On Discontinuities in the Sverdrup Interior

Toshihiro Sakamoto

Department of Earth and Planetary Physics, Graduate School of Science, The University of Tokyo, Tokyo, Japan

ABSTRACT

A formulation of Sverdrup dynamics is presented based on a reduced-gravity model in place of the standard approach using the vorticity equation. The integral conservation law of momentum is used to investigate solutions that may include discontinuities. A surfacing line is interpreted as a ''shock front'' across which the jump condition derived from this conservation law is satisfied.

1. Introduction

A reduced-gravity model is often used to describe the large-scale wind-driven ocean circulation. Consider a square basin on a β plane ranging $0 \leq x \leq 1$ and $-0.5 \leq y \leq 0.5$ in nondimensional units. The governing equations for a steady, planetary geostrophic flow in the oceanic interior can be written in dimensionless form as

$$-f\psi_x = -hh_x + \lambda\tau^x, \qquad (1)$$

$$-f\psi_y = -hh_y, \qquad (2)$$

where h is the layer thickness, ψ is the transport streamfunction, $f = 1 + \beta y$ is the Coriolis parameter, $\tau^x = \sin\pi y$ is the zonal wind forcing modeling the subtropics, and β and λ are nondimensional parameters. The boundary condition is assumed to be

$$\psi = 0, \qquad h = h_e = \text{const} \quad \text{at } x = 1. \qquad (3)$$

The standard procedure for solving this problem is to make the vorticity equation (the Sverdrup relation) using (1)–(2) and to integrate it afterward with respect to x subject to (3). The result is

$$\psi = \psi_s \equiv \frac{\lambda}{\beta}(1 - x)\tau_y^x, \qquad (4)$$

$$h^2 = h_s^2 \equiv h_e^2 + \frac{2f^2}{\beta}(1 - x)\left(\frac{\tau^x}{f}\right)_y \qquad (5)$$

(e.g., Welander 1966).

Corresponding author address: Dr. Toshihiro Sakamoto, Department of Earth and Planetary Physics, Graduate School of Science, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113, Japan.
E-mail: sakamoto@ocean.geoph.s.u-tokyo.ac.jp

Parsons (1969) and later Veronis (1973) and Huang and Flierl (1987) extended the above basic model to the case when layer outcropping occurs. Friction plays a crucial role in their models because they shed light on the separation of the western boundary current so that the governing equations are

$$-f\psi_x = -hh_x + \epsilon\psi_y/h + \lambda\tau^x, \qquad (6)$$

$$-f\psi_y = -hh_y - \epsilon\psi_x/h, \qquad (7)$$

where ϵ ($\ll 1$) is the dimensionless friction coefficient. They determined the position of the surfacing line by matching the Sverdrup interior (4)–(5) with the frictional interior layer, which is assumed to be created in relation to outcropping. Instead of an explicit form of the solution for the interior layer, they introduced a simple relation, the so-called (*semi*)*geostrophic condition,* which represents the leading-order balance in the interior layer. This is the reason why the expression for the surfacing line is independent of the friction coefficient. It seems paradoxical, however, that the assumption of the existence of the interior layer, however thin it may be, is responsible to the derivation leading to the geostrophic condition. Obviously, the same procedure no longer works for the inviscid equations (1)–(2). Is the surfacing line found by Parsons an intrinsic property of the latter equations?

The potential difficulty arising from Parsons' approach may stem from the use of the differential equations (6)–(7) to treat an outcrop that may give rise to jump discontinuities in the dependent variables. However, such discontinuous solutions are possible as ''weak solutions'' of the hyperbolic system (1)–(2) provided that a basic integral conservation law is employed appropriately. In this short note, we focus on a first integral of motion in order to outline Sverdrup dynamics when discontinuities are permitted. Of particular interest is another physical interpretation of the surfacing line from

the standpoint of the inviscid/hyperbolic problem. It should be noted that integral conservation laws have not been discussed extensively in large-scale ocean dynamics. In fact, the major theories on the wind-driven circulation are based on the vorticity equation. This is probably because one, except for Dewar (1991), usually assumes the Sverdrup interior to be continuous.

