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A NOTE ON GLOBAL IMPLICIT FUNCTION THEOREM

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ABSTRACT. We study the boundary behaviour of some certain maximal implicit function. We give estimates of the maximal balls on which some implicit functions are defined and we consider some cases when the implicit function is globally defined. We extend in this way an earlier result from [3] concerning an inequality satisfied by the partial derivatives $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ of the map h which verifies the global implicit function problem

$$h\left(t,x\right)=h\left(a,b\right),\quad x\left(a\right)=b.$$

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The implicit function theorem is a classical result in mathematical analysis. Local versions can be found in [1],[8], [10], [13], [15], [17], [18] and some papers deal with some global versions (see [2], [3], [9], [16]).

We first give some local versions of the implicit function theorem, using our local homeomorphism theorem from [4].

Theorem 1. *Let E, F be Banach spaces, $\dim F < \infty$, $U \subset E$ open, $V \subset F$ open, $h : U \times V \to F$ continuous such that there exists $K \subset U \times V$ countable such that h is differentiable on $(U \times V) \setminus K$ and $\frac{\partial h}{\partial y}(x,y) \in \operatorname{Isom}(F,F)$ for every $(x,y) \in (U \times V) \setminus K$ and let $A = \operatorname{Pr}_1 K \subset U$. Then, for every $(a,b) \in U \times V$ there exists $r, \delta > 0$ and a unique continuous map $\varphi : B(a,r) \to B(b,\delta)$ such that*

$$\varphi\left(a\right)=b \text{ and } h\left(x,\varphi\left(x\right)\right)=h\left(a,b\right)\ \text{ for every } x\in B\left(a,r\right)$$

and φ is differentiable on $B(a,r) \setminus A$.

Proof. Let $(a,b) \in U \times V$ be fixed and $f : U \times V \to E \times F$ be defined by

$$f\left(x,y\right)=\left(x,h\left(x,y\right)+b-h\left(a,b\right)\right)\ \text{ for }\ (x,y)\in U\times V.$$

128-06

Also, let $T : U \times V \to E \times F$,

$$T(x, y) = (0, y - h(x, y) + h(a, b) - b) \text{ for } (x, y) \in U \times V.$$

Then $\mathrm{Im}T \subset F$, hence T is compact and we see that $f = I - T$, f is differentiable on $(U \times V) \setminus K$ and $f'(x, y) \in \mathrm{Isom}(E \times F, E \times F)$ for every $(x, y) \in (U \times V) \setminus K$. Using the local inversion theorem from [4], we see that f is a local homeomorphism on $U \times V$. Let $W \in \mathcal{V}((a, b))$ and $\delta > 0$ be such that

$$f|B(a, \delta) \times B(b, \delta) : B(a, \delta) \times B(b, \delta) \to W$$

is a homeomorphism and let

$$g = (g_1, g_2) : W \to B(a, \delta) \times B(b, \delta)$$

be its inverse. We take $\ell > 0$ such that $Q = B(a, b) \times (b, \ell) \subset W$ and let $r = \min\{\ell, \delta\}$. We have

$$\begin{aligned}(x, z) &= f(g(x, z)) \\ &= f(g_1(x, z), g_2(x, z)) \\ &= (g_1(x, z), h(g_1(x, z), g_2(x, z)) + b - h(a, b))\end{aligned}$$

for every $(x, z) \in Q$, hence

$$x = g_1(x, z), \quad h(x, g_2(x, z)) = z + h(a, b) - b$$

for $x \in B(a, r),\ z \in B(b, \ell)$.

We define now $\varphi : B(a, r) \to B(b, \delta)$ by $\varphi(x) = g_2(x, b)$ for every $x \in B(a, r)$ and we see that $h(x, \varphi(x)) = h(a, b)$ for every $x \in B(a, r)$. We have $f(a, b) = (a, b) = f(a, \varphi(a))$ and using the injectivity of f on $B(a, \delta) \times B(b, \delta)$, we see that $\varphi(a) = b$. Also, if $\psi : B(a, r) \to B(b, \delta)$ is continuous and $\psi(a) = b, h(x, \psi(x)) = h(a, b)$ for every $x \in B(a, r)$, then $f(x, \varphi(x)) = (x, b) = f(x, \Psi(x))$ for every $x \in B(a, r)$ and using again the injectivity of the map f on $B(a, \delta) \times B(b, \delta)$, we find that $\varphi = \psi$ on $B(a, r)$.

