Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 3, Article 74, 2005

A NOTE ON THE BEZIER VARIANT OF CERTAIN BERNSTEIN DURRMEYER OPERATORS

M.K. GUPTA

DEPARTMENT OF MATHEMATICS
CH. CHARAN SINGH UNIVERSITY
MEERUT-250004, INDIA.
mkgupta2002@hotmail.com

Received 27 July, 2004; accepted 13 May, 2005
Communicated by R.N. Mohapatra

ABSTRACT. In the present note, we introduce a Bezier variant of a new type of Bernstein Durrmeyer operator, which was introduced by Gupta [3]. We estimate the rate of convergence by using the decomposition technique of functions of bounded variation and applying the optimum bound. It is observed that the analysis for our Bezier variant of new Bernstein Durrmeyer operators is different from the usual Bernstein Durrmeyer operators studied by Zeng and Chen [9].

Key words and phrases: Lebesgue integrable functions; Bernstein polynomials; Bezier variant; Functions of bounded variation.

2000 *Mathematics Subject Classification.* 41A30, 41A36.

1. INTRODUCTION

Durrmeyer [1] introduced the integral modification of Bernstein polynomials to approximate Lebesgue integrable functions on the interval $[0,1]$. The operators introduced by Durrmeyer are defined by

$$D_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt, \quad x\in[0,1], \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$.

Gupta [3] introduced a different Durrmeyer type modification of the Bernstein polynomials and estimated the rate of convergence for functions of bounded variation. The operators introduced in [3] are defined by

$$B_n(f,x) = n\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 b_{n,k}(t)f(t)dt, \quad x\in[0,1], \tag{1.2}$$

ISSN (electronic): 1443-5756

The author is thankful to the referee for his suggestions leading to substantial improvements in the paper.
140-04

where

$$p_{n,k}(x) = (-1)^k \frac{x^k}{k!} \phi_n^{(k)}(x), \quad b_{n,k}(t) = (-1)^{k+1} \frac{t^k}{k!} \phi_n^{(k+1)}(t)$$

and

$$\phi_n(x) = (1-x)^n.$$

It is easily verified that the values of $p_{n,k}(x)$ used in (1.1) and (1.2) are same. Also it is easily verified that

$$\sum_{k=0}^{n} p_{n,k}(x) = 1, \quad \int_0^1 b_{n,k}(t)dt = 1 \quad \text{and} \quad b_{n,n}(t) = 0.$$

By considering the integral modification of Bernstein polynomials in the form (1.2) some approximation properties become simpler in the analysis. So it is significant to study further on the different integral modification of Bernstein polynomials introduced by Gupta [3]. For $\alpha \geq 1$, we now define the Bezier variant of the operators (1.2), to approximate Lebesgue integrable functions on the interval $[0,1]$ as

$$B_{n,\alpha}(f,x) = \sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x) \int_0^1 b_{n,k}(t) f(t)dt, \quad x \in [0,1], \tag{1.3}$$

where

$$Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$$

and

$$J_{n,k}(x) = \sum_{j=k}^{n} p_{n,j}(x),$$

when $k \leq n$ and 0 otherwise.

Some important properties of $J_{n,k}(x)$ are as follows:

(i) $J_{n,k}(x) - J_{n,k+1}(x) = p_{n,k}(x), k = 0,1,2,3,\dots;$
(ii) $J'_{n,k}(x) = np_{n-1,k-1}(x), k = 1,2,3,\dots;$
(iii) $J_{n,k}(x) = n\int_0^x p_{n-1,k-1}(u)du, k = 1,2,3,\dots;$
(iv) $J_{n,0}(x) > J_{n,1}(x) > J_{n,2}(x) > \dots > J_{n,n}(x) > 0$, $0 < x < 1$.

For every natural number k, $J_{n,k}(x)$ increases strictly from 0 to 1 on $[0,1]$.

Alternatively we may rewrite the operators (1.3) as

$$B_{n,\alpha}(f,x) = \int_0^1 K_{n,\alpha}(x,t) f(t)dt, \quad 0 \leq x \leq 1, \tag{1.4}$$

where

$$K_{n,\alpha}(x,t) = \sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x) b_{n,k}(t).$$

It is easily verified that $B_{n,\alpha}(f,x)$ are linear positive operators, $B_{n,\alpha}(1,x) = 1$ and for $\alpha = 1$, the operators $B_{n,1}(f,x) \equiv B_n(f,x)$, i.e. the operators (1.3) reduce to the operators (1.2). For further properties of $Q_{n,k}^{(\alpha)}(x)$, we refer the readers to [3].

