*Journal of Mathematical Research & Exposition*
Feb., 2008, Vol. 28, No. 1, pp. 211–216
DOI:10.3770/j.issn:1000-341X.2008.01.027
Http://jmre.dlut.edu.cn

# An Algebraic System and Its Application

*WEI Yuan*[1], *ZHANG Yu-feng*[2]
(1. Department of Mathematics and Information, Binzhou University, Shandong 256600, China;
2. School of Information Science and Engineering, Shandong University of Science and Technology, Shandong 266510, China)
(E-mail: wei_-y79@126.com)

**Abstract** An algebraic system $X$ is constructed by using the known loop $\widetilde{A}_1$. Then a new isospectral problem is established by taking advantage of $X$, which is devoted to working out the well-known Volterra lattice hierarchy. And an extended algebraic system $\widetilde{X}$ of $X$ is presented, from which the integrable coupling systems of the Volterra lattice is obtained.

**Keywords** algebraic system; Volterra lattice hierarchy; integrable coupling system.

**Document code** A
**MR(2000) Subject Classification** 05A10; 60G50
**Chinese Library Classification** O157.1

## 1. Introduction

By making use of Tu scheme[1,2], one has obtained some continuous interesting integrable Hamiltonian hierarchies of soliton equation such as AKNS hierarchy, BPT hierarchy, KN hierarchy etc. In recent years, the nonlinear integrable lattice equations have also been extensively studied. The mathematical structures and physical applications of the discrete lattice systems, such as the bi-Hamiltonian structure, integrable symplectic maps, Bäcklund transformations and nonlinear superposition formulae as well as soliton solutions, master symmetries and so on were investigated in Refs. [3–8]. Searching for the new Lax integrable lattice equations is an important subject in the theory of nonlinear integrable lattice equations. The following loop algebra $\widetilde{A}_1$ is frequently used to construct the isospectral problems by Tu scheme:

$$h_1(n)=\begin{pmatrix}\lambda^n & 0\\ 0 & 0\end{pmatrix}, h_2(n)=\begin{pmatrix}0 & 0\\ 0 & \lambda^n\end{pmatrix}, e(n)=\begin{pmatrix}0 & \lambda^n\\ 0 & 0\end{pmatrix}, f(n)=\begin{pmatrix}0 & 0\\ \lambda^n & 0\end{pmatrix},$$
$$\begin{aligned}&[h_1(m),e(n)]=e(m+n),[h_1(m),f(n)]=-f(m+n),\\ &[h_2(m),e(n)]=-e(m+n),[h_2(m),f(n)]=f(m+n),\\ &[e(m),f(n)]=h_1(m+n)-h_2(m+n),\\ &\deg(h_i(n))=\deg(e(n))=\deg(f(n))=n,\quad i=1,2.\end{aligned}\tag{1}$$

In this paper, in terms of the loop algebra (1) and the different operators, we construct an algebraic system $X$ and its extended system $\overline{X}$, the discrete integrable coupling systems of Volterra lattice is worked out. Therefore, the method can be used generally.

**Received date**: 2005-09-15; **Accepted date**: 2006-07-16

## 2. The algebraic system and the Volterra lattice hierarchy

Above all, we construct an algebraic system based on the loop algebra (1). Let $X$ denote a linear space expanded by the linear independent vectors $h_1(n), h_2(n)$, $e(n)$ and $f(n)$,

$$X = \text{span}\{h_1(n), h_2(n), e(n), f(n)\}. \tag{2}$$

Define their product operation in $X$ as follows[9]:

$$\begin{aligned}
&h_1(m) * e(n) = e(m+n), e(m) * h_2(n) = e(m+n),\\
&f(m) * h_1(n) = f(m+n), h_2(m) * f(n) = f(m+n),\\
&e(m) * f(n) = h_1(m+n), f(m) * e(n) = h_2(m+n),\\
&h_1(m) * h_2(n) = \begin{cases} -h_2(m+n), \text{ when } \ h_1(m), h_2(n) \in \Gamma U,\\ -h_1(m+n), \text{when } \ h_1(m), h_2(n) \in U\Gamma,\end{cases}\\
&h_2(m) * h_1(n) = \begin{cases} -h_1(m+n), \text{when } \ h_1(m), h_2(n) \in \Gamma U,\\ -h_2(m+n), \text{when } \ h_1(m), h_2(n) \in U\Gamma.\end{cases}
\end{aligned} \tag{3}$$