2. The integral conservation of momentum

The momentum equations (1) and (2) constitute a hyperbolic system and are already in characteristic form with the characteristics $y = $ const and $x = $ const. In particular, (1) can be rewritten as

$$\frac{d}{dx}\left(\frac{h^2}{2} - f\psi\right) = \lambda\tau^x \quad \text{on } y = \text{const.} \qquad (8)$$

This equation can be integrated with respect to x from the eastern boundary to give

$$\frac{h^2}{2} - f\psi + \lambda(1 - x)\tau^x = \frac{h_e^2}{2}. \qquad (9)$$

This is a first integral of motion, the integral of momentum in the x direction, for the problem under consideration. If (8) is integrated from x_1 to x_2 ($0 < x_1 \leq x_2 \leq 1$ for definiteness), we have

$$\left[\frac{h^2}{2} - f\psi\right]_{x=x_1}^{x=x_2} = \lambda(x_2 - x_1)\tau^x. \qquad (10)$$

This relation may be regarded as the integral conservation law of momentum in the x direction. We expect that (9) and (10) remain valid even when discontinuities occur in the interior; for an outcrop region, however, a special treatment is needed as will be discussed later.

Here, the momentum flux in (10) is denoted by G:

$$G \equiv \frac{h^2}{2} - f\psi, \qquad (11)$$

which may be called tentatively the *geostrophic function*. From the integral (9), G can also be written as

$$G = \frac{h_e^2}{2} - \lambda(1 - x)\tau^x \qquad (12)$$

so that G is determined provided that the wind forcing and the boundary values of h are prescribed. Figure 1 shows the distribution of G for two pairs of (λ, h_e). It is found that negative values of G are seen when h_e^2/λ becomes small. If we set $G = g_c \equiv$ const, the isolines of G are represented in parametric form as

$$x = X(y) = 1 - \left(\frac{h_e^2}{2} - g_c\right)\frac{1}{\lambda\tau^x}. \qquad (13)$$

We note that $y = 0$ is also an isoline of G as confirmed from (9), although (13) is invalid there. Actually, the two straight lines $x = 1$ and $y = 0$ are the asymptotes of (13) on which $G = h_e^2/2$ (see Fig. 1).
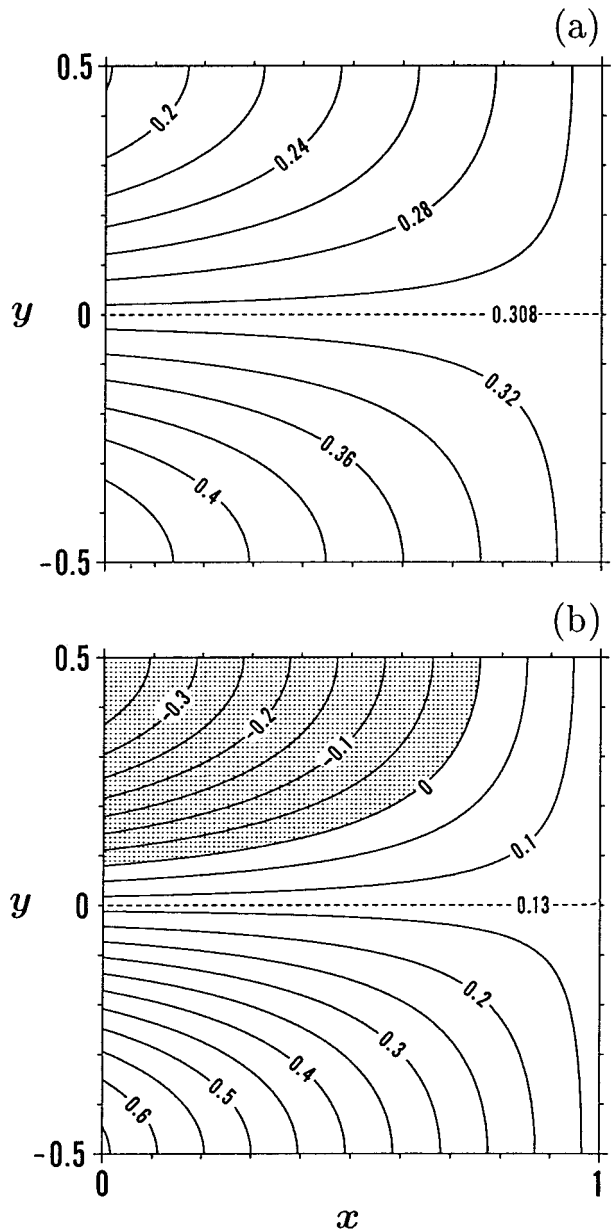

FIG. 1. Contours of G for (a) $\lambda = 0.13$, $h_e = 0.785$ and (b) $\lambda = 0.53$, $h_e = 0.51$ ($\beta = 0.55$ in common). Also shown is the asymptote $y = 0$ denoted by the dotted line. The region of negative values is shaded.

