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A NOTE ON THE ACCURACY OF RAMANUJAN'S APPROXIMATIVE FORMULA FOR THE PERIMETER OF AN ELLIPSE

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ABSTRACT. We present a detailed error analysis, with best possible constants, of Ramanujan's most accurate approximation to the perimeter of an ellipse.

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1. INTRODUCTION

Let a and b be the semi-major and semi-minor axes of an ellipse with perimeter p and whose eccentricity is k. The final sentence of Ramanujan's famous paper *Modular Equations and Approximations to* π, [6], says:

" *The following approximation for* p *[was] obtained empirically:*

$$p = \pi \left\{ (a+b) + \frac{3(a-b)^2}{10(a+b) + \sqrt{a^2 + 14ab + b^2}} + \varepsilon \right\} \tag{1.1}$$

where ε *is about* $\frac{3ak^{20}}{68719476736}$."

Ramanujan never explained his "empirical" method of obtaining this approximation, nor ever subsequently returned to this approximation, neither in his published work, nor in his Notebooks [4]. Indeed, although the notebooks do contain the above approximation (see Entry 3 of Chapter XVIII) the statement there does not even mention his asymptotic error estimate stated above.

Twenty years later Watson [7] claimed to have proven that Ramanujan's approximation is *in defect*, but he never published his proof.

In 1978, we established the following optimal version of Ramanujan's approximation:

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Theorem 1.1 (Ramanujan's Approximation Theorem). *Ramanujan's approximative perimeter*

$$p_R := \pi \left\{ (a+b) + \frac{3(a-b)^2}{10(a+b)+\sqrt{a^2+14ab+b^2}} \right\} \tag{1.2}$$

underestimates the true perimeter, p, by

$$\epsilon := \pi(a+b)\cdot\theta(\lambda)\cdot\lambda^{10}, \tag{1.3}$$

where

$$\lambda := \frac{a-b}{a+b}, \tag{1.4}$$

and where the function $\theta(\lambda)$ grows monotonically in $0 \le \lambda \le 1$ while at the same time it satisfies the optimal inequalities

$$\frac{3}{2^{17}} < \theta(\lambda) \le \frac{14}{11}\left(\frac{22}{7}-\pi\right). \tag{1.5}$$

Please take note of the striking form of the sharp upper bound since it involves the number $\left(\frac{22}{7}-\pi\right)$ which measures the accuracy of Archimedes' famous approximation, $\frac{22}{7}$, to the transcendental number π!

Corollary 1.2. *The error in defect, ϵ, as a function of λ, grows monotonically for $0 \le \lambda \le 1$.*

Corollary 1.3. *The error in defect, ϵ, as a function of the eccentricity, e, is given by*

$$\epsilon(e) := a\left\{\delta(e)\left(\frac{2}{1+\sqrt{1-e^2}}\right)^{19}\right\}e^{20}. \tag{1.6}$$

Moreover, $\epsilon(e)$ grows monotonically with e, $0 \le e \le 1$, while $\delta(e)$ satisfies the optimal inequalities

$$\frac{3\pi}{68719476736} < \delta(e) \le \frac{\frac{7}{11}\left(\frac{22}{7}-\pi\right)}{2^{18}}. \tag{1.7}$$

This Corollary 1.3 explains the significance of Ramanujan's own error estimate in (1.1). The latter is an asymptotic *lower bound* for $\epsilon(e)$ but it is not the optimal one. That is given in (1.7).

2. LATER HISTORY

We sent an (updated) copy of our 1978 preprint to Bruce Berndt in 1988 and he subsequently quoted its conclusions in his edition of Volume 3 of the Notebooks (see p. 150 [4]). However the details of our proofs have never been published and so we have decided to present them in this paper.