We define the product operation of $h_1(n)$ and $h_2(n)$ as follows if we need modified matrices $\Delta n$:

$$\begin{aligned}
&h_1(m) * h_1(n) = h_1(m+n), \quad h_2(m) * h_2(n) = h_2(m+n),\\
&h_1(m) * h_2(n) = h_2(m) * h_1(n) = 0,\\
&e(m) * h_1(n) = h_2(m) * e(n) = h_1(m) * f(n) = f(m) * h_2(n) = 0, \quad h_1(n), h_2(n) \in \Delta n,
\end{aligned} \tag{4}$$

$$h_1(n) = h_1(0)\lambda^n, \quad h_2(n) = h_2(0)\lambda^n, \quad e(n) = e(0)\lambda^n, \quad f(n) = f(0)\lambda^n. \tag{5}$$

According to the algebraic system (2) along with (3)–(5), we consider the following isospectral problem:

$$E\psi_n = U_n\psi_n, \quad \lambda_t = 0, \quad U_n = h_2(1) + u_1 e(0) + u_2 f(0) - u_3 h_2(0). \tag{6}$$

Set

$$\begin{aligned}
\Gamma &= \sum_{m\geq 0} [a_m(h_1(-m) - h_2(-m)) + b_m e(-m) + c_m f(-m)]\\
&= a(h_1(0) - h_2(0)) + be(0) + cf(0),
\end{aligned}$$

where $a = \sum_{m\geq 0} a_m\lambda^{-m}$, $b = \sum_{m\geq 0} b_m\lambda^{-m}$, $c = \sum_{m\geq 0} c_m\lambda^{-m}$. The discrete stationary zero curvature equation

$$(E\Gamma)U_n - U_n\Gamma = 0 \tag{7}$$

is equivalent to

$$\begin{aligned}
&(u_2 b^{(1)} - u_1 c)h_1(0) + (-\lambda a^{(1)} + u_3 a^{(1)} + u_1 c^{(1)} + \lambda a - u_2 b - u_3 a)h_2(0) +\\
&\quad (u_1 a^{(1)} + \lambda b^{(1)} + u_1 a - u_3 b^{(1)})e(0) + (-u_2 a^{(1)} - \lambda c - u_2 a + u_3 c)f(0) = 0.
\end{aligned} \tag{8}$$

Since $h_i(0)$ $(i = 1, 2)$, $e(0)$, $f(0)$ are linear independent, from Eq.(8) we have

$$\begin{cases}
u_2 b^{(1)} - u_1 c = 0,\\
-\lambda a^{(1)} + u_3 a^{(1)} + u_1 c^{(1)} + \lambda a - u_2 b - u_3 a = 0,\\
u_1 a^{(1)} + \lambda b^{(1)} + u_1 a - u_3 b^{(1)} = 0,\\
-u_2 a^{(1)} - \lambda c - u_2 a + u_3 c = 0.
\end{cases} \tag{9}$$

Inserting the expanding expressions $a=\sum_{m\geq 0} a_m\lambda^{-m}$, $b=\sum_{m\geq 0} b_m\lambda^{-m}$, $c=\sum_{m\geq 0} c_m\lambda^{-m}$ into (9) gives the recurrence relations as follows

$$\begin{cases} u_2 b_m^{(1)} - u_1 c_m = 0, \\ a_{m+1} - a_{m+1}^{(1)} - u_3 a_m + u_3 a_m^{(1)} + u_1 c_m^{(1)} - u_2 b_m = 0, \\ u_1 a_m + u_1 a_m^{(1)} + b_{m+1}^{(1)} - u_3 b_m^{(1)} = 0, \\ u_2 a_m + u_2 a_m^{(1)} + c_{m+1} - u_3 c_m = 0. \end{cases} \tag{10}$$

Taking $a_0=\frac{1}{2}$, $b_0=-u_1^{(1)}$, $c_0=-u_2$, we have

$$a_1=-u_1^{(-1)}u_2,\ b_1=-u_1^{(-1)}u_3^{(-1)}-u_1^{(-1)},\ c_1=-u_2u_3-u_2,\ldots.$$

