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The Convergence of Grünwald Interpolation Operator on the Zeros of Freud Orthogonal Polynomials

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Abstract Let $W_\beta(x)=\exp(-\frac{1}{2}|x|^\beta)$ be the Freud weight and $p_n(x)\in\Pi_n$ be the sequence of orthogonal polynomials with respect to $W_\beta^2(x)$, that is,

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)W_\beta^2(x)\mathrm{d}x=\begin{cases}0, & n\neq m,\\ 1, & n=m.\end{cases}$$

It is known that all the zeros of $p_n(x)$ are distributed on the whole real line. The present paper investigates the convergence of Grünwald interpolatory operators based on the zeros of orthogonal polynomials for the Freud weights. We prove that, if we take the zeros of Freud polynomials as the interpolation nodes, then

$$G_n(f,x)\to f(x),\quad n\to\infty$$

holds for every $x\in(-\infty,\infty)$, where $f(x)$ is any continous function on the real line satisfying $|f(x)|=O(\exp(\frac{1}{2}|x|^\beta))$.

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1. Introduction

The convergence of interpolation operators based on the zeros of orthogonal polynomials has been extensively studied and explored. In recent years, orthogonal polynomials with exponential weights have been investigated by many researchers (for example, see [1], [3], [4]). Lubinsky[1] gave a deliberate discussion on the properties of the orthogonal polynomials and their zeros. The present paper will apply these useful properties in proving convergence of Grünwald interpolation operators based on the zeros of orthogonal polynomials with exponential weights. In this paper we will deal with one type of exponential weights, the so-called Freud weight.

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Let $W_\beta(x)=\exp(-Q(x))=\exp(-\frac{1}{2}|x|^\beta)$, $\beta>7/6$, be the Freud weight, $p_n(x)=\gamma_n x^n+\cdots,\gamma_n>0$, denote the nth Freud orthogonal polynomial with respect to $W_\beta(x)$, so that

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)W_\beta^2(x)\mathrm{d}x=\begin{cases}0, & n\neq m,\\ 1, & n=m,\end{cases}$$

and $\{x_k\}_{k=1}^n$ be the zeros of $p_n(x)$ satisfying

$$-\infty<x_n<x_{n-1}<\cdots<x_1<\infty.$$

Then, taking zeros of $p_n(x)$ as the interpolation nodes, we define the Grünwald operators as follows:

$$G_n(f,x)=\sum_{k=1}^n f(x_k)l_k^2(x)=\sum_{k=1}^n f(x_k)(\frac{p_n(x)}{p_n'(x_k)(x-x_k)})^2. \tag{1}$$

The main result of this paper is the following

Theorem *Let $f\in C_{(-\infty,\infty)}$ satisfy $|f(x)|=O\left(\exp(\frac{1}{2}|x|^\beta)\right)$ and $\{x_k\}_{k=1}^n$ denote the zeros of the nth Freud polynomial. Then for any given point $x\in\mathbb{R}$, we have*

$$\lim_{n\to\infty}G_n(f,x)=f(x).$$

2. Lemmas

In the following lemmas, and elsewhere in the paper, C always denotes a positive constant, which may differ at each different occurrence.

Lemma 1 *Let $p_n(x)$ be the nth Freud orthogonal polynomial, and $\{x_k\}_{k=1}^n$ its zeros. Then we have the following properties of $p_n(x)$ and its zeros:*

(1) $\sup_{x\in\mathbf{R}}|p_n(x)|\cdot W_\beta(x)\sim a_n^{-1/2}n^{1/6}$;

(2) $\frac{a_n}{n}|p_n'(x_k)|\cdot W_\beta(x_k)\sim a_n^{-1/2}(\max\{n^{-2/3},1-\frac{|x_k|}{a_n}\})^{1/4}$;

(3) $|\frac{p_n''(x_k)}{p_n'(x_k)}|\leq C(1+|Q'(x_k)|)$;

(4) $x_j-x_{j+1}\sim\frac{a_n}{n}\max\{n^{-2/3},1-\frac{|x_j|}{a_n}\}^{-1/2}$;

(5) $a_n=Cn^{1/\beta},|x_k|\leq Ca_n$,

where a_n is the Mhskar-Rahmanov-Saff number defined as follows: Suppose $Q(x):\mathbb{R}\to\mathbb{R}$, is even and continuous in $\mathbb{R}$, Q'' is continuous in $(0,\infty)$ and $Q'>0$. Then a_μ is the Mhskar-Rahmanov-Saff number, i.e., the positive root of the equation

$$\mu=\frac{2}{\pi}\int_0^1 a_\mu tQ'(a_\mu t)(1-t^2)^{-\frac{1}{2}}\mathrm{d}t.$$

Proof Readers could find (1)–(4) in [1], and (5) in [3].