Guo [2] studied the rate of convergence for bounded variation functions for Bernstein Durrmeyer operators. Zeng and Chen [9] were the first to estimate the rate of convergence for the Bezier variant of Bernstein Durrmeyer operators. Several other Bezier variants of some summation-integral type operators were studied in [4], [6] and [8] etc. It is well-known that Bezier basis functions play an important role in computer aided design. Moreover the recent work on different Bernstein Bezier type operators motivated us to study further in this direction. The advantage of the operators $B_{n,\alpha}(f,x)$ over the Bernstein Durrmeyer operators considered in [9] is that one does not require the results of the type Lemma 3 and Lemma 4 of [9]. Consequently some approximation formulae become simpler. Further for $\alpha = 1$, these operators provide improved estimates over the main results of [2] and [3]. In the present paper, we estimate the rate of point wise convergence of the operators $B_{n,\alpha}(f,x)$ at those points $x \in (0,1)$ at which one sided limits $f(x-)$ and $f(x+)$ exist.

2. AUXILIARY RESULTS

In this section we give certain results, which are necessary to prove the main result.

Lemma 2.1 ([3]). *If n is sufficiently large, then*

$$\frac{x(1-x)}{n} \leq B_n((t-x)^2, x) \leq \frac{2x(1-x)}{n}.$$

Lemma 2.2 ([4]). *For every $0 \leq k \leq n$, $x \in (0,1)$ and for all $n \in \mathbb{N}$, we have*

$$p_{n,k}(x) \leq \frac{1}{\sqrt{2enx(1-x)}}.$$

Lemma 2.3. *For all $x \in (0,1)$, there holds*

$$Q_{n,k}^{(\alpha)}(x) \leq \alpha \cdot p_{n,k}(x) \leq \frac{\alpha}{\sqrt{2enx(1-x)}}.$$

Proof. Using the well known inequality $|a^\alpha - b^\alpha| \leq \alpha\,|a-b|$, $(0 \leq a, b \leq 1, \alpha \geq 1)$ and by Lemma 2.2, we obtain

$$Q_{n,k}^{(\alpha)}(x) \leq \alpha p_{n,k}(x) \leq \frac{\alpha}{\sqrt{2enx(1-x)}}.$$

□

Lemma 2.4. *Let $x \in (0,1)$ and $K_{n,\alpha}(x,t)$ be the kernel defined by (1.4). Then for n sufficiently large, we have*

$$\lambda_{n,\alpha}(x,y) := \int_0^y K_{n,\alpha}(x,t)dt \leq \frac{2\alpha \cdot x(1-x)}{n(x-y)^2}, \quad 0 \leq y < x, \tag{2.1}$$

and

$$1 - \lambda_{n,\alpha}(x,z) := \int_z^1 K_{n,\alpha}(x,t)dt \leq \frac{2\alpha \cdot x(1-x)}{n(z-x)^2}, \quad x < z < 1. \tag{2.2}$$

Proof. We first prove (2.1), as follows

$$\begin{aligned}
\int_0^y K_{n,\alpha}(x,t)dt &\leq \int_0^y K_{n,\alpha}(x,t)\frac{(x-t)^2}{(x-y)^2}dt \\
&\leq \frac{1}{(x-y)^2} B_{n,\alpha}((t-x)^2, x) \\
&\leq \frac{\alpha \cdot B_{n,1}((t-x)^2, x)}{(x-y)^2} \leq \frac{2\alpha \cdot x(1-x)}{n(x-y)^2},
\end{aligned}$$

by Lemma 2.1. The proof of (2.2) is similar. □

3. MAIN RESULT

In this section we prove the following main theorem.

