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A SUFFICIENT CONDITION FOR THE INTEGRAL VERSION OF MARTINS' INEQUALITY

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ABSTRACT. We prove that if f is a nondecreasing, positive, twice differentiable function on $\mathbb{R}^+$ such that $t(\ln f(t))'' + (\ln f(t))' \geq 0$ for all $t > 0$, then f satisfies the integral version of Martins' inequality.

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Recently a number of papers have appeared on Martins' inequality:

$$\left(\frac{1}{n}\sum_{i=1}^{n} i^r \Bigg/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{\frac{1}{r}} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}\,, \tag{1}$$

which holds for $r > 0$ and $n \in \mathbb{N}$ (see [2]). For example, in [1] it is proved that

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \Bigg/ \frac{1}{n+1}\sum_{i=1}^{n+1} a_i^r\right)^{\frac{1}{r}} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+1]{a_{n+1}!}}\,, \tag{2}$$

where $\{a_i\}$ is an increasing non-constant sequence of positive numbers satisfying (1) $a_\ell/a_{\ell+1} \geq a_{\ell-1}/a_\ell$ and (2) $(a_{\ell+1}/a_\ell)^\ell \geq (a_\ell/a_{\ell-1})^{\ell-1}$ for $\ell > 1$ (and where it is agreed that $a_n!$ stands for $\prod_{i=1}^n a_i$). In particular, the authors show that the sequence $a_i = ci + d$ gives a generalization of Martins' result whenever $c > 0$ and $d \geq 0$.

On a parallel path, continuous versions of the inequality have been investigated, and in [4] F. Qi and B.-N. Guo ask under which conditions the following holds:

$$\left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,dx}{\frac{1}{b+\delta-a}\int_a^{b+\delta} f^r(x)\,dx}\right)^{\frac{1}{r}} \leq \frac{\exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right)}{\exp\left(\frac{1}{b+\delta-a}\int_a^{b+\delta} \ln f(x)\,dx\right)}, \tag{3}$$

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whenever f is a positive, increasing and integrable function on the closed interval $[a, b+\delta]$ (with $b > a$ and $\delta > 0$) and $r > 0$ is arbitrary. In a related result, in [3] N. Towghi and F. Qi prove that for all $r > 0$ and any non-negative, integrable f we have

$$\frac{\sup_{x\in[a,b]} f(x)}{\sup_{x\in[a,b+\delta]} f(x)} \leq \left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,dx}{\frac{1}{b+\delta-a}\int_a^{b+\delta} f^r(x)\,dx} \right)^{\frac{1}{r}} \tag{4}$$

(note that the l.h.s. in (4) is the limit for $r \to \infty$ of the r.h.s.). In another remark, they note that (3) itself fails without extra assumptions. The issue, then, is at least to identify a sufficient hypothesis, and this is the aim of the present paper. While logarithmic convexity of f has been identified as sufficient in related inequalities, our result below requires a strictly weaker hypothesis:

Theorem 1. *Let f be a nondecreasing, positive, twice differentiable function on $\mathbb{R}^+$ such that*

$$t(\ln f(t))'' + (\ln f(t))' \geq 0 \tag{5}$$

for all $t > 0$. Then

$$F(t) := \frac{\frac{1}{t-a}\int_a^t f(x)\,dx}{\exp\left(\frac{1}{t-a}\int_a^t \ln f(x)\,dx\right)} \tag{6}$$

is non-decreasing on $[a,\infty]$ for every $a \geq 0$ and therefore inequality (3) *holds for f and every choice of $0 \leq a < b$, and $r, \delta > 0$.*

Proof. It is plain that if f satisfies (5) then f^r also does (for every $r > 0$), and so the last statement is a trivial consequence of function F being non-decreasing.

Fix $a \geq 0$ and in the following always assume that $t \geq a$. Note that condition (5) implies

$$(t-a)(\ln f(t))'' + (\ln f(t))' \geq 0 \tag{7}$$

for all $t \geq a$ (we are assuming that $f(t)$ is non-decreasing, and therefore $(\ln f(t))' \geq 0$). Computing the derivatives in (7) gives

$$(t-a)\frac{f(t)f''(t) - (f'(t))^2}{f^2(t)} + \frac{f'(t)}{f(t)} \geq 0 \tag{8}$$

which is in turn equivalent to

$$\left(-\frac{(t-a)f(t)}{1+(t-a)f'(t)/f(t)} + \int_a^t f(x)dx \right)' \geq 0 \tag{9}$$

(if you apply the quotient rule to differentiate the first summand in (9), and collect the l.h.s. over the common denominator, then the numerator is seen to be $(t-a)f(t)$ times the l.h.s. in (8)). Now, (9) implies

$$\frac{(t-a)f(t)}{1+(t-a)f'(t)/f(t)} \leq \int_a^t f(x)dx \leq (t-a)f(t)\,, \tag{10}$$

where the second inequality is due to f' being non-decreasing. Next, considering the left hand side of the following inequality as a quadratic polynomial in $\int_a^t f(x)dx$, (10) is seen to be equivalent to

$$\begin{aligned} &\left(\int_a^t f(x)dx\right)^2 \left(1 + \frac{(t-a)f'(t)}{f(t)}\right) \\ &\quad - \left(\int_a^t f(x)dx\right)\left(2(t-a)f(t) + (t-a)^2 f'(t)\right) + (t-a)^2 f^2(t) \leq 0 \end{aligned} \tag{11}$$

(inequality (11) says that $\int_a^t f(x)\,dx$ must lie between the two solutions of the quadratic polynomial, and the quadratic formula says that these two solutions are the l.h.s. and the r.h.s. of (10)).

Dividing both sides of (11) by $\left(\int_a^t f(x)\,dx\right)^2$ and rearranging the terms we then obtain the equivalent form

$$\left(\frac{(t-a)^2 f(t)}{\int_a^t f(x)dx}\right)' \geq \left((t-a)+(t-a)\ln f(t) - \int_a^t \ln f(x)dx\right)', \tag{12}$$

which clearly implies

$$\frac{(t-a)^2 f(t)}{\int_a^t f(x)dx} \geq (t-a)+(t-a)\ln f(t) - \int_a^t \ln f(x)dx \tag{13}$$

(since both sides vanish when $t=a$). Finally, if we divide (13) by $(t-a)^2$ we obtain

$$\frac{f(t)}{\int_a^t f(x)dx} - \frac{1}{t-a} - \frac{1}{t-a}\ln f(t) + \frac{1}{(t-a)^2}\int_a^t \ln f(x)dx \geq 0\,, \tag{14}$$

which is equivalent to

$$\left(\ln \int_a^t f(x)dx - \ln(t-a) - \frac{1}{t-a}\int_a^t \ln f(x)dx\right)' \geq 0\,. \tag{15}$$

But this amounts to saying that the derivative of the natural logarithm of

$$\frac{\frac{1}{t-a}\int_a^t f(x)dx}{\exp\left(\frac{1}{t-a}\int_a^t \ln f(x)dx\right)} \tag{16}$$

is non-negative: the latter function of t must therefore be non-increasing. Sufficiency of condition (5) is thus proved. □

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