In the continuously differentiable part of the basin, the familiar Sverdrup interior can be reproduced from the integral (9) straightforwardly. Differentiating (9) with respect to y leads to

$$hh_y - f\psi_y - \beta\psi = -\lambda(1 - x)\tau_y^x. \qquad (14)$$

Comparing this with (2), we have

$$\psi = \frac{\lambda}{\beta}(1 - x)\tau_y^x, \qquad (15)$$

which is indeed the Sverdrup function ψ_s given in (4). The corresponding expression for h is immediately obtained again from (9) with (15) as

$$h^2 = h_e^2 - 2\lambda(1 - x)\tau^x + \frac{2f\lambda}{\beta}(1 - x)\tau_y^x \quad (16)$$

and hence proves to be the same as h_s in (5).

3. The possibility of discontinuities

From the definition (11) with (13), we have the conservation law of G:

$$\frac{dG}{dy} = 0 \quad \text{on } x = X(y), \quad (17)$$

or equivalently,

$$h[h_y + X'(y)h_x] - f[\psi_y + X'(y)\psi_x] - \beta\psi = 0, \quad (18)$$

where

$$X'(y) = \left(\frac{h_e^2}{2} - g_c\right)\frac{\tau_y^x}{\lambda(\tau^x)^2} \quad (19)$$

corresponds to the characteristic speed. Equation (17) is formally in characteristic form in the sense that the characteristic $x = X(y)$ carries information *from a latitude circle* on which boundary values must be prescribed [cf. (8)]. Using (19) and (13), the Sverdrup function (15) may be represented as

$$\psi = \frac{\lambda}{\beta}X'(y)\tau^x. \quad (20)$$

Substitution of (20) into (18) yields

$$X'(y)(hh_x - f\psi_x - \lambda\tau^x) + (hh_y - f\psi_y) = 0, \quad (21)$$

which is the linear combination of the original equations (1)–(2); the multiplier $(X', 1)$ is obviously the direction vector for the trajectory $x = X(y)$. It turns out, therefore, that the conservation equation (17) is consistent with the original system (1)–(2) under the Sverdrup constraint (15).

The characteristic equation (17) suggests an important hyperbolic property of the Sverdrup interior: Discontinuous derivatives of G and hence discontinuities in h and ψ are possible on the characteristic $x = X(y)$ (e.g., Whitham 1974, chap. 5). This means that, when discontinuities occur at a point $x = X_1(y_i)$ along a particular *initial* latitude circle $y = y_i$, information on those discontinuities propagates along the characteristic curve $x = X_1(y)$. From the integral conservation law (10), the required "jump condition" across this characteristic is found to be

$$[G]_{X_1(y)-}^{X_1(y)+} = 0, \quad (22)$$

where y is treated as a parameter. Keeping this idea in mind, we can include an outcrop in the Sverdrup interior straightforwardly as will be shown in the next section.

It should be pointed out, however, that discontinuities in the flow must be associated with a delta-function structure of the wind forcing in the 1.5-layer model. Here, we avoid this difficulty by implicitly assuming that compensating transport occurs in the lower layer; a similar analysis of the 2-layer equations is presented in section 5.

4. Inclusion of an outcrop

Since over the outcrop $h = \psi = 0$ indicating that $g_c = 0$ in (13), the surfacing line must be the zero contour of G:

$$x = X_0(y) \equiv 1 - \frac{h_e^2}{2\lambda\tau^x} \quad (23)$$

(see Fig. 1). Similarly, a surfacing line for a zonal wind of the general form $\tau^x(x, y)$ is represented implicitly by

$$\int_{X_0(y)}^{1} \tau^x(x, y) \, dx = \frac{h_e^2}{2\lambda}. \quad (24)$$

From this expression, we can confirm some well-known qualitative features of a surfacing line; that is, it shifts eastward as h_e^2/λ decreases and does not exist where $\tau^x \leq 0$. We note that the equation for the surfacing line may be derived from the zonal momentum equation (1).