Berndt's discussion of Ramanujan's approximation includes Almkvist's very plausible suggestion that Ramanujan's "empirical process" was to develop a *continued fraction expansion* of Ivory's infinite series for the perimeter ([1]) as well as a proof, due independently to Almkvist and Askey, of our fundamental lemma (see §3). However, their proof is different from ours.

The most recent work on the subject includes that by R. Barnard, K. Pearce, and K. Richards in [3], published in the year 2000, and the paper by H. Alzer and Qui, S.-L. (see [2]), which was published in the year 2004. The former also prove the major conclusion in our fundamental lemma, but their methods too are quite different from ours. The latter includes a sharp lower bound for elliptical arc length in terms of a power-mean type function. But their methods are also quite different from ours.

3. FUNDAMENTAL LEMMA

Lemma 3.1 (Fundamental Lemma)**.** *Define the functions $\mathbf{A}(x)$ and $\mathbf{B}(x)$ and the coefficients A_n and B_n by:*

$$\mathbf{A}(x) := 1 + \frac{3x}{10+\sqrt{4-3x}} := 1 + A_1x + A_2x^2 + \cdots , \tag{3.1}$$

$$\mathbf{B}(x) := \sum_{n=0}^{\infty} \left\{ \frac{1}{2n-1} \frac{1}{4^n} \binom{2n}{n} \right\}^2 x^n := 1 + B_1x + B_2x^2 + \cdots . \tag{3.2}$$

Then:

$$A_1 = B_1,\ A_2 = B_2,\ A_3 = B_3,\ A_4 = B_4 \tag{3.3}$$

$$A_5 < B_5,\ A_6 < B_6, \dots, A_n < B_n, \dots , \tag{3.4}$$

where the strict inequalities in (3.4) are valid for all $n \geq 5$.

Proof. First we prove (3.3). We read this off directly from the numerical values of the expansion:

$$A_1 = B_1 = \frac{1}{4}$$
$$A_2 = B_2 = \frac{1}{16}$$
$$A_3 = B_3 = \frac{1}{64}$$
$$A_4 = B_4 = \frac{25}{4096}.$$

Now we prove (3.4). For A_5, B_5, A_6, and B_6 we verify (3.4) directly from their explicit numerical values. Namely,

$$A_5 = \frac{47\frac{1}{2}}{2^{14}},\ B_5 = \frac{49}{2^{14}} \ \Rightarrow A_5 - B_5 = \frac{-\frac{3}{2}}{2^{14}} < 0$$
$$A_6 = \frac{803}{2^{21}},\ B_6 = \frac{882}{2^{21}} \ \Rightarrow A_6 - B_6 = \frac{-79}{2^{21}} < 0.$$

Therefore it is sufficient to prove

$$A_n < B_n \tag{3.5}$$

for all

$$n \geq 7. \tag{3.6}$$

Now the *explicit* formula for A_n is

$$A_n = a_{n-1} + a_{n-2} + a_{n-3} + \cdots + a_1 + a_0 \tag{3.7}$$

where

$$
\begin{aligned}
a_{n-1} &:= \frac{1}{2n-3} \cdot \frac{1}{16^n} \binom{2n-2}{n-1} 3^{n-1} \\
a_{n-2} &:= \frac{1}{2n-5} \cdot \frac{1}{16^{n-1}} \binom{2n-4}{n-2} 3^{n-2} \left(\frac{-1}{2^5}\right) \\
\vdots &\ \vdots \\
a_1 &:= \frac{1}{2 \cdot 1 - 1} \frac{1}{16^2} \binom{2}{1} 3^1 \left(\frac{-1}{2^5}\right)^{n-2} \\
a_0 &:= \frac{4}{16} \left(\frac{-1}{2^5}\right)^{n-1}.
\end{aligned} \tag{3.8}
$$

Next we write

$$
A_n = a_{n-1} \left(1 + \frac{a_{n-2}}{a_{n-1}} + \frac{a_{n-3}}{a_{n-1}} + \frac{a_{n-4}}{a_{n-1}} + \cdots + \frac{a_1}{a_{n-1}} + \frac{a_0}{a_{n-1}}\right) \tag{3.9}
$$

and assert:

Claim 1. *The ratios* $\frac{a_{n-k-1}}{a_{n-k}}$ *increase monotonically in absolute value as* k *increases from* $k = 1$ *to* $k = n - 1$.

