The Bolus Velocity in the Stochastic Theory of Ocean Turbulent Tracer Transport

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ABSTRACT

A stochastic theory of tracer transport in compressible turbulence has recently been developed and then applied to the ocean case because stratified flow in isopycnal coordinates is analogous to compressible flow with the isopycnal layer thickness playing the role of density. The results generalize the parameterization of Gent and McWilliams in the sense that the eddy-induced transport velocity (i.e., the bolus velocity, which is directly related to the thickness–velocity correlation) is given by downgradient Fickian diffusion of thickness with a general mixing tensor K. This result is however dependent on an imprecise postulate relating the Lagrangian to the Eulerian mean velocities. In this paper the authors show that this postulate is unnecessary in a certain flow regime. This regime exists whenever the effect of thickness–velocity correlations dominates over the effect of relative vorticity–velocity correlations. The primary example is planetary geostrophic turbulence, but it may also exist in homogeneous quasigeostrophic turbulence. The bolus velocity is modified and becomes equivalent to upgradient diffusion of potential vorticity along isopycnals with the same mixing tensor K. This changes the conceptual basis of the bolus velocity parameterization of Gent and McWilliams at least in this regime, but practically, the effective change is of rather small magnitude.

1. Introduction

The bolus velocity, defined in isopycnal coordinates as

$$\mathbf{u}^* = \frac{\widetilde{z'_\rho \mathbf{u}'}}{\tilde{z}_\rho}, \qquad (1)$$

where $\mathbf{u}$ is the horizontal velocity, z_ρ is the isopycnal thickness, and the tilde represents an average along an isopycnal surface, plays an important role in turbulent tracer transport. It is important because average tracer quantities are advected not just by the *Eulerian mean velocity* $\tilde{\mathbf{u}}$ but by the total transport velocity given by

$$\hat{\mathbf{u}} = \tilde{\mathbf{u}} + \mathbf{u}^*.$$

[We note, however, that only that part of $\mathbf{u}^*$ associated with the divergent component of thickness flux is effective in tracer transport—see Greatbatch (1998) and the comments below regarding gauge fields.] This was originally realized in the atmospheric community [for a review, see Andrews et al. (1987)] but not highlighted in the ocean case until Gent et al. (1995), when the idea was introduced in connection with the mesoscale turbulence parameterization of Gent and McWilliams (1990, henceforth referred to as GM90).

The bolus velocity is a turbulence correlation and therefore needs to be specified by some type of turbulence theory. In the absence of a theory, GM90 have parameterized it as

$$\mathbf{u}^* = -\frac{1}{\tilde{z}_\rho}\partial_\rho(\kappa \nabla_\rho \tilde{z}), \qquad (2)$$

where ∇_ρ is the horizontal gradient in isopycnal coordinates and κ is a scalar diffusivity coefficient. In the case that κ is constant, or at least independent of ρ, this amounts to simple Fickian diffusion of thickness along isopycnal surfaces, which is a plausible choice from the point of view of baroclinic instability. To go beyond this one needs some sort of turbulence theory; the simplest type is embodied in a stochastic model of turbulence (Monin and Yaglom 1971). Such a model is behind simple one-point or mixing-length theories, which are still important in practice and which form the foundation for more elaborate turbulence closures. A stochastic model is of interest because it makes minimal assump-

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tions about the nature of the turbulence: It only assumes that turbulence exists and that it is random and Markovian, that is, that turbulence statistics are independent of its past history. These assumptions are expected to apply in fluid turbulence provided time averages are taken over timescales much longer than the Lagrangian integral timescale and the turbulence is not strongly inhomogeneous. Such minimal dependence on the details of the turbulence is important in view of the large uncertainties prevailing in geostrophic turbulence models. However, until recently it was not possible to apply a stochastic theory to the case of ocean mesoscale turbulence because the theory assumed incompressible flow or a divergence-free velocity field. Because the ocean is strongly stratified, it is believed that mesoscale turbulence is highly nonisotropic, with mixing confined to isopycnal layers. It is therefore most natural to deal with the turbulence in isopycnal coordinates, in which case the two-dimensional velocity field is generally divergent and this requires a compressible stochastic theory of turbulence.

Such a theory, applicable to tracer transport, was recently developed by Dukowicz and Smith (1997, henceforth referred to as DS) and it validated the parameterization of GM90 in general terms. The weak point of the theory is a postulate that relates Lagrangian to Eulerian mean velocities. The present paper presents a further development of the theory for the ocean case, which avoids the need for the aforementioned postulate under certain conditions, and this allows us to go beyond the conceptual basis of the bolus velocity parameterization of GM90 when these conditions prevail.

2. Stochastic theory

The stochastic theory (DS) assumes that infinitesimal fluid parcels (particles) may be represented by a general conditional probability density function $p(\mathbf{x}, t\,|\,\mathbf{y}, t_0)$ such that the probability of finding a particle in volume $d\mathbf{x}$ centered around point $\mathbf{x}$ at time t, given that it was in volume $d\mathbf{y}$ centered around point $\mathbf{y}$ at time t_0, $t \geq t_0$, is $p(\mathbf{x}, t\,|\,\mathbf{y}, t_0)\, d\mathbf{x}$. The assumption of Markovian statistics allows one to write down a Fokker–Planck equation for the time evolution of the probability density function. Specifying the appropriate choice for the probability density function (DS) leads to a theory of turbulent transport in a compressible medium. We apply the theory to the continuity and tracer equations of interest for the ocean, in which case the equations in isopycnal coordinates take the form