Denote

$$\begin{aligned}(\lambda^n\Gamma)_+ &= \sum_{m=0}^{n}[a_m(h_1(n-m)-h_2(n-m))+b_m e(n-m)+c_m f(n-m)],\\ (\lambda^n\Gamma)_- &= \lambda^n\Gamma-(\lambda^n\Gamma)_+.\end{aligned}$$

Then Eq. (7) can be written as

$$-U_n(\lambda^n\Gamma)_+ \ + \ (E(\lambda^n\Gamma)_+)U_n = U_n(\lambda^n\Gamma)_- \ - \ (E(\lambda^n\Gamma)_-)U_n. \tag{11}$$

It is easy to find that the terms on the left-hand side in (11) contain powers $\lambda^k$, $k\geqslant 0$, while the terms on the right-hand side in (11) contain powers $\lambda^k$, $k\leqslant 0$. Therefore, we obtain

$$(E(\lambda^n\Gamma)_+)U_n - U_n(\lambda^n\Gamma)_+ = (a_{n+1}^{(1)}-a_{n+1})h_2(0) - b_{n+1}^{(1)}e(0) + c_{n+1}f(0).$$

Taking $V^{(n)}=(\lambda^n\Gamma)_+$, we have

$$(EV^{(n)})U_n - U_nV^{(n)} = (a_{n+1}^{(1)}-a_{n+1})h_2(0) - b_{n+1}^{(1)}e(0) + c_{n+1}f(0). \tag{12}$$

From Eq.(12), we obtain the following lattice equation hierarchy

$$\begin{cases} u_{1tn} = -b_{n+1}^{(1)}, \\ u_{2tn} = c_{n+1}, \\ u_{3tn} = a_{n+1} - a_{n+1}^{(1)}. \end{cases} \tag{13}$$

When $u_1=u_2=u_3$, and $n=0$, (13) becomes $u_{1t0}=u_1(u_1^{(1)}-u_1^{(-1)})$, which is the Volterra lattice equation. Therefore, we call (13) the generalized Volterra lattice hierarchy.

## 3. Integrable coupling system of the Volterra lattice hierarchy

In terms of the theory on continuous integrable couplings[10,11], some integrable hierarchies, such as KN hierarchy, TC hierarchy etc, have been obtained in Refs. [12–14]. In this paper we firstly extend the algebra system (3) into the following:

$$\widetilde{X} = \mathrm{span}\{h_1(n), h_2(n), e(n), f(n), \widetilde{e}(n), \widetilde{f}(n)\} \tag{14}$$

with

$$h_1(m)*\widetilde{e}(n)=\widetilde{e}(m+n),\ h_2(m)*\widetilde{f}(n)=\widetilde{f}(m+n),$$

$$\begin{aligned}
e(m)*\widetilde{f}(n) &= \widetilde{e}(m+n),\ f(m)*\widetilde{e}(n)=\widetilde{f}(m+n),\\
\widetilde{e}(m)*h_1(n) &= \widetilde{f}(m)*h_2(n)=\widetilde{f}(m)*e(n)=\widetilde{e}(m)*f(n)\\
&= h_1(m)*\widetilde{f}(n)=\widetilde{f}(m)*h_1(n)=h_2(m)*\widetilde{e}(n)\\
&= \widetilde{e}(m)*h_2(n)=e(m)*\widetilde{e}(n)=\widetilde{e}(m)*e(n)\\
&= f(m)*\widetilde{f}(n)=\widetilde{f}(m)*f(n)=\widetilde{e}(m)*\widetilde{f}(n)\\
&= \widetilde{f}(m)*\widetilde{e}(n)=0,\\
\widetilde{e}(n) &= \widetilde{e}(0)\lambda^n,\ \widetilde{f}(n)=\widetilde{f}(0)\lambda^n,
\end{aligned}$$

where the operation relations among $h_1(n), h_2(n), e(n), f(n)$ are the same as those in (3)–(4).