Proof. For $k = 1, \ldots, n-2$,

$$
\begin{aligned}
\left|\frac{a_{n-k-1}}{a_{n-k}}\right| &= \left(1 + \frac{2}{2n-2k-3}\right)\left(\frac{1}{2} + \frac{1}{4n-4k-2}\right)\frac{1}{12} \\
&\le \frac{1}{6} \quad \text{(which is the worst case and occurs when } k = n-2\text{)} \\
&< 1
\end{aligned}
$$

For $k = n - 1$,

$$
\left|\frac{a_0}{a_1}\right| = \frac{1}{3} < 1.
$$

This completes the proof. □

Claim 2. *The ratios* $\dfrac{a_{n-k-1}}{a_{n-k}}$*alternate in sign.*

Proof. This is a consequence of the definition of the a_k. □

By Claim 1 and Claim 2 we can write (3.9) in the form

$$
\begin{aligned}
A_n &= a_{n-1}\,(1 - \text{something positive and smaller than } 1) \\
&< a_{n-1}.
\end{aligned}
$$

Therefore, to prove (3.8) for $n \ge 7$, it suffices to prove that

$$
a_{n-1} < B_n \tag{3.10}
$$

for all $n \ge 7$.

By (3.8) and the definition of B_n, this last afirmation is equivalent to proving

$$
\frac{1}{2n-3} \cdot \frac{1}{16^n} \binom{2n-2}{n-1} 3^{n-1} < \left\{\frac{1}{2n-1} \cdot \frac{1}{4^n} \binom{2n}{n}\right\}^2,
$$

which, after some algebra, reduces to proving the implication

$$
n \ge 7 \Rightarrow \frac{\frac{n}{2} \cdot \frac{2n-1}{2n-3}}{\binom{2n}{n}} \cdot 3^{n-1} < 1.
$$

If we define for all integers $n \geq 7$

$$f(n) := \frac{\frac{n}{2} \cdot \frac{2n-1}{2n-3}}{\binom{2n}{n}} \cdot 3^{n-1} \tag{3.11}$$

then the affirmation (3.10) turns out to be equivalent to

$$n \geq 7 \Rightarrow f(n) < 1 \tag{3.12}$$

This latter affirmation is a consequence of the following two conditions:

Condition 1. $f(7) < 1.$

Condition 2. $f(7) > f(8) > f(9) > \cdots > f(k) > f(k+1) > \cdots$

□

Proof of Condition 1. By direct numerical computation,

$$f(7) = \frac{1701}{1936} < 1.$$

□

Proof of Condition 2. We must show that

$$k \geq 7 \Rightarrow f(k) > f(k+1).$$

If we define

$$g(k) := \frac{f(k)}{f(k+1)}, \tag{3.13}$$

then we must show that

$$k \geq 7 \Rightarrow g(k) > 1. \tag{3.14}$$

Using the definition (3.11) of $f(n)$ and the definition (3.14) of $g(n)$, and reducing algebraically we find

$$g(k) = \frac{2k}{6k-9}\left(\frac{2k-1}{k+1}\right)^2,$$

and we must show that

$$k \geq 7 \Rightarrow \frac{2k}{6k-9}\left(\frac{2k-1}{k+1}\right)^2 > 1. \tag{3.15}$$