Theorem 3.1. *Let f be a function of bounded variation on the interval $[0,1]$ and suppose $\alpha \geq 1$. Then for every $x \in (0,1)$ and n sufficiently large, we have*

$$\left| B_{n,\alpha}(f,x) - \left[\frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-) \right] \right| \leq \frac{\alpha}{\sqrt{2enx(1-x)}} |f(x+) - f(x-)| + \frac{5\alpha}{nx(1-x)} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x),$$

where

$$g_x(t) = \begin{cases} f(t) - f(x-), & \text{for } 0 \leq t < x \\ 0, & \text{for } t = x \\ f(t) - f(x+), & \text{for } x < t \leq 1 \end{cases}$$

and $\bigvee_a^b (g_x)$ is the total variation of g_x on $[a,b]$.

Proof. Clearly

$$\left| B_{n,\alpha}(f,x) - \left\{ \frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-) \right\} \right| \leq |B_{n,\alpha}(g_x,x)| + \frac{1}{2} \left| B_{n,\alpha}(\operatorname{sgn}(t-x),x) + \frac{\alpha-1}{\alpha+1} \right| |f(x+) - f(x-)|. \tag{3.1}$$

First, we have

$$\begin{aligned} B_{n,\alpha}(\operatorname{sgn}(t-x),x) &= \int_x^1 K_{n,\alpha}(x,t)dt - \int_0^x K_{n,\alpha}(x,t)dt \\ &= \int_0^1 K_{n,\alpha}(x,t)dt - 2\int_0^x K_{n,\alpha}(x,t)dt \\ &= 1 - 2\int_0^x K_{n,\alpha}(x,t)dt = -1 + 2\int_x^1 K_{n,\alpha}(x,t)dt. \end{aligned}$$

Using Lemma 2.2, Lemma 2.3 and the fact that

$$\sum_{j=0}^{k} p_{n,j}(x) = \int_x^1 b_{n,k}(t)dt,$$

we have

$$\begin{aligned} B_{n,\alpha}(\operatorname{sgn}(t-x),x) &= -1 + 2\sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x) \int_x^1 b_{n,k}(t)dt \\ &= -1 + 2\sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x) \sum_{j=0}^{k} p_{n,j}(x) \end{aligned}$$

$$= -1 + 2\sum_{j=0}^{n} p_{n,j}(x) \sum_{k=j}^{n} Q_{n,k}^{(\alpha)}(x)$$

$$= -1 + 2\sum_{j=0}^{n} p_{n,j}(x) J_{n,j}^{\alpha}(x).$$

Since

$$\sum_{j=0}^{n} Q_{n,j}^{(\alpha+1)}(x) = 1,$$

therefore we have

$$B_{n,\alpha}(\operatorname{sgn}(t-x),x) + \frac{\alpha-1}{\alpha+1} = 2\sum_{j=0}^{n} p_{n,j}(x) J_{n,j}^{\alpha}(x) - \frac{2}{\alpha+1}\sum_{j=0}^{n} Q_{n,j}^{(\alpha+1)}(x).$$

By the mean value theorem, it follows

$$Q_{n,j}^{(\alpha+1)}(x) = J_{n,j}^{\alpha+1}(x) - J_{n,j+1}^{\alpha+1}(x) = (\alpha+1) p_{n,j}(x)\gamma_{n,j}^{\alpha}(x),$$

where

$$J_{n,j+1}^{\alpha}(x) < \gamma_{n,j}^{\alpha}(x) < J_{n,j}^{\alpha}(x).$$

Hence

$$\left| B_{n,\alpha}(sgn(t-x),x) + \frac{\alpha-1}{\alpha+1} \right| \le 2\sum_{j=0}^{n} p_{n,j}(x)(J_{n,j}^{\alpha}(x) - \gamma_{n,j}^{\alpha}(x))$$

$$\le 2\sum_{j=0}^{n} p_{n,j}(x)(J_{n,j}^{\alpha}(x) - J_{n,j+1}^{\alpha}(x))$$

$$\le 2\alpha \sum_{j=0}^{n} p_{n,j}^{2}(x),$$

where we have used the inequality $b^\alpha - a^\alpha < \alpha(b-a)$, $0 \le a, b \le 1$ and $\alpha \ge 1$. Applying Lemma 2.2, we get

$$\left| B_{n,\alpha}(\operatorname{sgn}(t-x),x) + \frac{\alpha-1}{\alpha+1} \right| = \frac{2\alpha}{\sqrt{2enx(1-x)}}, \qquad x \in (0,1). \tag{3.2}$$