Lemma 2 *For $n\geq 1$ and $|x|\leq\delta a_n,\delta\in(0,1)$, we have*

(1) $|p_n(x)|W_\beta(x)\leq Ca_n^{-\frac{1}{2}}$;

(2) $|p_n'(x)|W_\beta(x)\leq Cna_n^{-\frac{3}{2}}$,

while for an arbitrary $x\in\mathbb{R}$, *(3)* $|p_n^{'}(x)|W_\beta(x) \le Cn^{7/6-1/\beta}a_n^{-1/2}$.

Proof Readers could find (1), (2) in [4]. Here we give the proof of (3). By [1], we have

$$\| p_n^{'}(x)W_\beta(x) \|_{L_\infty(R)} \le C\rho_n(1) \| p_n(x)W_\beta(x) \|_{L_\infty(R)},$$

where

$$\rho_n(x) = \int_{Q(\max\{1,|x|\})}^{Cn} \frac{\mathrm{d}s}{Q^{[-1]}(s)}.$$

It is easy to find $\rho_n(1) = Cn^{1-\frac{1}{\beta}}$. Now by Lemma 1, we obtain that

$$\begin{aligned}|p_n^{'}(x)|W_\beta(x) &\le \| p_n^{'}(x)W_\beta(x) \|_{L_\infty(R)} \le C\rho_n(1) \| p_n(x)W_\beta(x) \|_{L_\infty(R)} \\ &\le Cn^{1-1/\beta}a_n^{-1/2}n^{1/6} \le Cn^{7/6-1/\beta}a_n^{-1/2}.\end{aligned}$$

In the sequel, without loss of generality, we suppose that $x \in [x_{j+1}, x_j)$ for some $1 \le j \le n-1$, while the case $x \in (-\infty, x_n)$ or $x \in (x_1, +\infty)$ can be treated similarly.

Lemma 3 *Let $\{x_k\}_{k=1}^n$ be the zeros of nth Freud orthogonal polynomial and write $E = \{k : k \ne j, j+1\}$. We have the following*

$$\sum_{k\in E} \frac{1}{|x-x_k|} \le C\frac{n}{a_n}\log n.$$

Proof By the definition of set E, we obtain that

$$\sum_{k\in E} \frac{1}{|x-x_k|} = \sum_{k=1}^{j-1} \frac{1}{|x-x_k|} + \sum_{k=j+2}^{n} \frac{1}{|x-x_k|} \le \sum_{k=1}^{j-1} \frac{1}{|x_j-x_k|} + \sum_{k=j+2}^{n} \frac{1}{|x_{j+1}-x_k|}.$$

It is easily deduced from Lemma 1 (4) that

$$|x_j - x_{j+1}| \ge \frac{a_n}{n}.$$

Thus

$$\sum_{k\in E} \frac{1}{|x-x_k|} \le C\frac{n}{a_n}\sum_{k=1}^{n}\frac{1}{k} \le C\frac{n}{a_n}\log n.$$

Lemma 3 is proved. □

Lemma 4 *Let $l_k(x)$ be the interpolating fundamental polynomials. Then for any given point $x \in (-\infty, \infty)$, we have*

$$\sum_{k=1}^{n} l_k^2(x) \to 1, \quad n \to \infty.$$

Proof Suppose without loss of generality that $x \in [x_{j+1}, x_j)$. By the definition of Hermite-Fejér operators, we have

$$H_n(f,x) = \sum_{k=1}^{n} f(x_k)[1-(x-x_k)\frac{p_n^{''}(x_k)}{p_n^{'}(x_k)}]l_k^2(x).$$

Let $f(x) \equiv 1$, whence $H_n(f,x) \equiv 1$. We have

$$1 = \sum_{k=1}^{n}[1-(x-x_k)\frac{p_n^{''}(x_k)}{p_n^{'}(x_k)}]l_k^2(x).$$

Thus

$$|1-\sum_{k=1}^{n} l_k^2(x)| \le \sum_{k=1}^{n} \frac{p_n^2(x)|p_n^{''}(x_k)|}{|p_n^{'}(x_k)|^3|x-x_k|}.$$