Since there exists only one surfacing line over the basin (see Fig. 1) and since $h \neq 0$ at the eastern boundary, the outcrop lies west of the surfacing line. Therefore, we have the following useful rule for determining the location of an outcrop geometrically: *For the hyperbolic system (1)–(2), outcropping may occur where $G \leq 0$. In the domain $G \leq 0$, the values of h, ψ, and G must be replaced by 0.* Accordingly, the double-valued parts of the solutions for h and ψ are replaced by corresponding jump discontinuities. These discontinuities are acceptable for the reason mentioned in the preceding section. In this way, we may avoid the breakdown of the original equations (1)–(2) over the outcrop. In fact, using (15), (16), and (23), we can verify that

$$\left(\frac{h_s^2}{2} - f\psi_s\right) - \left(\frac{0^2}{2} - f \cdot 0\right) = 0 \quad \text{on } x = X_0(y), \quad (25)$$

or concisely

$$[G]_{X_0(y)-}^{X_0(y)+} = 0. \quad (26)$$

That is, the jump condition (22) is satisfied across the surfacing line. Now the formal correspondence between the present analysis and the matching procedure developed by Parsons (1969) becomes clear.

Finally, to get a physically consistent solution, the constant h_e may be determined from the prescribed total volume (Parsons 1969). In Fig. 1, the values of h_e have been chosen so that the total volume is unity.

Therefore, we are successful in reproducing the surfacing line in Parsons' model using the inviscid equations (1)–(2) subject to (3) without further approximations. The integral conservation law of momentum (10)

is employed in order to take into account discontinuities in the solution. We have shown that the surfacing line must lie on one of the G contours and appears as a ''shock front'' across which the jump condition (26) is satisfied.

5. The 2-layer model

The 2-layer planetary geostrophic equations corresponding to (1)–(2) are

$$h_1 p_x - f\psi_{1x} = \lambda\tau^x, \tag{27}$$

$$h_1 p_y - f\psi_{1y} = 0, \tag{28}$$

$$h_2(p - h_1)_x - f\psi_{2x} = 0, \tag{29}$$

$$h_2(p - h_1)_y - f\psi_{2y} = 0, \tag{30}$$

where p is the depth-independent pressure and an obvious notation is used for the quantities for each layer. The boundary condition is

$$\psi_1 = \psi_2 = 0, \qquad p = p_e = \text{const},$$

$$h_1 = h_e = \text{const} \quad \text{at } x = 1. \tag{31}$$

With the moving lower layer, the Sverdrup function (4) means the barotropic transport, that is,

$$\psi_1 + \psi_2 = \psi_s, \tag{32}$$

so that the smoothness of the Ekman pumping is guaranteed.

Adding the upper-layer equations, (27) and (28), to the lower-layer equations, (29) and (30), gives

$$H(p - h_1)_x + h_1 h_{1x} - f\psi_{sx} = \lambda\tau^x, \tag{33}$$

$$H(p - h_1)_y + h_1 h_{1y} - f\psi_{sy} = 0, \tag{34}$$

respectively, where H is the total depth. These equations are already in characteristic form, as in the case of the reduced-gravity equations, leading to the following integral conservation laws of momentum:

$$[P_2]_{x=x_1}^{x=x_2} = \frac{\lambda}{H}(x_2 - x_1)\left(\tau^x - \frac{f}{\beta}\tau_y^x\right), \tag{35}$$

$$[P_2]_{y=y_1}^{y=y_2} = -\frac{\lambda}{H}(1 - x)\left[\tau^x - \frac{f}{\beta}\tau_y^x\right]_{y=y_1}^{y=y_2}, \tag{36}$$

where

$$P_2 \equiv p - h_1 + \frac{h_1^2}{2H}. \tag{37}$$

In the nonoutcrop region, p_x and p_y may be eliminated from (27) and (28) using (33) and (34), respectively. Integrating the resulting equations, we have

$$[G_2]_{x=x_1}^{x=x_2} = \int_{x_1}^{x_2}\left(\frac{h_2}{H}\lambda\tau^x - \frac{h_1}{H}f\psi_{sx}\right) dx, \tag{38}$$

$$[G_2]_{y=y_1}^{y=y_2} = \int_{y_1}^{y_2}\left(-\beta\psi_1 - \frac{h_1}{H}f\psi_{sy}\right) dy, \tag{39}$$

where

$$G_2 \equiv \frac{h_1^2}{2} - \frac{h_1^3}{3H} - f\psi_1, \tag{40}$$

which is the 2-layer analog of G in (11). Again, we assume that these integral conservation laws hold even for discontinuous solutions. If it is also assumed that h_1 and ψ_1 are at most as singular as the step function, the above integral constraints yield the following jump conditions:

$$[p] = \left[h_1 - \frac{h_1^2}{2H}\right], \tag{41}$$

$$f[\psi_1] = \left[\frac{h_1^2}{2} - \frac{h_1^3}{3H}\right]. \tag{42}$$

An outcrop may be included in the present 2-layer system by defining a surfacing line as a front across which (42) is satisfied.