$$\partial_t h + \boldsymbol{\nabla}_\rho \cdot h\mathbf{u} = 0, \tag{3}$$

$$\partial_t h\phi + \boldsymbol{\nabla}_\rho \cdot h\phi\mathbf{u} = 0, \tag{4}$$

where the thickness is written as h ($\equiv z_\rho$) for convenience and ϕ represents any tracer concentration per unit volume satisfying the equation

$$D_t\phi = \partial_t\phi + \mathbf{u}\cdot\boldsymbol{\nabla}_\rho\phi = 0, \tag{5}$$

where $\mathbf{u}$ is the two-dimensional horizontal velocity on the isopycnal surface. These equations manifestly have the form of two-dimensional compressible equations to which the stochastic theory is applicable. The theory gives equations for the evolution of averaged quantities, which are defined as

$$\tilde{h}(\mathbf{x}, t)\hat{\phi}(\mathbf{x}, t) = \int d\mathbf{y}\; h(\mathbf{y}, t_0)\phi(\mathbf{y}, t_0)p(\mathbf{x}, t\,|\,\mathbf{y}, t_0), \tag{6}$$

where the tilde indicates that the average is taken on an isopycnal surface. Applying the theory to Eqs. (3) and (4), we obtain

$$\partial_t\tilde{h} + \boldsymbol{\nabla}_\rho\cdot\tilde{h}(\mathbf{v} - \tilde{h}^{-1}\boldsymbol{\nabla}_\rho\cdot\mathsf{K}\tilde{h}) = 0, \tag{7}$$

$$\partial_t\hat{\phi} + (\mathbf{v} - \tilde{h}^{-1}\boldsymbol{\nabla}_\rho\cdot\mathsf{K}\tilde{h})\cdot\boldsymbol{\nabla}_\rho\hat{\phi} = \tilde{h}^{-1}\boldsymbol{\nabla}_\rho\cdot\mathsf{K}\tilde{h}\cdot\boldsymbol{\nabla}_\rho\hat{\phi}, \tag{8}$$

where for convenience we have defined $\hat{\phi}$ to be a thickness-weighted average given by

$$\hat{\phi} = \widetilde{h\phi}/\tilde{h}, \tag{9}$$

and the vector $\mathbf{v}$, which we will refer to as the *Lagrangian mean velocity*, is defined by

$$\mathbf{v}(\mathbf{x}, t) = \lim_{\Delta t\to 0}\Delta t^{-1}\int d\mathbf{z}\;(\mathbf{z} - \mathbf{x})p(\mathbf{z}, t + \Delta t\,|\,\mathbf{x}, t), \tag{10}$$

and where K is a symmetric tensor given by

$$\mathsf{K}(\mathbf{x}, t) = \frac{1}{2}\lim_{\Delta t\to 0}\Delta t^{-1}\int d\mathbf{z}\;(\mathbf{z} - \mathbf{x})(\mathbf{z} - \mathbf{x})p(\mathbf{z}, t + \Delta t\,|\,\mathbf{x}, t). \tag{11}$$

In the above equations $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are to be interpreted as two-dimensional position vectors in the horizontal projection of an isopycnal surface. The limit as $\Delta t \to 0$ makes sense only for timescales much longer than the Lagrangian integral timescale, in which case K is also positive definite and therefore has the properties of a turbulent diffusivity.

Alternatively, we may simply construct equations for averaged quantifies by decomposing all quantities into average and fluctuating components, such that for any quantity φ

$$\varphi = \tilde{\varphi} + \varphi', \qquad \tilde{\phi}' = 0,$$

$$\varphi = \hat{\varphi} + \varphi'', \qquad \hat{\phi}'' = 0,$$

$$\varphi^* \equiv \widetilde{h'\varphi'}/\tilde{h} = \varphi' - \varphi'',$$

$$\hat{\varphi} = \tilde{\varphi} + \varphi^*,$$

substituting into Eqs. (3) and (4), and averaging the resulting equations, to obtain

$$\partial_t\tilde{h} + \boldsymbol{\nabla}_\rho\cdot\tilde{h}(\tilde{\mathbf{u}} + \mathbf{u}^*) = 0, \tag{12}$$

$$\partial_t\hat{\phi} + (\tilde{\mathbf{u}} + \mathbf{u}^*)\cdot\boldsymbol{\nabla}_\rho\hat{\phi} = -\tilde{h}^{-1}\boldsymbol{\nabla}_\rho\cdot\tilde{h}\widehat{\mathbf{u}''\phi''}, \tag{13}$$

where we have made use of (1). Comparing (12)–(13) to (7)–(8), we observe that

$$\tilde{h}\mathbf{u}^* = \tilde{h}(\mathbf{v} - \boldsymbol{\nabla}_\rho \cdot \mathsf{K} - \tilde{\mathbf{u}}) - \mathsf{K} \cdot \boldsymbol{\nabla}_\rho \tilde{h} + \mathbf{k} \times \boldsymbol{\nabla}_\rho \psi, \tag{14}$$

$$\tilde{h}\widehat{\mathbf{u}''\phi''} = -\tilde{h}\mathsf{K} \cdot \boldsymbol{\nabla}_\rho \hat{\phi} + \psi \mathbf{k} \times \boldsymbol{\nabla}_\rho \hat{\phi} + \mathbf{k} \times \boldsymbol{\nabla}_\rho \chi, \tag{15}$$