Let now $x_0 \in B(a, r) \setminus A$. Then $(x_0, b) = f(x_0, \beta)$, with $(x_0, \beta) \in (B(a, r) \times B(b, \delta) \setminus K)$, hence f is differentiable in (x_0, β), $f'(x_0, \beta) \in \mathrm{Isom}(E \times F, E \times F)$ and since f is a homeomorphism on $B(a, r) \times B(b, \delta)$, it results that g is also differentiable in $(x_0, b) = f(x_0, \beta)$ and $g'(x_0, b) = [f'(x_0, \beta)^{-1}]$, and we see that φ is differentiable in x_0. □

Theorem 2. *Let E be an infinite dimensional Banach space, $\dim F < \infty$, $U \subset E$, $V \subset F$ be open sets, $h : U \times V \to F$ be continuous such that there exists $K \subset U \times V$,*

$$K = \bigcup_{p=1}^{\infty} K_p$$

with K_p compact sets for $p \in \mathbb{N}$ such that h is differentiable on $(U \times V) \setminus K$, there exists $\frac{\partial h}{\partial y}$ on $U \times V$ and $\frac{\partial h}{\partial y}(x, y) \in \mathrm{Isom}(F, F)$ for every $(x, y) \in U \times V$ and let $A = P_{r_1} K$. Then, for every $(a, b) \in U \times V$ there exists $r, \delta > 0$ and a unique continuous implicit function $\varphi : B(a, r) \to B(b, \delta)$ differentiable on $B(a, r) \setminus A$ such that $\varphi(a) = b$ and $h(x, \varphi(x)) = h(a, b)$ for every $x \in B(a, r)$.

Proof. We apply Theorem 11 of [8]. We see that in an infinite dimensional Banach space E, a set K which is a countable union of compact sets is a "thin" set, i.e., $\mathrm{int}\, K = \phi$ and $B \setminus K$ is connected and simply connected for every ball B from E. Also, since A is a countable union of compact sets, we see that $\mathrm{int}\, A = \phi$.

If E, F are Banach spaces and $A \in \mathcal{L}(E, F)$, we let

$$\|A\| = \sup_{\|x\|=1} \|A(x)\|$$

and

$$\ell(A) = \inf_{\|x\|=1} \|A(x)\|$$

and if $D \subset E,\ \lambda > 0$, we let

$$\lambda D = \{x \in E |\text{ there exists } y \in D \text{ such that } x = \lambda y\}.$$

If X, Y are Banach spaces, $D \subset X$ is open, $x \in D$ and $f : D \to Y$ is a map, we let

$$D^{+} f(x) = \limsup_{y \to x} \frac{\|f(y) - f(x)\|}{\|y - x\|}$$

and we say that f is a light map if for every $x \in D$ and every $U \in \mathcal{V}(x)$, there exists $Q \in \mathcal{V}(x)$ such that $\overline{Q} \subset U$ and $f(x) \notin f(\partial Q)$. □

Remark 3. We can replace in Theorem 1 and Theorem 2 the condition "$\dim F < \infty$" by "There exists $\frac{\partial h}{\partial y}$ on $U \times V$ and it is continuous on $U \times V$ and $\frac{\partial h}{\partial y}(x, y) \in \operatorname{Isom}(F, F)$ for every $(x, y) \in U \times V$" to obtain the same conclusion, and this is the classical implicit function theorem. Also, keeping the notations from Theorem 1 and Theorem 2, we see that if $(\alpha, \beta) \in B(a, r) \times B(b, \delta)$ is such that $h(\alpha, \beta) = h(a, b)$, then $\beta = \varphi(\alpha)$.

We shall use the following lemma from [7].

Lemma 4. *Let $a > 0,\ f : [0, a] \to [0, \infty)$ be continuous and let $\omega : [0, \infty) \to [0, \infty)$ be continuous such that $\omega > 0$ on $(0, \infty)$ and*

$$|f(b) - f(c)| \le \int_b^c \omega(f(t))\, dt \text{ for every } 0 < b < c \le a.$$

Then, if

$$m = \inf_{t \in [0,a]} f(t), \quad M = \sup_{t \in [0,a]} f(t),$$

it results that

$$\int_m^M \frac{ds}{\omega(s)} \le a.$$

We obtain now the following characterization of the boundary behaviour of the solutions of some differential inequalities.