Denoting $\widetilde{X}_1=\mathrm{span}\{h_1(n),h_2(n),e(n),f(n)\}$, $\widetilde{X}_2=\mathrm{span}\{\widetilde{e}(n),\widetilde{f}(n)\}$, we find

$$\text{(i) } \widetilde{X}=\widetilde{X}_1\oplus\widetilde{X}_2,\quad \text{(ii) } \widetilde{X}_1*\widetilde{X}_2\subset\widetilde{X}_2, \tag{15}$$

where the symbol $\oplus$ stands for a direct summation and

$$\widetilde{X}_1*\widetilde{X}_2=\{x_1(m)*x_2(n)|x_1(m)\in\widetilde{X}_1,x_2(n)\in\widetilde{X}_2\}.$$

Then from Eq.(14), we consider an isospectral problem

$$\begin{aligned}
E\psi_n &= \overline{U}_n\psi_n,\ \lambda_t=0,\\
\overline{U}_n &= h_2(1)+u_1e(0)+u_2f(0)-u_3h_2(0)+u_4\widetilde{e}(0)+u_5\widetilde{f}(0).
\end{aligned} \tag{16}$$

Set

$$\begin{aligned}
\overline{\Gamma} &= \sum_{m\geq 0}[a_m(h_1(-m)-h_2(-m))+b_me(-m)+c_mf(-m)+d_m\widetilde{e}(-m)+h_m\widetilde{f}(-m)]\\
&= a(h_1(0)-h_2(0))+be(0)+cf(0)+d\widetilde{e}(0)+h\widetilde{f}(0),
\end{aligned}$$

where

$$\begin{aligned}
&a=\sum_{m\geq 0}a_m\lambda^{-m},\ b=\sum_{m\geq 0}b_m\lambda^{-m},\ c=\sum_{m\geq 0}c_m\lambda^{-m},\\
&d=\sum_{m\geq 0}d_m\lambda^{-m},\ h=\sum_{m\geq 0}h_m\lambda^{-m}.
\end{aligned}$$

A direct calculation gives

$$\begin{aligned}
(E\overline{\Gamma})\overline{U}_n-\overline{U}_n\overline{\Gamma}=&(u_2b^{(1)}-u_1c)h_1(0)+(-\lambda a^{(1)}+u_3a^{(1)}+u_1c^{(1)}+\lambda a-u_2b-u_3a)h_2(0)+\\
&u_1a^{(1)}+\lambda b^{(1)}+u_1a-u_3b^{(1)})e(0)+(-u_2a^{(1)}-\lambda c-u_2a+u_3c)f(0)+\\
&(u_4a^{(1)}+u_5b^{(1)}-u_1h)\widetilde{e}(0)+(-u_5a^{(1)}+u_4c^{(1)}-\lambda h+u_3h-u_2d)\widetilde{f}(0).
\end{aligned}$$

The discrete stationary zero curvature equation $(E\overline{\Gamma})\overline{U}_n-\overline{U}_n\overline{\Gamma}=0$ admits the recurrence relations:

$$\begin{cases} u_2 b_m^{(1)} - u_1 c_m = 0, \\ a_{m+1} - a_{m+1}^{(1)} - u_3 a_m + u_3 a_m^{(1)} + u_1 c_m^{(1)} - u_2 b_m = 0, \\ u_1 a_m + u_1 a_m^{(1)} + b_{m+1}^{(1)} - u_3 b_m^{(1)} = 0, \\ u_2 a_m + u_2 a_m^{(1)} + c_{m+1} - u_3 c_m = 0, \\ u_4 a_m^{(1)} + u_5 b_m^{(1)} - u_1 h_m = 0, \\ u_5 a_m^{(1)} - u_4 c_m^{(1)} + h_{m+1} - u_3 h_m + u_2 d_m = 0, \end{cases} \tag{17}$$

where $d$ is an arbitrary constant or function. Note that

$$\begin{aligned} (\lambda^n \Gamma)_+ = & \sum_{m=0}^{n} [a_m(h_1(n-m) - h_2(n-m)) + b_m e(n-m) + c_m f(n-m) + \\ & d_m \widetilde{e}(n-m) + h_m \widetilde{f}(n-m)], \\ (\lambda^n \Gamma)_- = & \lambda^n \Gamma - (\lambda^n \Gamma)_+, \end{aligned}$$

then a direct calculation gives rise to

$$(E(\lambda^n \Gamma)_+)\overline{U}_n - \overline{U}_n(\lambda^n \Gamma)_+ = (a_{n+1}^{(1)} - a_{n+1})h_2(0) - b_{n+1}^{(1)} e(0) + c_{n+1} f(0) + h_{n+1} \widetilde{f}(0).$$