where $\mathbf{k}$ is the unit vector in the vertical direction and ψ, χ are arbitrary scalar gauge fields, which are not specified by the stochastic theory. We will not focus on these fields henceforth since they play no role in tracer transport, except in section 7 where we will indicate a possible relationship among them. It is important, however, to keep the gauge fields in mind when comparing parameterizations to corresponding experimentally or computationally derived turbulence correlations. Note also that we have implicitly assumed that it is permissible to directly compare Eqs. (12)–(13) to (7)–(8). This may not be the case in general since the averages in (12) and (13) cannot all be the same as those specified in (6) and (9), namely, probability averages on an isopycnal surface, since such averages apply only to quantities satisfying the advection equation (5) [e.g., $\tilde{\mathbf{u}}$ in (12) and (13) cannot be a probability average since $\mathbf{u}$ does not satisfy (5)]. We have therefore implicitly invoked a form of the ''ergodic hypothesis''; that is, we have made the assumption that isopycnal probability averages are approximately equivalent to isopycnal Eulerian averages.

To make use of (7)–(8) we must specify the Lagrangian mean velocity $\mathbf{v}$. In the case of divergence-free flow, $\boldsymbol{\nabla}_\rho \cdot \hat{\mathbf{u}} = 0$, Monin and Yaglom (1971) introduced the postulate

$$\mathbf{v} = \tilde{\mathbf{u}} + \boldsymbol{\nabla}_\rho \cdot \mathsf{K}. \tag{16}$$

It is argued in DS that (16) is also the simplest plausible postulate in the case of divergent flow, $\boldsymbol{\nabla}_\rho \cdot \hat{\mathbf{u}} \neq 0$. Therefore, substituting (16) into (14), one obtains

$$\mathbf{u}^* = -\frac{\mathsf{K} \cdot \boldsymbol{\nabla}_\rho \tilde{h}}{\tilde{h}} + \mathrm{gt}, \tag{17}$$

a generalized downgradient Fickian diffusion of thickness, in general agreement with GM90, and where we have indicated by the notation ''gt'' that gauge terms may be present. Note that this prediction of the bolus velocity (17) does not come directly from stochastic theory but is dependent on the accuracy of the postulate (16). Equation (15), however, appears to be a robust prediction of the stochastic theory because it does not depend on the postulate (16).

3. Potential vorticity

We now take advantage of the fact that the potential vorticity,

$$q = \frac{(f + \zeta)}{h}, \tag{18}$$

where f is the Coriolis parameter (planetary vorticity) and $\zeta = \mathbf{k} \cdot (\boldsymbol{\nabla}_\rho \times \mathbf{u})$ is the relative vorticity, obeys (5) and therefore may be viewed as a tracer for the purpose of the stochastic theory (see DS for a comment on q as an active tracer). Therefore, according to (15), we have

$$\tilde{h}\widehat{\mathbf{u}''q''} = -\tilde{h}\mathsf{K} \cdot \boldsymbol{\nabla}_\rho \hat{q} + \mathrm{gt}, \tag{19}$$

where the thickness-weighted mean potential vorticity has the special form $\hat{q} = \tilde{h}^{-1}(f + \tilde{\zeta})$. We now make use of the following identity relating thickness-weighted to unweighted correlations (which, to our knowledge, appears here for the first time),

$$\hat{q}\widetilde{h'\mathbf{u}'} \equiv -\tilde{h}\widehat{\mathbf{u}''q''} + \widetilde{\zeta'\mathbf{u}'}, \tag{20}$$

together with (1) and (19), to give

$$\begin{aligned}\mathbf{u}^* &= \frac{\mathsf{K} \cdot \boldsymbol{\nabla}_\rho \hat{q}}{\hat{q}} + \frac{\widetilde{\zeta'\mathbf{u}'}}{\tilde{h}\hat{q}} + \mathrm{gt} \\ &= -\frac{\mathsf{K} \cdot \boldsymbol{\nabla}_\rho \tilde{h}}{\tilde{h}} + \frac{\mathsf{K} \cdot \boldsymbol{\nabla}_\rho (f + \tilde{\zeta})}{(f + \tilde{\zeta})} + \frac{\widetilde{\zeta'\mathbf{u}'}}{\tilde{h}\hat{q}} + \mathrm{gt}.\end{aligned} \tag{21}$$

Note that we have in effect substituted the identity (20) for the Monin and Yaglom postulate (16) to obtain a revised version of the bolus velocity originally given by (17). In other words, instead of the postulate (16), we have an alternative expression consistent with the stochastic theory for the Lagrangian mean velocity,

$$\mathbf{v} = \tilde{\mathbf{u}} + \boldsymbol{\nabla}_\rho \cdot \mathsf{K} + \frac{\mathsf{K} \cdot \boldsymbol{\nabla}_\rho \tilde{h}}{\tilde{h}} + \frac{\mathsf{K} \cdot \boldsymbol{\nabla}_\rho \hat{q}}{\hat{q}} + \frac{\widetilde{\zeta'\mathbf{u}'}}{\tilde{h}\hat{q}} + \mathrm{gt}, \tag{22}$$

which is obtained by combining (14) and (21).

Seemingly, we have not gained much in expressing the bolus velocity as in (21) because it appears that we have traded one turbulent correlation for another. However, as we will see next, there exist important parameter regimes in which we expect $\hat{q}\widetilde{h'\mathbf{u}'}$ to dominate over $\widetilde{\zeta'\mathbf{u}'}$, namely, in the planetary geostrophic regime, and in homogeneous mesoscale (quasigeostrophic) turbulence. In these situations there is an approximate balance

$$\hat{q}\widetilde{h'\mathbf{u}'} \approx -\tilde{h}\widehat{\mathbf{u}''q''},$$

and we are justified in neglecting terms containing $\widetilde{\zeta'\mathbf{u}'}$ in (21) and (22). This is the situation to which the theory that follows will apply.