By Lemma 1(3) and the above inequality, noting that $Q(x)=\frac{1}{2}|x|^\beta$ in this paper, we have

$$\begin{aligned}|1-\sum_{k=1}^{n} l_k^2(x)| &\le C\sum_{k=1}^{n} \frac{p_n^2(x)(1+|Q^{'}(x_k)|)}{|p_n^{'}(x_k)|^2|x-x_k|}\\ &\le C(\sum_{k\in E}\frac{p_n^2(x)(1+\frac{1}{2}\beta|x_k|^{\beta-1})}{|p_n^{'}(x_k)|^2|x-x_k|}+\frac{p_n^2(x)(1+\frac{1}{2}\beta|x_j|^{\beta-1})}{|p_n^{'}(x_j)|^2|x-x_j|}+\\ &\quad \frac{p_n^2(x)(1+\frac{1}{2}\beta|x_{j+1}|^{\beta-1})}{|p_n^{'}(x_{j+1})|^2|x-x_{j+1}|})\\ &=: C(I+I_j+I_{j+1}).\end{aligned}$$

Write

$$I=(\sum_{k\in E_1\cap E}+\sum_{k\in E_2\cap E})\frac{p_n^2(x)W_\beta^2(x)(1+\frac{1}{2}\beta|x_k|^{\beta-1})W_\beta^2(x_k)}{|p_n^{'}(x_k)|^2W_\beta^2(x_k)|x-x_k|}W_\beta^{-2}(x),$$

where

$$E_1=\{k:|x_k|\le n^{\frac{1}{6\beta(\beta-1)}}\},\quad E_2=\{k:|x_k|> n^{\frac{1}{6\beta(\beta-1)}}\}.$$

Applying Lemma 1(1),(2),(5) and Lemmas 2-3 yields that

$$\begin{aligned}I \le & C\sum_{k\in E_1\cap E}\frac{(a_n^{-1/2})^2 n^{1/(6\beta)}W_\beta^2(x_k)}{(\frac{n}{a_n}a_n^{-1/2})^2|x-x_k|}W_\beta^{-2}(x)+\\ & C\sum_{k\in E_2\cap E}\frac{(a_n^{-1/2}n^{1/6})^2a_n^{\beta-1}W_\beta^2(x_k)}{(\frac{n}{a_n}a_n^{-1/2}n^{-1/6})^2|x-x_k|}W_\beta^{-2}(x)\\ \le & Cn^{\frac{7}{6\beta}-1}\log n\cdot\exp(|x|^\beta)+Cn^{2/3}\exp\left(-n^{\frac{1}{6(\beta-1)}}\right)\log n\cdot \exp(|x|^\beta)\to 0,\quad n\to\infty.\end{aligned}$$

Next we estimate I_j and I_{j+1}:

$$\begin{aligned}I_j &= \frac{p_n^2(x)(1+\frac{1}{2}\beta|x_j|^{\beta-1})}{|p_n^{'}(x_j)|^2|x-x_j|}\\ &= \frac{|p_n(x)|W_\beta(x)|(p_n(x)-p_n(x_j))|(1+\frac{1}{2}\beta|x_j|^{\beta-1})W_\beta(x)W_\beta^2(x_j)}{|p_n^{'}(x_j)|^2W_\beta^2(x_j)|x-x_j|}W_\beta^{-2}(x).\end{aligned}$$

If $j\in E_1$, by Lemma 1(2), Lemma 2(1) and the mean value theorem, we have

$$\begin{aligned}I_j &\le C\frac{a_n^{-1/2}n^{1/(6\beta)}|p_n^{'}(\xi_j)|W_\beta(\xi_j)}{(\frac{n}{a_n}a_n^{-1/2})^2}W_\beta^{-1}(\xi_j)W_\beta^2(x_j)W_\beta(x)W_\beta^{-2}(x)\\ &\le Cn^{\frac{8}{3\beta}-2}\left(|p_n^{'}(\xi_j)|W_\beta(\xi_j)\right)W_\beta^{-1}(\xi_j)W_\beta^2(x_j)W_\beta(x)W_\beta^{-2}(x),\end{aligned}$$

where $\xi_j\in(x,x_j)$. Applying Lemma 2(2) again gives

$$\begin{aligned}I_j &\le Cn^{\frac{8}{3\beta}-2}na_n^{-3/2}W_\beta^{-1}(\xi_j)W_\beta^2(x_j)W_\beta(x)W_\beta^{-2}(x)\\ &\le Cn^{\frac{7}{6\beta}-1}\exp(|x|^\beta)\to 0,\quad n\to\infty,\end{aligned}$$

where the last inequality comes from $W_\beta^{-1}(\xi_j)\cdot W_\beta(x_j)W_\beta(x)\le 1$, $\xi_j\in(x,x_j)$.