Define the rational function of the real variable x:

$$g(x) := \frac{2x}{6x-9}\left(\frac{2x-1}{x+1}\right)^2. \tag{3.16}$$

Then the graph of $y = g(x)$ has a vertical asymptote at $x = \frac{3}{2}$ and

$$\lim_{x \to \frac{3}{2}^+} g(x) = +\infty. \tag{3.17}$$

Moreover, the derivative of $g(x)$ is given by:

$$g'(x) = \frac{2(2x^2 - 7x + 1)}{x(x+1)(2x-1)(2x+3)},$$

which implies that

$$g'(x)\begin{cases} <0 & \text{if } \frac{3}{2}<x<\frac{7+\sqrt{41}}{4}\ , \\ =0 & \text{if } x=\frac{7+\sqrt{41}}{4}, \\ >0 & \text{if } x>\frac{7+\sqrt{41}}{4}. \end{cases}$$

Therefore, for $x \geq \frac{3}{2}$, $g(x)$ *decreases* from "$+\infty$" at $x=\frac{3}{2}$ (see (3.17)) to an *absolute minimum value* (in $\frac{3}{2} \leq x < \infty$)

$$g\left(\frac{7+\sqrt{41}}{4}\right) = 1+\frac{37-\sqrt{41}}{399+69\sqrt{41}} = 1.0363895208\ldots$$

and then *increases monotonically* as $x \to \infty$ to its asymptotic limit $y=\frac{4}{3}$ and this is enough to complete the proof of the Fundamental Lemma. □

4. IVORY'S IDENTITY

In 1796, J. Ivory [5] published the following identity (in somewhat different notation):

Theorem 4.1 (Ivory's Identity)**.** *If* $0 \leq x \leq 1$ *then the following formula for* $\mathbf{B}(x)$ *is valid:*

$$\frac{1}{\pi}\int_0^{\pi}\sqrt{1+2\sqrt{x}\cos(2\phi)+x}\,d\phi = \sum_{n=0}^{\infty}\left\{\frac{1}{2n-1}\frac{1}{4^n}\binom{2n}{n}\right\}^2 x^n \equiv \mathbf{B}(x). \tag{4.1}$$

Proof. We sketch his elegant proof.

$$\begin{aligned}
&\frac{1}{\pi}\int_0^{\pi}\sqrt{1+2\sqrt{x}\cos(2\phi)+x}\,d\phi \\
&\quad= \frac{1}{\pi}\int_0^{\pi}\sqrt{1+\sqrt{x}e^{2i\phi}}\sqrt{1+\sqrt{x}e^{-2i\phi}}\,d\phi \\
&\quad= \frac{1}{\pi}\int_0^{\pi}\sum_{m=0}^{\infty}\left\{\frac{1}{2m-1}\cdot\frac{1}{4^m}\binom{2m}{m}(\sqrt{x})^m e^{2\pi i m\phi}\right\} \\
&\qquad\qquad \times\sum_{n=0}^{\infty}\left\{\frac{1}{2n-1}\cdot\frac{1}{4^n}\binom{2n}{n}(\sqrt{x})^n e^{-2\pi i n\phi}\right\}\,d\phi \\
&\quad= \frac{1}{\pi}\sum_{m=0}^{\infty}\left\{\frac{1}{2m-1}\cdot\frac{1}{4^m}\binom{2m}{m}(\sqrt{x})^m\right\} \\
&\qquad\qquad \times\sum_{n=0}^{\infty}\left\{\frac{1}{2n-1}\cdot\frac{1}{4^n}\binom{2n}{n}(\sqrt{x})^n\right\}\int_0^{\pi}e^{2\pi i(m-n)\phi}\,d\phi \\
&\quad= \sum_{n=0}^{\infty}\left\{\frac{1}{2n-1}\cdot\frac{1}{4^n}\binom{2n}{n}\right\}^2 x^n
\end{aligned}$$

□

We will need the following evaluation in our investigation of the accuracy of Ramanujan's approximation.