4. Planetary geostrophic regime

Consider the planetary geostrophic (PG) regime in which the flow is on the gyre or planetary scale and is characterized by geostrophic balance and the conservation of planetary potential vorticity, $q_\infty = f/h$. An illuminating discussion of this regime is given by Pedlosky (1984). Furthermore, de Verdière (1986) demonstrates that this regime is subject to baroclinic instability, and therefore contains turbulence. Given that we have exact geostrophy:

$$f\mathbf{k} \times \mathbf{u} = -\frac{1}{\rho_0}\nabla_\rho \pi, \tag{23}$$

where π is the Montgomery potential and ρ_0 is a constant reference density, we may follow an argument in Greatbatch (1998) comparing thickness-weighted and unweighted averages of (23), to obtain

$$\hat{q}_\infty \widetilde{h'\mathbf{u}'} = -\tilde{h}\widehat{\mathbf{u}''q''_\infty}, \tag{24}$$

where $\hat{q}_\infty = f/\tilde{h}$. Alternatively, it is easy to show that (20) reduces to (24) when $q = q_\infty$. Comparing (24) to (20), we see that in the PG limit we can neglect $\widetilde{\zeta'\mathbf{u}'}$ in (20), which is the condition for the validity of our theory. This result is independent of the type of averaging used. Using this result, (21) becomes

$$\begin{aligned}\mathbf{u}^* &= \frac{\mathsf{K}\cdot\nabla_\rho\hat{q}_\infty}{\hat{q}_\infty} + \text{gt},\\ &= -\frac{\mathsf{K}\cdot\nabla_\rho\tilde{h}}{\tilde{h}} + \frac{\mathsf{K}\cdot\nabla_\rho f}{f} + \text{gt},\\ &= -\frac{\mathsf{K}\cdot\nabla_\rho\tilde{h}}{\tilde{h}} + \frac{\beta}{f}\begin{Bmatrix}K_{12}\\ K_{22}\end{Bmatrix} + \text{gt},\end{aligned} \tag{25}$$

where $\beta = \partial_y f$. We observe that this is the same as (17) except that the bolus velocity is augmented by a velocity contribution proportional to β. In the case of isotropic mixing, $\mathsf{K} = \kappa\mathsf{I}$, $\kappa > 0$, this term corresponds to a poleward component of bolus velocity in each hemisphere. We also have the following result for the Lagrangian mean velocity:

$$\mathbf{v} = \tilde{\mathbf{u}} + \nabla_\rho\cdot\mathsf{K} + \frac{\mathsf{K}\cdot\nabla_\rho f}{f} + \text{gt}, \tag{26}$$

which is a modified form of the Monin and Yaglom postulate, exact in the PG regime.

5. Mesoscale regime

According to the multiscale analysis of Pedlosky (1984), synoptic-scale (mesoscale) dynamics are given by the quasigeostrophic (QG) equations, modified by weak coupling to the gyre-scale dynamics (planetary geostrophy), depending on the scale separation between the gyre scale and the mesoscale. In quasigeostrophy we have

$$\begin{aligned}\mathbf{u}' &= \frac{1}{f_0\rho_0}\mathbf{k}\times\nabla_\rho\pi',\\ \zeta' &= \frac{1}{f_0\rho_0}\nabla_\rho\cdot\nabla_\rho\pi',\end{aligned} \tag{27}$$

where f_0 is the ''local'' Coriolis parameter, taken to be constant. Gill (1982) shows that quasigeostrophy is valid provided (i) $\varepsilon_L = \beta L/f_0 \ll 1$, (ii) $\tau f_0 \gg 1$, and (iii) $R_o = U/f_0 L \ll 1$, where L is a horizontal length scale such as the Rossby radius, and τ and U are characteristic time and velocity scales, respectively.

Assume solutions of the form $\pi' = \Pi e^{i(kx+ly-\omega t)}$, as in Killworth (1997), where $i = \sqrt{-1}$, k, l are the horizontal wavenumbers, ω is the frequency, in general complex to allow for growth of instabilities, and Π is a complex amplitude. The spatial average, in the limit of averaging over very many wavelengths, is given by

$$\begin{aligned}\widetilde{\zeta'\mathbf{u}'} &= \frac{1}{2}\text{Re}(\zeta'\mathbf{u}'^*) = \text{Re}\left(-\frac{i\Pi^*\Pi(k^2+l^2)}{2f_0^2\rho_0^2}\begin{Bmatrix}-l\\ k\end{Bmatrix}\right)\\ &= 0,\end{aligned} \tag{28}$$

where the asterisk here denotes a complex conjugate. Thus, the quasigeostrophic vorticity and velocity fluctuations tend to be uncorrelated in a spatial average. This will be true for both a single wave and a linear combination of these waves. Moreover, since we are taking only spatial averages, the same result will be true for a general Fourier transform of the vorticity and velocity fluctuations with respect to the spatial coordinates at any instant in time.