From the definition (40) with (38), we have the conservation law of G_2:

$$\frac{dG_2}{dy} = 0 \quad \text{on } x = X(y), \tag{43}$$

where $X(y)$ satisfies

$$\frac{\lambda}{H}\left(\tau^x - \frac{f}{\beta}\tau_y^x\right)\int_1^{X(y)} h_1\, dx + \lambda[1 - X(y)]\tau^x = g_c \equiv \text{const}. \tag{44}$$

Hence, when H is finite, characteristic curves for G_2 cannot always be determined in advance because (44) depends on the solution h_1.

As a simple example, we consider the case when the lower layer is assumed to be in no-motion unless otherwise it outcrops. That is, the solution is given by

$$\psi_1 = \psi_s \equiv \frac{\lambda}{\beta}(1 - x)\tau^x, \tag{45}$$

$$h_1^2 = h_s^2 \equiv h_e^2 - 2\lambda\left(\tau^x - \frac{f}{\beta}\tau_y^x\right)(1 - x) \tag{46}$$

in the compensated region [cf. (4) and (5)] and

$$\psi_1 = 0, \qquad h_1 = 0 \tag{47}$$

in the outcrop. We must replace the double-valued part of this solution by a suitable front, that is, a surfacing line $x = X_0(y)$. Applying (45)–(47) to the jump condition (42), we have
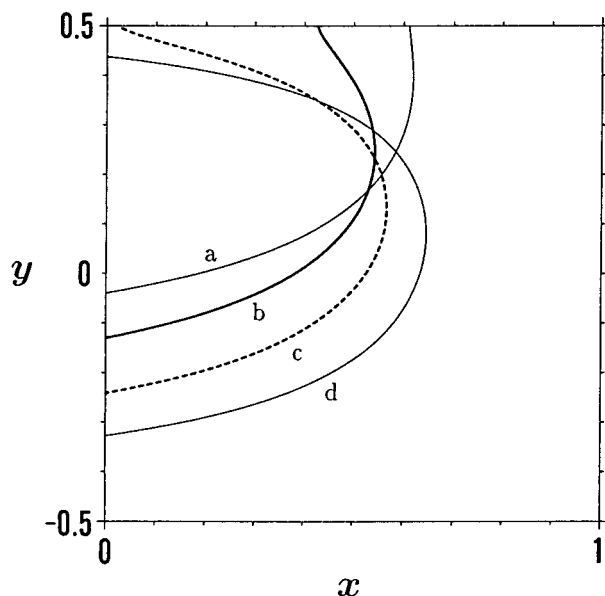
FIG. 2. Surfacing lines in the two-layer model for different values of H when the lower layer is perfectly compensated: (a) $H = 20$, (b) $H = 10$, (c) $H = 5$, and (d) $H = 3$.

$$\frac{h_s^2}{2} - \frac{h_s^3}{3H} - f\psi_s = 0 \quad \text{on } x = X_0(y), \qquad (48)$$

hence the surfacing line is represented by

$$\frac{h_e^2}{2} - \lambda(1 - x)\tau^x - \frac{1}{3H}\left[h_e^2 - 2\lambda\left(\tau^x - \frac{f}{\beta}\tau_y^x\right)(1 - x)\right]^{3/2}$$
$$= 0. \qquad (49)$$

Figure 2 illustrates some surfacing lines for different values of H calculated numerically using (49). The value of h_e for each case is chosen so that the total volume of the upper layer is unity under the assumption that the outcrop lies west of the surfacing line, as before.

For $H = 20$, the configuration of the surfacing line is in good qualitative agreement with that for the reduced-gravity model as expected. As H decreases, the outcrop shifts southward and can even be detached from the northern boundary. Kamenkovich and Reznik (1972) carried out perturbation analysis of 2-layer planetary geostrophic equations in order to obtain asymptotic solutions in powers of $1/H$. They correctly predicted the northwestward deflection of the separated boundary current near the northern boundary. We note that (49) is *exact* for any H provided that the submerged portion of the lower layer is completely compensated, although compensation may be imperfect in reality when H is small.

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