For $j\in E_2$, by Lemma 1(1),(2), Lemma 2(3) and the mean value theorem, we obtain that

$$\begin{aligned} I_j &\le C\frac{a_n^{-1/2}n^{1/6}a_n^{\beta-1}n^{1-1/\beta}a_n^{-1/2}n^{1/6}}{(\frac{n}{a_n}a_n^{-1/2}n^{-1/6})^2}\cdot\exp\{-\frac{1}{2}n^{\frac{1}{6(\beta-1)}}\}\exp(\frac{1}{2}|x|^\beta)\\ &\le Cn^{\frac{2}{3}}\exp\{-\frac{1}{2}n^{\frac{1}{6(\beta-1)}}\}\exp(\frac{1}{2}|x|^\beta)\to 0,\quad n\to\infty. \end{aligned}$$

Proceeding in a similar argument to I_{j+1} and noticing $W_\beta^{-1}(\xi_{j+1})\cdot W_\beta(x_{j+1})W_\beta(x)\le 1$, we also have $I_{j+1}\to 0$ as $n\to\infty$.

Combining all the above estimates, we complete the proof of Lemma 4.

It is easy to deduce from Lemma 4 that, for any given point $x\in(-\infty,\infty)$, $\sum_{k=1}^n l_k^2(x)\le C$.

Lemma 5 *For large enough n, we have*

$$\sum_{k=1}^n |x-x_k|l_k^2(x)\le Cn^{\frac{1}{\beta}-1}\log n\exp(|x|^\beta).$$

Proof Let $x\in[x_{j+1},x_j)$ without loss of generality. We proceed similarly as in Lemma 4,

$$\sum_{k=1}^n |x-x_k|l_k^2(x)=\sum_{k\in E}|x-x_k|l_k^2(x)+|x-x_j|l_j^2(x)+|x-x_{j+1}|l_{j+1}^2(x).$$

By Lemmas 1,2 and 3 , for large enough n, we obtain that

$$\begin{aligned} &\sum_{k\in E}|x-x_k|l_k^2(x)\\ &=\Big(\sum_{k\in E_1\cap E}+\sum_{k\in E_2\cap E}\Big)\frac{p_n^2(x)W_\beta^2(x)W_\beta^2(x_k)}{(p_n'(x_k)W_\beta(x_k))^2|x-x_k|}W_\beta^{-2}(x)\\ &\le C\frac{(a_n^{-1/2})^2}{(\frac{n}{a_n}a_n^{-1/2})^2}\sum_{k\in E_1\cap E}\frac{\exp(-|x_k|^\beta)}{|x-x_k|}\exp(|x|^\beta)+\\ &\quad C\frac{(a_n^{-1/2}n^{1/6})^2}{(\frac{n}{a_n}a_n^{-1/2}n^{-1/6})^2}\sum_{k\in E_2\cap E}\frac{\exp(-|x_k|^\beta)}{|x-x_k|}\exp(|x|^\beta)\\ &\le Cn^{\frac{2}{\beta}-2}\frac{n}{a_n}\log n\cdot\exp(|x|^\beta)+Cn^{\frac{2}{\beta}-\frac{4}{3}}\frac{n}{a_n}\log n\cdot\exp(-n^{\frac{1}{6(\beta-1)}})\exp(|x|^\beta)\\ &\le Cn^{\frac{1}{\beta}-1}\log n\exp(|x|^\beta). \end{aligned}$$

For $j\in E_1$, by Lemmas 1, 2 and the mean value theorem, we get

$$\begin{aligned} &|x-x_j|l_j^2(x)\\ &\le C\frac{|p_n(x)|W_\beta(x)}{|p_n'(x_j)|^2W_\beta^2(x_j)}|p_n'(\phi_j)|W_\beta(\phi_j)[W_\beta^{-1}(\phi_j)W_\beta(x)W_\beta(x_j)]\exp(-\frac{|x_j|^\beta}{2})\exp(|x|^\beta)\\ &\le Cn^{\frac{1}{\beta}-1}\exp(|x|^\beta)\quad(\text{where }\phi_j\in(x,x_j)), \end{aligned}$$

for $j\in E_2$, for sufficiently large n, by Lemmas 1 and 2, we have the following inequality

$$|x-x_j|l_j^2\le Cn^{\frac{1}{\beta}-\frac{1}{3}}\exp\{-\frac{1}{2}n^{\frac{1}{6(\beta-1)}}\}\exp(|x|^\beta)\le Cn^{\frac{1}{\beta}-1}\exp(|x|^\beta).$$

Similarly, $|x-x_{j+1}|l_{j+1}^2 \le Cn^{\frac{1}{\beta}-1}\exp(|x|^\beta)$.

Combining all the above estimates, we have proved Lemma 5.