A more general form of this result may be obtained without resorting to Fourier transforms. Consider the general spatial average for the correlation:

$$\widetilde{\zeta'\mathbf{u}'} = \frac{1}{f_0^2\rho_0^2 A}\mathbf{k}\times\int_A dx\,dy\,(\nabla_\rho\cdot\nabla_\rho\pi')\nabla_\rho\pi', \tag{29}$$

where A is some arbitrary convex averaging area on an isopycnal surface. Noting the identity

$$(\nabla_\rho\cdot\nabla_\rho\pi')\nabla_\rho\pi' = \nabla_\rho\cdot\mathsf{T}, \tag{30}$$

where T is the traceless tensor given by

$$\mathsf{T} = \nabla_\rho\pi'\nabla_\rho\pi' - \frac{1}{2}(\nabla_\rho\pi'\cdot\nabla_\rho\pi')\mathsf{I},$$

and I is the two-dimensional unit tensor, we observe that the area integral in (29) may be transformed by the divergence theorem into a line integral around the boundary of the area A. Since there is no contribution from the interior, the correlation will scale with the ratio of the periphery length to the area, which may always be made small by choosing a suitably large averaging area A. That is, in quasigeostrophic flow,

$$\lim_{A\to\infty}\widetilde{\zeta'\mathbf{u}'} = 0, \tag{31}$$

in a spatial average. This is analogous to the average in (28) except that in this case no assumption of linear waves is required, and it demonstrates that vorticity and velocity fluctuations are essentially uncorrelated provided the averaging scale is sufficiently large. The averaging scale is not specified but it should clearly be large in comparison to the eddy scale and small compared to the domain size. Since we expect the eddy scale to be set by the internal radius of deformation, there should be no difficulty in choosing an averaging scale that is large compared to the eddy scale but small compared to the basin scale. In general, the correlation will not be exactly zero but it will be small in the sense that it can always be reduced as much as necessary as the averaging area is increased.

The question becomes, is this result useful for QG turbulence? Typically, turbulence correlations are most meaningful in an ensemble average. However, ensemble averages are mainly a theoretical concept, and typically space or time averages are substituted for practical reasons. This is justified by the ''ergodic hypothesis'' (Monin and Yaglom 1971), which allows substitution of time averages for ensemble averages in stationary turbulence and spatial averages for ensemble averages in homogeneous turbulence. Therefore, the result (31) may be taken as an indication that in homogeneous quasigeostrophic turbulence the correlation $\widetilde{\zeta'\mathbf{u}'}$ will likely be small in comparison with $\hat{q}\widetilde{h'\mathbf{u}'}$, and therefore that our theory will be valid in such a situation.

There is at least one type of quasigeostrophic flow where our theory will not apply. Consider a very strongly stratified flow (gravity $g \to \infty$) or a constant-depth barotropic flow with a rigid-lid boundary condition, described by the barotropic vorticity equation. For such a flow (i.e., where the rigid-lid approximation is valid), the thickness h is nearly constant and $\widetilde{h'\mathbf{u}'}$ will be very nearly zero in any average. Therefore, the dominant balance assumed for (20) in this paper will collapse, and instead we will have

$$\widetilde{\zeta'\mathbf{u}'} \approx \tilde{h}\widehat{\mathbf{u}''q''},$$

which shows that the relative vorticity–velocity correlation need not be zero, in general, in a time or ensemble average. In view of (31), which shows that the spatial average is expected to be zero, this is tantamount to saying that such barotropic turbulence must be inhomogeneous since we cannot relate the spatial average to the ensemble average. It is possible to apply the stochastic theory result (15) to this new balance. However, this will imply the need for constraints on the diffusivity tensor K under certain circumstances. Consider a reentrant zonal channel in a statistically steady state where all the mean quantities must be zonally uniform. It is then easy to show that $\widetilde{\zeta' v'} = -\partial(\widetilde{u'v'})/\partial y$, and therefore that the cross-channel integral of $\widetilde{\zeta' v'}$ must always vanish because of the boundary condition $v' = 0$ on the north and south boundaries. This, therefore, implies the existence of a global constraint on any parameterization, whereas the present parameterization, like most others, is local when K is prescribed a priori.

In summary, using (21) we obtain a result that is expected to be valid in PG turbulence and in homogeneous QG turbulence:

$$\begin{aligned} \mathbf{u}^* &= \frac{\mathsf{K}\cdot\nabla_\rho \hat{q}}{\hat{q}} + \text{gt} \\ &= -\frac{\mathsf{K}\cdot\nabla_\rho \tilde{h}}{\tilde{h}} + \frac{\mathsf{K}\cdot\nabla_\rho (f + \tilde{\zeta})}{(f + \tilde{\zeta})} + \text{gt}, \end{aligned} \tag{32}$$

where $f + \tilde{\zeta} \to f$ in the PG case. This is our principal result. It is derived using only (15), a robust prediction of stochastic theory, the identity (20), and the fact that $|\widetilde{\zeta'\mathbf{u}'}| \ll |\hat{q}\widetilde{h'\mathbf{u}'}|$ in certain situations, as discussed above. The transformation of this result to z coordinates is useful for practical applications and is given in the appendix.

The bolus velocity in this form represents a flux of tracer up the potential vorticity gradient, which results from a modification of (17) given by the second term on the right-hand side of (32), the so-called β velocity. However, this modification is typically of very small magnitude (as will be seen in section 6) and will therefore have little practical impact, except for the fact that it importantly changes the conceptual basis for the parameterization of bolus velocity, from being based on the mixing of thickness as in GM90 to being based on the mixing of potential vorticity, as previously suggested by Treguier et al. (1997).

6. The β velocity

The β velocity,

$$\mathbf{u}^*_\beta = \frac{\mathsf{K}\cdot\nabla_\rho f}{f}, \tag{33}$$

predicted in (25) and appearing in (26), and in a more general form in (32), is a striking consequence of the present theory. It is a rigorous prediction in situations where $|\widetilde{\zeta'\mathbf{u}'}| \ll |\hat{q}\widetilde{h'\mathbf{u}'}|$, assuming that the stochastic theory is valid. We will now discuss it in more detail.