Corollary 4.2.

$$\mathbf{B}(1) = \frac{4}{\pi}. \tag{4.2}$$

Proof. By Ivory's identity,

$$\begin{aligned}\mathbf{B}(1) &= \frac{1}{\pi}\int_0^\pi \sqrt{1+2\sqrt{1}\cos(2\phi)+1}\,d\phi \\ &= \frac{1}{\pi}\int_0^\pi \sqrt{2+2\cos(2\phi)}\,d\phi \\ &= \frac{1}{\pi}\int_0^\pi \sqrt{4\cos^2(\phi)}\,d\phi \\ &= \frac{4}{\pi}.\end{aligned}$$

□

5. THE ACCURACY LEMMA

Theorem 5.1 (Accuracy Lemma). *For* $0 \le x \le 1$*, the function*

$$\mathbf{A}(x) := 1 + \frac{3x}{10+\sqrt{4-3x}} \tag{5.1}$$

underestimates the function

$$\mathbf{B}(x) := \sum_{n=0}^{\infty} \left\{ \frac{1}{2n-1}\frac{1}{4^n}\binom{2n}{n} \right\}^2 x^n \tag{5.2}$$

by a discrepancy, $\Delta(x)$ *which is never more than* $\left(\frac{4}{\pi}-\frac{14}{11}\right)x^5$ *and which is always more than* $\frac{3}{2^{17}}x^5$*:*

$$\frac{3}{2^{17}}x^5 < \Delta(x) \le \left(\frac{4}{\pi}-\frac{14}{11}\right)x^5. \tag{5.3}$$

Moreover, the constants $\left(\frac{4}{\pi}-\frac{14}{11}\right)$ *and* $\frac{3}{2^{17}}x^5$ *are the best possible.*

Proof. By the definition of $\mathbf{A}(x)$ and $\mathbf{B}(x)$ given in Theorem 1.1, the discrepancy $\Delta(x)$ is given by the series

$$\begin{aligned}\Delta(x) &:= \mathbf{B}(x) - \mathbf{A}(x) \\ &= (B_5 - A_5)x^5 + (B_6 - A_6)x^6 + \cdots \\ &:= \delta_5 x^5 + \delta_6 x^6 + \cdots ,\end{aligned}$$

where, again by Theorem 1.1,

$$\delta_k > 0 \quad \text{for } k = 5, 6, \dots .$$

On the one hand

$$\begin{aligned}\Delta(x) &= x^5(\delta_5 + \delta_6 x + \cdots) \\ &\le x^5(\delta_5 + \delta_6 + \delta_7 + \cdots) \\ &= x^5\Delta(1) \\ &= x^5\{\mathbf{B}(1) - \mathbf{A}(1)\} \\ &= x^5\left(\frac{4}{\pi}-\frac{14}{11}\right)\end{aligned}$$

where we used Corollary 1.2 of Ivory's identity in the last equality. Therefore

$$\Delta(x) \le \left(\frac{4}{\pi} - \frac{14}{11}\right) x^5.$$

This is half of the accuracy lemma. Moreover, the constant $\left(\frac{4}{\pi} - \frac{14}{11}\right)$ is assumed for $x = 1$ and thus cannot be replaced by anything smaller, i.e., it is the best possible constant.

On the other hand, we can write

$$\Delta(x) = x^5\{\delta_5 + G(x)\},$$

where

$$G(x) := \delta_6 x + \delta_7 x^2 + \cdots \Rightarrow \begin{cases} G(x) \ge 0 & \text{for all } 0 \le x \le 1 \ , \\ G(x) \to 0 & \text{as } x \to 0. \end{cases}$$

This shows that

$$\Delta(x) > \delta_5 x^5 = \frac{3}{2^{17}} x^5$$

and that

$$\lim_{x\to 0} \frac{\Delta(x)}{x^5} = \frac{3}{2^{17}}.$$

This proves both the other inequality in the theorem and the optimality of the constant $\delta_5 = \frac{3}{2^{17}}$, i.e., that it cannot be replaced by any larger constant.

