligned}&\left(B_m(\cdot - x)^4(* - y)^2\right)(x,y) \\ &\quad \leq \frac{6(3m^2-26m+24)}{4m^5} + \frac{6m^2-42m+36}{4m^5} \\ &\qquad + \frac{3m^2-17m+14}{4m^5} + \frac{10m-12}{4m^5} + \frac{14m-16}{m^5} + \frac{m-2}{4m^5} + 1 \\ &\quad = \frac{27m^2-148m+170}{m^5}\,,\end{aligned}$$

from where (2.25) results. □

Theorem 2.4. *Let the function $f \in C_b(\Delta_2)$. Then, for any $(x,y) \in \Delta_2$, any natural number m, $m \geq 2$, we have*

$$|f(x,y) - (UB_m f)(x,y)| \leq \left(1 + \delta_1^{-1}\frac{1}{2\sqrt{m}} + \delta_2^{-1}\frac{1}{2\sqrt{m}} + \delta_1^{-1}\delta_2^{-1}\frac{3}{2m}\right)\omega_{\textit{mixed}}(f;\delta_1,\delta_2) \tag{2.28}$$

for any δ_1, $\delta_2 > 0$ and

$$|f(x,y) - (UB_m f)(x,y)| \leq \frac{7}{2}\omega_{\textit{mixed}}\left(f;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{m}}\right). \tag{2.29}$$

Proof. For the first inequality we apply Theorem 1.1 and Lemma 2.3. The inequality (2.29) is obtained from (2.28) by choosing $\delta_1 = \delta_2 = \frac{1}{\sqrt{m}}$. □

Corollary 2.5. *If $f \in C_b(\Delta_2)$, then*

$$\lim_{m\to\infty}(UB_m f)(x,y) = f(x,y) \tag{2.30}$$

uniformly on Δ_2.

Proof. Because $f \in C_b(\Delta_2)$, there results that f is uniform B-continuous on Δ_2 and then $\lim_{m\to\infty}\omega_{\text{mixed}}\left(f;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{m}}\right) = 0$ (see [2] or [3]). From (2.29), there results the conclusion. □

Theorem 2.6. *Let the function $f \in D_b(\Delta_2)$ with $D_B f \in B(\Delta_2)$. Then for any $(x,y) \in \Delta_2$, any natural number m, $m \geq 8$, we have*

$$|f(x,y) - (UB_m f)(x,y)| \leq \frac{9}{2m}\|D_b f\|_\infty + \left(\frac{3}{2m} + \delta_1^{-1}\frac{3}{m\sqrt{m}} + \delta_2^{-1}\frac{3}{m\sqrt{m}} + \delta_1^{-1}\delta_2^{-1}\frac{9}{4m^2}\right)\omega_{\textit{mixed}}(D_B f;\delta_1,\delta_2) \tag{2.31}$$

for any $\delta_1, \delta_2 > 0$ and

$$|f(x,y) - (UB_m f)(x,y)| \leq \frac{3}{4m}\left(6\|D_B f\|_\infty + 13\omega_{\textit{mixed}}\left(D_B f;\frac{1}{\sqrt{m}}\,\frac{1}{\sqrt{m}}\right)\right). \tag{2.32}$$

Proof. It results from Theorem 1.2 and Lemma 2.3. □

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