We can understand the appearance of this velocity component from elementary considerations by adapting an argument due to Rhines (Rhines and Holland 1979). Consider a fluid particle moving in a turbulent velocity field. Since the potential vorticity is conserved along a trajectory, at any time t along the trajectory we have

$$q(t) = \hat{q}(t) + q''(t) = \hat{q}(0) + q''(0), \tag{34}$$

where the $q''(0)$, for example, indicates the initial condition. For a sufficiently small time increment, this may be rewritten as follows:

$$q''(t) = \hat{q}(0) - \hat{q}(t) + q''(0) \approx -\Delta\mathbf{x}\cdot\nabla\hat{q}, \tag{35}$$

where $\Delta\mathbf{x} = \mathbf{x}(t) - \mathbf{x}^0(0)$ is the displacement of the particle, and $q''(0)$ has been absorbed into the definition of $\mathbf{x}^0(0)$. We now multiply both sides by $h\mathbf{u}''$ and take some appropriate average (we retain the tilde sign to indicate that this is still an isopycnal average) to obtain

$$\tilde{h}\widehat{\mathbf{u}''q''} = -\tilde{h}\widehat{\mathbf{u}''\Delta\mathbf{x}}\cdot\nabla\hat{q}. \tag{36}$$

Rhines and Holland (1979) identify the correlation on the right-hand side with the Lagrangian diffusivity tensor:

$$\tilde{h}\mathsf{K} \equiv \tilde{h}\widehat{\mathbf{u}''\Delta\mathbf{x}}. \tag{37}$$

From experience we expect this to be a positive-definite quantity (consider the turbulent dispersion of a dye released from a concentrated source). Note that (36) and

(37) are equivalent to (19), but obtained more transparently and with less rigorous arguments. In the PG regime, where $q = f/h$, and therefore $\hat{q} = f/\tilde{h}$ (we have dropped the subscript ∞ for simplicity), Eq. (35) may be written out as $q'' \approx -\beta\Delta y/\tilde{h} + (\Delta\mathbf{x}\cdot\nabla\tilde{h})/\tilde{h}^2$. Thus, there are two components to the potential vorticity fluctuation: the first one is due to the change in the planetary vorticity f, and the second one is due to a change in mean thickness $\tilde{h}$ along the trajectory. We will now assume that the mean thickness is constant since we wish to focus on the first component. This implies that meridional fluid parcel displacements are always negatively correlated with potential vorticity fluctuations. Alternatively, since $q'' = -(h'/h)(f/\tilde{h})$, fluid parcels displaced poleward will be stretched while parcels displaced away from the poles will be flattened relative to their mean thickness. Now, because the right-hand side of (37) is positive-definite and the diffusivity tensor is likely to be diagonally dominant, velocities v'' will be positively correlated with displacements Δy. Therefore, the meridional potential vorticity–velocity correlation will always be negative. Because of Eq. (24), this implies that the associated bolus velocity will always be poleward. This is the content of Eq. (33); it shows that the β velocity is inherent in the PG regime provided that turbulence is present. Note that the apparent singularity at the equator is merely due to the fact that the β velocity given in (33) is valid only in the PG regime, which excludes the equator.

The existence of the β velocity has been specifically confirmed by Lee et al. (1997) in an eddy-resolving channel experiment performed with a three-layer isopycnal model. Using an appropriate buoyancy forcing, they obtained essentially constant mean thickness in the middle layer so that the predominant contribution to bolus velocity was due to (33). Using the value $\kappa \sim 1000$ m^2 s^{-1}, which they measure in the top and bottom layers and the values $\beta \sim 2 \times 10^{-11}$ m^{-1} s^{-1} and $f \sim 0.83 \times 10^{-4}$ s^{-1} characteristic of the center of the channel, (33) predicts a northward velocity $v^*_\beta \sim 0.024$ cm s^{-1}, in reasonable agreement with the value $v^*_\beta \sim 0.05$ cm s^{-1}, which they find in the experiment. As pointed out by Killworth (1997), this is a rather small velocity [even compared to the total bolus velocity—we typically find that $\mathbf{u}^* \sim O(1$ cm s$^{-1})$ in eddy-resolving calculations].

There is an alternative but equivalent way to arrive at the β velocity. Welander (1973), Rhines (1979), Rhines and Holland (1979), Tung (1986), and Greatbatch (1998) propose closely related parameterizations of the momentum equations based on the turbulent flux of potential vorticity. In the u-momentum equation there then appears a ‘‘friction force that is everywhere directed westward,’’ equal to $-\beta K_{22}$ (Welander 1973). When balanced against the Coriolis term, this results in the β velocity of Eqs. (25), (33). From the atmospheric perspective, Tung (1986) points out that the presence of this term has important physical implications. Without this term, the winter stratospheric westerly jet would reach unrealistically large velocities and the temperature near the winter pole would be too low. It is of interest to note that the β velocity is associated with the presence of diabatic heating in the experiments of Lee et al. (1997) and in the atmospheric context of Tung (1986). Without diabatic heating it is likely that a statistical equilibrium consistent with the continuity and momentum equations would not be possible, and a poleward thickness flux would be prohibited.