Lemma 6 *Let $f(x)$ satisfy conditions of the Theorem. Then for any given point $x\in(-\infty,\infty)$, we have*

$$\sum_{k=1}^{n}|f(x)-f(x_k)|\cdot l_k^2(x)\to 0,\quad n\to\infty.$$

Proof Let $\omega(f,t)$ be the modulus of continuity of $f(x)$. Then

$$\begin{aligned}&\sum_{k=1}^{n}|f(x)-f(x_k)|\cdot l_k^2(x)\\ &\le \sum_{|x_k|\le 2|x|}\omega(f,|x-x_k|)\cdot l_k^2(x)+\sum_{|x_k|>2|x|}|f(x)-f(x_k)|\cdot l_k^2(x)\\ &\le \omega(f,1/\log n)_{[-4|x|,4|x|]}\sum_{|x_k|\le 2|x|}(1+\log n|x-x_k|)\cdot l_k^2(x)+\\ &\quad\sum_{|x_k|>2|x|}|f(x)-f(x_k)|\cdot l_k^2(x)\\ &=: S_1+S_2.\end{aligned}$$

Applying Lemmas 4 and 5 yields that $S_1\to 0$ as $n\to\infty$. Now we estimate S_2:

$$\begin{aligned}S_2&=\sum_{|x_k|>2|x|}|f(x)-f(x_k)|\cdot l_k^2(x)\\ &\le C\Big(\sum_{2|x|<|x_k|\le x^2n^{1/(6\beta)}}+\sum_{|x_k|>x^2n^{1/(6\beta)}}\Big)\exp(\frac{1}{2}|x_k|^\beta)\frac{(p_n(x)W_\beta(x))^2W_\beta^2(x_k)}{(p_n'(x_k)W_\beta(x_k))^2(x-x_k)^2}\exp(|x|^\beta)\\ &=: S_{21}+S_{22}.\end{aligned}$$

Zeros of Freud orthogonal polynomials in Lemma 1 satisfy that for $1\le k\le n-1$,

$$x_k-x_{k+1}\sim\frac{a_n}{n}(\max\{n^{-\frac{2}{3}},1-\frac{|x_k|}{a_n}\})^{-\frac{1}{2}}.$$

When $2|x|<|x_k|\le x^2n^{1/(6\beta)}$, we get

$$x_k-x_{k+1}\sim\frac{a_n}{n}\sim n^{\frac{1}{\beta}-1}$$

for sufficiently large n. Thus

$$\Big(\sum_{2|x|<|x_k|\le x^2n^{1/(6\beta)}}1\Big)\le C\frac{x^2n^{1/(6\beta)}}{n^{\frac{1}{\beta}-1}}.$$

Therefore, it follows from Lemma 2 that

$$\begin{aligned}S_{21}&\le C\frac{(a_n^{-1/2})^2}{(\frac{n}{a_n}a_n^{-1/2})^2}\cdot\frac{1}{x^2}\cdot\frac{x^2n^{1/(6\beta)}}{n^{\frac{1}{\beta}-1}}\exp(|x|^\beta)\\ &\le Cn^{\frac{7}{6\beta}-1}\exp(|x|^\beta)\to 0,\quad n\to\infty.\end{aligned}$$

Applying Lemma 1(1)(2) together with Lemma 3 yields that

$$S_{22} \le C\frac{(a_n^{-1/2}n^{1/6})^2}{(\frac{n}{a_n}a_n^{-1/2}n^{-1/6})^2}\exp(|x|^\beta)\sum_{|x_k|>x^2n^{1/(6\beta)}}\frac{\exp(-\frac{1}{2}|x_k|^\beta)}{(x-x_k)^2}$$
$$\le Cn^{\frac{2}{\beta}-\frac{4}{3}}\cdot\frac{n}{x^2}\exp(-\frac{1}{2}x^{2\beta}n^{1/6}) \to 0, \quad n\to\infty.$$

Combining all the above estimates, we thus have proved Lemma 6.

3. Proof of the Theorem

Write

$$\begin{aligned}|G_n(f,x)-f(x)| &= |\sum_{k=1}^n f(x_k)l_k^2(x)-f(x)| \\ &= |\sum_{k=1}^n[f(x_k)-f(x)]l_k^2(x)-f(x)[1-\sum_{k=1}^n l_k^2(x)]| \\ &\le \sum_{k=1}^n|f(x_k)-f(x)|l_k^2(x)+|f(x)||1-\sum_{k=1}^n l_k^2(x)|.\end{aligned}$$

Applying Lemmas 4 and 6 implies that

$$|G_n(f,x)-f(x)|\to 0, \quad n\to\infty$$

for any given point x. This completes the proof of the Theorem. □

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