This completes the proof of the Accuracy Lemma. □

6. THE ACCURACY OF RAMANUJAN'S APPROXIMATION

Now we can achieve the main goal of this paper, namely to prove *Ramanujan's Approximation Theorem.*

First we express the perimeter of an ellipse and Ramanujan's approximative perimeter in terms of the functions $\mathbf{A}(x)$ and $\mathbf{B}(x)$.

Theorem 6.1. *If p is the perimeter of an ellipse with semimajor axes a and b, and if p_R is Ramanujan's approximative perimeter, then:*

$$\begin{aligned} p &= \pi(a+b) \cdot \mathbf{B}\left\{\left(\frac{a-b}{a+b}\right)^2\right\} \\ p_R &= \pi(a+b) \cdot \mathbf{A}\left\{\left(\frac{a-b}{a+b}\right)^2\right\}. \end{aligned} \tag{6.1}$$

Proof. We begin with *Ivory's Identity* (§4) and in it we substitute $x := \left(\frac{a-b}{a+b}\right)^2$. Then the integral becomes

$$\frac{1}{\pi}\int_0^{\pi} \sqrt{1 + 2\sqrt{\left(\frac{a-b}{a+b}\right)^2}\cos(2\phi) + \left(\frac{a-b}{a+b}\right)^2}\, d\phi$$
$$= \frac{4}{\pi(a+b)}\int_0^{\frac{\pi}{2}} (a^2\sin^2\phi + b^2\cos^2\phi)\, d\phi$$

and therefore

$$\mathbf{B}\left\{\left(\frac{a-b}{a+b}\right)^2\right\} = \frac{4}{\pi(a+b)}\int_0^{\frac{\pi}{2}} (a^2\sin^2\phi + b^2\cos^2\phi)\, d\phi.$$

But, it is well known (Berndt [4]) that the perimeter, p, of an ellipse with semi-axes a and b is given by

$$p = 4 \int_0^{\frac{\pi}{2}} (a^2 \sin^2 \phi + b^2 \cos^2 \phi)\, d\phi,$$

and thus

$$p = \pi(a+b) \cdot \mathbf{B} \left\{ \left(\frac{a-b}{a+b} \right)^2 \right\}. \tag{6.2}$$

Moreover, some algebra shows us that

$$\begin{aligned} \mathbf{A} \left\{ \left(\frac{a-b}{a+b} \right)^2 \right\} &= 1 + \frac{3 \left(\frac{a-b}{a+b}\right)^2}{10 + \sqrt{4 - 3 \left(\frac{a-b}{a+b}\right)^2}} \\ &= \frac{1}{a+b} \left\{ (a+b) + \frac{3(a-b)^2}{10(a+b) + \sqrt{a^2 + 14ab + b^2}} \right\} \end{aligned}$$

and we conclude that Ramanujan's approximative formula, p_R is given by

$$p_R = \pi(a+b) \mathbf{A} \left\{ \left(\frac{a-b}{a+b} \right)^2 \right\}. \tag{6.3}$$

□

The formula for p above was the object of Ivory's original paper [5].

Now we complete the proof of Theorem 1.1.

Proof. Writing

$$\lambda := \frac{a-b}{a+b},$$

and using the notation of the statement of Theorem 1.1. we conclude that

$$\begin{aligned} \epsilon &:= \pi(a+b) \cdot \theta(\lambda) \cdot \lambda^{10} \\ &= \pi(a+b) \cdot \frac{\Delta(\lambda^2)}{\lambda^{10}} \cdot \lambda^{10}, \end{aligned}$$

where

$$\theta(\lambda) \equiv \frac{\Delta(\lambda^2)}{\lambda^{10}} = \delta_5 + \delta_6 \lambda^2 + \cdots . \tag{6.4}$$

Now we apply the Accuracy Lemma and the proof is complete. □

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