From a theoretical perspective, a remarkable result by Killworth (1997) gives a bolus velocity that is equivalent to (25), and therefore also predicts the β velocity of (33), but from consideration of linear waves alone, in a perturbation theory about a slowly varying mean velocity. This in effect extends to turbulence a result derived from linear stability theory. However, a justification for the role of linear terms in turbulent mixing problems has been given by G. M. Lilley in Morris et al. (1990) for the case of incompressible flow. This theory, therefore, lends powerful independent support for the existence of the β velocity. It is not the object of this paper to discuss the pros and cons of the Killworth (1997) theory and the present theory. There is, however, an important difference between these two theories that is worth pointing out. The Killworth theory derives a form of the diffusivity κ that is proportional to the instability growth rate and therefore requires that the underlying mean flow be unstable for κ to exist; the present theory merely assumes that fully developed turbulence is present, in which case κ is formally given by (11). It should be noted that, when there is no turbulence, then κ is zero. In other words, the present theory, unlike the Killworth theory, makes no statement about the origin of the turbulence. It does require an independent specification of κ, in common with other parameterizations of its type, but then it leaves open the possibility that a future, more general type of turbulence theory will become available that will predict κ prognostically. Until that time, it is quite possible to have seemingly paradoxical situations arising from an ignorance of what the value of κ may be.

Nevertheless, there is a connection with baroclinic instability in the present theory, in common with the Killworth (1997) theory. Let us consider the linearized eddy continuity equation:

$$\frac{\tilde{d}}{dt}h' + \mathbf{u}'\cdot\nabla_\rho\tilde{h} + \tilde{h}\nabla_\rho\cdot\mathbf{u}' + h'\nabla_\rho\cdot\tilde{\mathbf{u}} = 0, \quad (38)$$

where $\tilde{d}/dt \equiv \partial_t + \tilde{\mathbf{u}}\cdot\nabla_\rho$. This is the same as the equation in the Killworth theory except that the last term on the right-hand side was neglected. In the PG regime, geostrophic balance implies

$$f\nabla_\rho\cdot\tilde{\mathbf{u}} + \beta\tilde{v} = 0, \qquad f\nabla_\rho\cdot\mathbf{u}' + \beta v' = 0, \quad (39)$$

where v' is the meridional fluctuating velocity component. Multiplying the continuity equation by h', making use of (39), and taking an average we obtain

$$\frac{\tilde{d}}{dt}\frac{\widetilde{h'^2}}{2} = -\widetilde{h'\mathbf{u}'}\cdot\nabla_\rho\tilde{h} + \widetilde{h'v'}\frac{\tilde{h}\beta}{f} + \widetilde{h'^2}\frac{\tilde{v}\beta}{f}$$
$$= \widetilde{h'\mathbf{u}'}\cdot\mathbf{x} + \widetilde{h'^2}\frac{\tilde{v}\beta}{f}, \qquad (40)$$

where

$$\mathbf{x} = \frac{\tilde{h}\nabla_\rho\hat{q}_\infty}{\hat{q}_\infty} = \begin{Bmatrix} -\partial_x\tilde{h} \\ -\partial_y\tilde{h} + \dfrac{\tilde{h}\beta}{f}\end{Bmatrix}.$$

Substituting from (1) and (25), we obtain

$$\frac{\tilde{d}}{dt}\frac{\widetilde{h'^2}}{2} = \mathbf{x}\cdot\mathsf{K}\cdot\mathbf{x} + \widetilde{h'^2}\frac{\tilde{v}\beta}{f}, \qquad (41)$$

where we observe that the first term on the right-hand side is always positive because, as stated previously, K is positive-definite. Neglecting the second term on the right-hand side, as in the initial phase of the growth of an instability, is consistent with the idea that eddy variance will increase as a result of the instability, and it therefore associates the bolus velocity parameterization in the form given by (25), that is, including the β-velocity correction, with the presence of baroclinic instability, consistent with Killworth (1997).

Finally, it might be tempting to ascribe the existence of the β velocity to Rossby waves because of its β dependence. This is clearly incorrect, as is obvious from the above analysis, but it is possible to demonstrate this more directly. Let us consider the case of pure Rossby waves when the gradient of mean thickness vanishes. Equation (40) in the limit of slowly varying mean velocities then becomes

$$\frac{\tilde{d}}{dt}\frac{\widetilde{h'^2}}{2} - \frac{\tilde{h}\beta}{f}\widetilde{h'v'} \approx 0. \qquad (42)$$

Since Rossby waves in this regime are stable and are neither dissipative nor dispersive, we expect the variance of the thickness to be unchanged, and therefore

$$\widetilde{h'v'} \approx 0.$$

Thus the Rossby wave contribution to the bolus velocity is negligible in this case. This therefore demonstrates that planetary Rossby waves are not responsible for the β velocity.

7. The gauge

It is now possible to restore the gauge. Since the vorticity–velocity correlation $\widetilde{\zeta'\mathbf{u}'}$ is assumed small compared to other terms in (20), we neglect it in that equation, and using (14) and (15) for the case of potential vorticity, we must have

$$\mathbf{v} - \frac{\nabla_\rho\cdot\mathsf{K}\tilde{h}}{\tilde{h}} = \tilde{\mathbf{u}} + \frac{\mathsf{K}\cdot\nabla_\rho\hat{q}}{\hat{q}} - \frac{\mathbf{k}\times\nabla_\rho(\chi + \hat{q}\psi)}{\tilde{h}\hat{q}}. \qquad (43)$$

One way to view the gauge fields is as arbitrary degrees of freedom arising in Eqs. (7) and (8), as well as in (12) and (13). The left-hand side of Eq. (43) represents the total tracer transport velocity in Eq. (8). If we substitute this in (7) and (8) we have the rather unsatisfactory situation that our equations depend on the gauge. The simplest postulate that allows the total tracer transport velocity, as well as the continuity and tracer transport equations, to be independent of gauge is

$$\chi + \hat{q}\psi = \text{const}, \qquad (44)$$

which we will henceforth adopt. Relating the two gauges, χ and ψ, as in (44) implies that the potential vorticity system contains only a single gauge.