Taking $\triangle n = h_1(0) + h_2(0)$, $\overline{V}^{(n)} = (\lambda^n \overline{\Gamma})_+ + \triangle n$, we obtain

$$\begin{aligned} (E\overline{V}^{(n)})\overline{U}_n - \overline{U}_n \overline{V}^{(n)} = & (a_{n+1}^{(1)} - a_{n+1})h_2(0) - b_{n+1}^{(1)} e(0) + \\ & c_{n+1} f(0) + u_4 \widetilde{e}(0) + (h_{n+1} + u_5)\widetilde{f}(0). \end{aligned}$$

Hence, the discrete zero curvature equation

$$\overline{U}_{tn} = (E\overline{V}^{(n)})\overline{U}_n - \overline{U}_n \overline{V}^{(n)} \tag{18}$$

leads to

$$\begin{cases} u_{1tn} = -b_{n+1}^{(1)}, \\ u_{2tn} = c_{n+1}, \\ u_{3tn} = a_{n+1} - a_{n+1}^{(1)}, \\ u_{4tn} = u_4, \\ u_{5tn} = u_5 + h_{n+1}. \end{cases} \tag{19}$$

In terms of the definition of integrable couplings[10,11], we conclude that the integrable discrete system (19) is the discrete integrable coupling of the Volterra lattice hierarchy.

## References


[1] TU Gui-zhang. *The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems* [J]. J. Math. Phys., 1989, **30**(2): 330–339.

[2] MA Wen-xiu. *A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction* [J]. Chinese J. Contemp. Math., 1992, **13**(1): 79–89.

[3] HU Xing-biao, TAM H W. *Some recent results on integrable bilinear equations* [J]. J. Nonlinear Math. Phys., 2001, **8**(suppl): 149–155.

[4] HU Xing-biao, TAM H W. *Application of Hirota's bilinear formalism to a two-dimensional lattice by Leznov* [J]. Phys. Lett.A, 2000, **276**(1-4): 65–72.

[5] HU Xing-biao, CLARKSON P A. *Rational solutions of an extended Lotka-Volterra equation* [J]. J. Nonlinear Math. Phys., 2002, **9**(suppl): 75–86.

[6] MA Wen-xiu. *Symmetry constraint of MKdV equations by binary nonlinearation* [J]. J. Phys. A, 1995, **219**: 1083–1091.

[7] HU Xing-biao. *A powerful approach to generate new integrable systems* [J]. J. Phys. A, 1994, **27**(7): 2497–2514.

[8] HU Xing-biao. *An approach to generate superextensions of integrable systems* [J]. J. Phys. A, 1997, **30**(2): 619–632.

[9] TAM H, ZHANG Yu-feng. *A new algebraic system and its applications* [J]. Chaos Solitons and Fractals. 2005, **23**: 151–157.

[10] MA Wen-xiu, FUCHSSTEINER B. *Integrable theory of the perturbation equations* [J]. Chaos, Solitons and Fractals, 1996, **7**: 1227–1250.

[11] FUCHSSTEINER B. *Coupling of Completely Integrable Systems: the Perturbation Bundle* [M]. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 413, Kluwer Acad. Publ., Dordrecht, 1993.

[12] ZHANG Yu-feng, ZHANG Hong-qing. *A direct method for integrable couplings of TD hierarchy* [J]. J. Math. Phys., 2002, **43**(1): 466–472.

[13] ZHANG Yu-feng, YAN Qing-you. *Integrable couplings of the hierarchies of evolution equations* [J]. Chaos Solitons Fractals, 2003, **16**(2): 263–269.

[14] ZHANG Yu-feng, ZHANG Hong-qing, YAN Qing-you. *Integrable couplings of Botie-Pempinelli-Tu (BPT) hierarchy* [J]. Phys. Lett. A, 2002, **299**(5-6): 543–548.