Next we estimate $B_{n,\alpha}(g_x, x)$. By a Lebesgue-Stieltjes integral representation, we have

$$\begin{aligned} B_{n,\alpha}(g_x,x) &= \int_0^1 K_{n,\alpha}(x,t) g_x(t)\,dt \\ &= \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right) K_{n,\alpha}(x,t) g_x(t)\,dt \\ &= E_1 + E_2 + E_3, \qquad \text{say,} \end{aligned} \tag{3.3}$$

where $I_1 = [0, x - x/\sqrt{n}]$, $I_2 = [x - x/\sqrt{n}\ , x + (1-x)/\sqrt{n}]$ and $I_3 = [x + (1-x)/\sqrt{n}, 1]$. We first estimate E_1. Writing $y = x - x/\sqrt{n}$ and using Lebesgue-Stieltjes integration by parts, we have

$$E_1 = \int_0^y g_x(t)\,d_t(\lambda_{n,\alpha}(x,t)) = g_x(y+)\lambda_{n,\alpha}(x,y) - \int_0^y \lambda_{n,\alpha}(x,t)\,d_t(g_x(t)).$$

Since $|g_x(y+)| \leq \bigvee_{y+}^{x}(g_x)$, it follows that

$$|E_1| \leq \bigvee_{y+}^{x}(g_x)\lambda_{n,\alpha}(x,y) + \int_0^y \lambda_{n,\alpha}(x,t)d_t\left(-\bigvee_{t}^{x}(g_x)\right).$$

By using (2.1) of Lemma 2.4, we get

$$|E_1| \leq \bigvee_{y+}^{x}(g_x)\frac{2\alpha \cdot x(1-x)}{n(x-y)^2} + \frac{2\alpha \cdot x(1-x)}{n}\int_0^y \frac{1}{(x-t)^2}d_t\left(-\bigvee_{t}^{x}(g_x)\right).$$

Integrating by parts the last term we have after simple computation

$$|E_1| \leq \frac{2\alpha \cdot x(1-x)}{n}\left[\frac{\bigvee_0^x(g_x)}{x^2} + 2\int_0^y \frac{\bigvee_t^x(g_x)}{(x-t)^3}dt\right].$$

Now replacing the variable y in the last integral by $x - x/\sqrt{t}$, we obtain

$$|E_1| \leq \frac{2\alpha(1-x)}{nx}\left[\bigvee_{0}^{x}(g_x) + \sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x}(g_x)\right] \leq \frac{4\alpha}{nx(1-x)}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x}(g_x). \tag{3.4}$$

Using a similar method and (2.2) of Lemma 2.4, we get

$$|E_3| \leq \frac{4\alpha}{nx(1-x)}\sum_{k=1}^{n}\bigvee_{x}^{x+(1-x)/\sqrt{k}}(g_x). \tag{3.5}$$

Finally we estimateE_2. For $t \in [x - x/\sqrt{n}, x + (1-x)/\sqrt{n}]$, we have

$$|g_x(t)| = |g_x(t) - g_x(x)| \leq \bigvee_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}}(g_x),$$

and therefore

$$|E_2| \leq \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)\int_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} d_t(\lambda_{n,\alpha}(x,t))$$

Since $\int_a^b d_t\lambda_n(x,t) \leq 1$, for all $(a,b) \subseteq [0,1]$, therefore

$$|E_2| \leq \bigvee_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}}(g_x). \tag{3.6}$$

Collecting the estimates (3.3) – (3.6), we have

$$|B_{n,\alpha}(g_x,x)| \leq \frac{5\alpha}{nx(1-x)}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x). \tag{3.7}$$

Combining the estimates of (3.1), (3.2) and (3.7), our theorem follows. □

For $\alpha = 1$, we obtain the following corollary, which is an improved estimate over the main results of [2] and [3].

Corollary 3.2. *Let f be a function of bounded variation on the interval $[0,1]$. Then for every $x \in (0,1)$ and n sufficiently large, we have*

$$\left| B_n(f,x) - \frac{1}{2}[f(x+) + f(x-)] \right| \leq \frac{1}{\sqrt{2enx(1-x)}} \left| f(x+) - f(x-) \right| + \frac{5}{nx(1-x)} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x).$$

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