8. Summary of results

Using (43) and (44), (14) and (15) may be put in the form

$$\mathbf{u}^* = \frac{\mathsf{K}\nabla_\rho\hat{q}}{\hat{q}} + \frac{\mathbf{k}\times\nabla_\rho\psi}{\tilde{h}}, \qquad (45)$$

$$\tilde{h}\widehat{\mathbf{u}''q''} = -\tilde{h}\mathsf{K}\cdot\nabla_\rho\hat{q} - \mathbf{k}\times\hat{q}\nabla_\rho\psi. \qquad (46)$$

Therefore, the bolus flux $\tilde{h}\mathbf{u}^*$ retains an arbitrary rotational component specified by the gauge ψ in addition to the largely irrotational potential vorticity term [see (32), where the dominant thickness-mixing term is irrotational for constant, isotropic K], as seems to be observed in various simulations (e.g., Bryan et al. 1999). The total transport velocity becomes

$$\mathbf{v} - \frac{\nabla_\rho\cdot\mathsf{K}\tilde{h}}{\tilde{h}} = \tilde{\mathbf{u}} + \frac{\mathsf{K}\cdot\nabla_\rho\hat{q}}{\hat{q}}. \qquad (47)$$

Substituting the above in (12) and (13), with $\hat{\phi}$ taken to be $\hat{q}$, we have

$$\partial_t\tilde{h} + \nabla_\rho\cdot\tilde{h}\left(\tilde{\mathbf{u}} + \frac{\mathsf{K}\cdot\nabla_\rho\hat{q}}{\hat{q}}\right) = 0, \qquad (48)$$

$$\partial_t\hat{q} + \left(\tilde{\mathbf{u}} + \frac{\mathsf{K}\cdot\nabla_\rho\hat{q}}{\hat{q}}\right)\cdot\nabla_\rho\hat{q} = \tilde{h}^{-1}\nabla_\rho\cdot\tilde{h}\mathsf{K}\cdot\nabla_\rho\hat{q}, \qquad (49)$$

and similarly, substituting (45) and (15) in (13), we have for an arbitrary tracer

$$\partial_t\hat{\phi} + \left(\tilde{\mathbf{u}} + \frac{\mathsf{K}\cdot\nabla_\rho\hat{q}}{\hat{q}}\right)\cdot\nabla_\rho\hat{\phi} = \tilde{h}^{-1}\nabla_\rho\cdot\tilde{h}\mathsf{K}\cdot\nabla_\rho\hat{\phi}. \qquad (50)$$

Therefore, the total transport velocity and the thickness and tracer transport equations are all independent of gauge, as remarked previously.

In summary, assuming the validity of the stochastic theory, we have deduced that under certain conditions the bolus velocity corresponds within a gauge to a flux directed up the potential vorticity gradient and is characterized by a general symmetric diffusivity tensor K. This is expected to be true in planetary geostrophic turbulence and also in homogeneous quasigeostrophic turbulence, that is, whenever $|\widetilde{\zeta'\mathbf{u}'}| \ll |\hat{q}\widetilde{h'\mathbf{u}'}|$. This result is corroborated by the calculations of Lee et al. (1997) and supports their thesis regarding the role of gradients of potential vorticity in determining the bolus velocity. The direct relationship of the bolus velocity with potential vorticity substantially modifies the conceptual basis of the GM90 parameterization, and agrees with the theoretical proposals of Killworth (1997) and Treguier et al. (1997), although only within the regimes discussed above. However, the resulting modification of the bolus velocity is quite small in magnitude and we expect that it will have little practical effect in simulations. Furthermore, we have postulated that gauge fields are related according to (44), which renders the total transport velocity and the thickness and tracer transport equations to be independent of gauge, a result that is highly desirable from a theoretical point of view.

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APPENDIX

Transformation to *z* Coordinates

This transformation is greatly facilitated by the formulas given by de Szoeke and Bennett (1993). Defining $\tilde{z}$ to be the average depth of an isopycnal surface and $\tilde{\rho}$ to be the inverse function to $\tilde{z}\,[\tilde{\rho}(x, y, \tilde{z}(x, y, \rho, t) = \rho]$, such that $\tilde{h} = \partial_\rho \tilde{z} = 1/\partial_z \tilde{\rho}$, Eq. (32) transforms into

$$\mathbf{u}^* = \frac{1}{\hat{q}\partial_z\tilde{\rho}} \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \begin{bmatrix} \partial_z\tilde{\rho}\partial_x\hat{q} - \partial_x\tilde{\rho}\partial_z\hat{q} \\ \partial_z\tilde{\rho}\partial_y\hat{q} - \partial_y\tilde{\rho}\partial_z\hat{q} \end{bmatrix} + \mathrm{gt}, \quad \text{(A1)}$$

where

$$\hat{q} = \partial_z\tilde{\rho}(f + \partial_x\tilde{v} - \partial_y\tilde{u}) - \partial_x\tilde{\rho}\partial_z\tilde{u} + \partial_y\tilde{\rho}\partial_z\tilde{v}. \quad \text{(A2)}$$

Additional details regarding the transformation are available in DS.

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