Theorem 5. *Let E, F be Banach spaces, $U \subset E$ a domain, $K \subset U$ at most countable, $\varphi : U \to F$ continuous on U and differentiable on $U \backslash K$ such that there exists $\omega : [0, \infty) \to [0, \infty)$ continuous with $\|\varphi'(x)\| \le \omega(\|\varphi(x)\|)$ for every $x \in U \backslash K$. Then, if $\alpha \in \partial U$ and $C \subset U$ is convex such that $\alpha \in \overline{C}$, either there exists*

$$\lim_{\substack{x \to \alpha \\ x \in C}} \varphi(x) = \ell \in F \text{ or } \lim_{\substack{x \to \alpha \\ x \in C}} \|\varphi(x)\| = \infty$$

or, if $\omega > 0$ on $(1, \infty)$ and

$$\int_1^\infty \frac{ds}{\omega(s)} = \infty,$$

there exists

$$\lim_{\substack{x \to \alpha \\ x \in C}} \varphi(x) = \ell \in F.$$

If $\omega > 0$ *on* $(0,1)$ *and*

$$\int_0^1 \frac{ds}{\omega(s)} = \infty$$

and there exists $\alpha \in U$ *such that* $\varphi(\alpha) = 0$, *it results that* $\varphi(x) = 0$, *for every* $x \in U$.

Proof. Replacing, if necessary, ω by $\omega + \lambda$ for some $\lambda > 0$, we can suppose that $\omega > 0$ on $[0, \infty)$. Let $\alpha \in \partial U,\ C \subset U$ convex such that $\alpha \in \overline{C}$ and let $q : [0,1) \to C$ be a path such that

$$\lim_{t \to 1} q(t) = \alpha$$

and there exists $L > 0$ such that $D_q^+(t) \le L$ for every $t \in [0,1)$. Then

$$\|q(s) - q(t)\| \le L \cdot (s - t)$$

for every $0 \le t < s < 1$ and let $0 \le c < d < 1$ be fixed,

$$A = \operatorname{co}(q([c,d]))$$

and $\varepsilon > 0$. Let $g : A \to \mathbb{R}$ be defined by

$$g(z) = \omega(\|\varphi(z)\|) \text{ for every } z \in A.$$

Then A is compact and convex and g is uniformly continuous on A, hence we can find $\delta'_\varepsilon > 0$ such that

$$|g(z_1) - g(z_2)| \le \varepsilon \text{ for } z_1, z_2 \in A$$

with $\|z_1 - z_2\| \le \delta'_\varepsilon$. Since $q : [c,d] \to C$ is uniformly continuous, we can find $\delta_\varepsilon > 0$ such that $\|q(t) - q(s)\| \le \delta'_\varepsilon$ if $s, t \in [c,d]$ are such that $|s - t| \le \delta_\varepsilon$. Let now

$$\Delta = (c = t_0 < t_1 < \cdots < t_m = d) \in \mathcal{D}([c,d])$$

be such that $\|\Delta\| \le \delta_\varepsilon$. Using Denjoi-Bourbaki's theorem we have

$$\begin{aligned}
|\|\varphi(q(d))\| - \|\varphi(q(c))\|| &\le \|\varphi(q(d)) - \varphi(q(c))\| \\
&\le \left\| \sum_{k=0}^{m-1} \varphi(q(t_{k+1})) - \varphi(q(t_k)) \right\| \\
&\le \sum_{k=0}^{m-1} \|(q(t_{k+1}) - q(t_k))\| \cdot \sup_{z \in [q(t_k), q(t_{k+1})] \setminus K} \|\varphi'(z)\| \\
&\le L \cdot \sum_{k=0}^{m-1} (t_{l+1} - t_k) \cdot \sup_{z \in [q(t_k), q(t_{k+1})]} \omega(\|\varphi(z)\|) \\
&\le L \cdot \sum_{k=0}^{m-1} (t_{l+1} - t_k) \cdot (\omega(\|\varphi(q(t_k)\| + \varepsilon).
\end{aligned}$$

Letting $\|\Delta\| \to 0$ and then $\varepsilon \to 0$, we obtain

$$\begin{aligned}
|\ \|\varphi(q(d))\| - \|\varphi(q(c))\|| &\le \|\varphi(q(d)) - \varphi(q(c))\| \\
&\le L \cdot \int_c^d \omega(\|\varphi(q(t))\|)\, dt \text{ for } 0 \le c < d < 1. \qquad (1)
\end{aligned}$$

If

$$m = \inf_{t \in [0,1)} \|\varphi(q(t))\|, \quad M = \sup_{t \in [0,1)} \|\varphi(q(t))\|,$$

we obtain from Lemma 4 and (1)

$$\int_m^M \frac{ds}{\omega(s)} \leq L. \tag{2}$$

Let now $z_p \to \alpha$ be such that

$$\|z_p - \alpha\| \leq \frac{1}{2^P}, \quad z_p \in C \text{ for } p \in \mathbb{N}$$

and suppose that there exists $\rho > 0$ such that $\|\varphi(z_p)\| \leq \rho$ for every $p \in \mathbb{N}$. We take

$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots < 1$$

such that $t_k \nearrow 1$ and we define $q : [0, 1) \to C$ by

$$q(t) = \frac{z_k(t_{k+1} - t) + z_{k+1}(t - t_k)}{t_{k+1} - t_k} \text{ for } t \in [t_k, t_{k+1}], \quad k \in \mathbb{N}.$$

Then

$$D^+ q(t) = c_k = \frac{\|z_{k+1} - z_k\|}{t_{k+1} - t_k} \text{ for } t \in [t_k, t_{k+1}]$$

and taking $t_k = \frac{k}{k+1}$ for $k \in \mathbb{N}$, we see that $c_k \to 0$. Then

$$\alpha_p = \sup_{t \in [t_p, 1)} D^+ q(t) = \sup_{k \geq p} c_k \to 0$$

and let

$$a_p = \inf_{t \in [t_p, 1)} \|\varphi(q(t))\|, \quad b_p = \sup_{t \in [t_p, 1)} \|\varphi(q(t))\| \text{ for } p \in \mathbb{N}.$$

Using (2) we obtain that

$$\int_{a_p}^{b_p} \frac{ds}{\omega(s)} \leq \alpha_p \text{ for } p \in \mathbb{N}$$

and let $p_0 \in \mathbb{N}$ be such that

$$\alpha_p < \int_\rho^\infty \frac{ds}{\omega(s)} \text{ for } p \geq p_0.$$

Suppose that there exists $p \geq p_0$ such that $b_p = \infty$. Then, since $q(t_k) = z_k$, we see that

$$a_k \leq \|\varphi(q(t_k))\| = \|\varphi(z_k)\| \leq \rho \text{ for } k \in \mathbb{N},$$

hence

$$0 < \int_\rho^\infty \frac{ds}{\omega(s)} \leq \int_{a_p}^{b_p} \frac{ds}{\omega(s)} \leq \alpha_p < \int_\rho^\infty \frac{ds}{\omega(s)}$$

and we have reached a contradiction.

It results that $b_p < \infty$ for $p \geq p_0$ and let

$$K_p = \sup_{t \in [0, b_p]} \omega(t)$$

for $p \geq p_0$. Then $K_p < \infty$ and we see from (1) that

$$\|\varphi(q(d)) - \varphi(q(c))\| \leq \alpha_p \cdot K_p \cdot |d - c|$$

for $t_p \leq c < d < 1$ and $p \geq p_0$, and this implies that

$$\lim_{t \to 1} \varphi(q(t)) = \ell \in F.$$

It results that

$$\lim_{p \to \infty} \varphi(z_p) = \lim_{p \to \infty} \varphi(q(t_p)) = \ell \in F.$$

Now, if the case

$$\lim_{\substack{x \to \alpha \\ x \in C}} \|\varphi(x)\| = \infty$$

does not hold, there exists $\rho > 0$ and $x_p \to \alpha,\ x_p \in C$ with

$$||\varphi(x_p)|| \le \rho \text{ and } ||x_p - \alpha|| \le \frac{1}{2^p}$$

for every $p \in \mathbb{N}$, and from what we have proved before, it results that

$$\lim_{p \to \infty} \varphi(x_p) = \ell \in F.$$

If $a_p \in C,\ \|a_p - \alpha\| \le \frac{1}{2^p}$ for every $p \in \mathbb{N}$, then

$$\lim_{p \to \infty} \varphi(a_p) = \ell_1 \in F.$$

Let $z_{2p} = x_p,\ z_{2p+1} = a_p$ for $p \in \mathbb{N}$. We see that

$$\lim_{p \to \infty} \varphi(z_p) = \ell_2 \in F,$$

hence

$$\ell = \lim_{p \to \infty} \varphi(x_p) = \lim_{p \to \infty} \varphi(z_{2p}) = \ell_2$$

and

$$\ell_1 = \lim_{p \to \infty} \varphi(a_p) = \lim_{p \to \infty} \varphi(z_{2p+1}) = \ell_2,$$

hence $\ell = \ell_1 = \ell_2$. We have proved that if $a_p \in C,\ \|a_p - \alpha\| \le \frac{1}{2^p}$ for every $p \in \mathbb{N}$, then

$$\lim_{p \to \infty} \varphi(a_p) = \ell.$$

We show now that if $a_p \in C, a_p \to \alpha$, then

$$\lim_{p \to \infty} \varphi(a_p) = \ell.$$

Indeed, if this is false, there exists $\varepsilon > 0$ and $(a_{p_k})_{k \in \mathbb{N}}$ such that

$$\|\varphi(a_{p_k}) - \ell\| > \varepsilon$$

for every $k \in \mathbb{N}$. Let $\left(a_{p_{k_q}}\right)_{q \in \mathbb{N}}$ be a subsequence such that

$$\left\|a_{p_{k_q}} - \alpha\right\| < \frac{1}{2^q}$$

for every $q \in \mathbb{N}$. From what we have proved before it results that

$$\lim_{q \to \infty} \varphi\left(a_{p_{k_q}}\right) = \ell$$

and we have reached a contradiction.

We have therefore proved that either

$$\lim_{\substack{x \to \alpha \\ x \in C}} \varphi(x) = \ell \in F,$$

or

$$\lim_{\substack{x \to \alpha \\ x \in C}} \|\varphi(x)\| = \infty.$$

Suppose now

$$\int_1^\infty \frac{ds}{\omega(s)} = \infty$$

and let $\alpha \in \partial U$. We take $x \in C$ and let $q : [0,1) \to C$ be defined by

$$q(t) = (1-t)x + t\alpha$$

for $t \in [0,1)$ and

$$m = \inf_{t\in[0,1)} \|\varphi(q(t))\|, \quad M = \sup_{t\in[0,1)} \|\varphi(q(t))\|.$$

Since $D^+q(t) = \|x-\alpha\|$ for every $t \in (0,1)$, we see from (2)

$$\int_{\|x\|}^{M} \frac{ds}{\omega(s)} \le \int_{m}^{M} \frac{ds}{\omega(s)} \le \|x-\alpha\|$$

and this implies that $M < \infty$. Let

$$b = \sup_{t\in[0,M]} \omega(t).$$

Then $b < \infty$ and using (1), we see that

$$\|\varphi(q(d)) - \varphi(q(c))\| \le b \cdot \|x-\alpha\| \cdot |d-c|$$

for $0 \le c < d < 1$ and this implies that

$$\lim_{\substack{x\to\alpha \\ x\in\mathbb{C}}} \varphi(x) = \ell \in F.$$

It results that the case

$$\lim_{\substack{x\to\alpha \\ x\in\mathbb{C}}} \|\varphi(x)\| = \infty$$

cannot hold, hence

$$\lim_{x\to\alpha} \varphi(x) = \ell \in F.$$

Suppose now that

$$\int_0^1 \frac{ds}{\omega(s)} = \infty$$

and that there exists $\alpha \in U$ such that $\varphi(\alpha) = 0$. Let $r = d(\alpha, \partial U)$, $y \in B(\alpha, r)$ and $q : [0,1] \to B(\alpha, r)$, $q(t) = (1-t)\alpha + ty$ for $t \in [0,1]$. Then $D^+q(t) = \|y-\alpha\|$ for $t \in [0,1]$ and let

$$m = \inf_{t\in[0,1]} \|\varphi(q(t))\|$$

and

$$M = \sup_{t\in[0,1]} \|\varphi(q(t))\|.$$

Then $m = 0$ and we see from (2) that

$$\int_0^M \frac{ds}{\omega(s)} \le \|y-\alpha\|.$$

This implies that $M = 0$ and hence $\varphi(y) = 0$. We proved that $\varphi \equiv 0$ on $B(\alpha, r)$ and since U is a domain, we see that $\varphi \equiv 0$ on U. □

Remark 6. We proved that if φ is as in Theorem 2 and

$$\int_1^\infty \frac{ds}{\omega(s)} = \infty,$$

then φ has angular limits in every point $\alpha \in \partial U$.

We now obtain the following characterization of the boundary behaviour of some implicit function.

Theorem 7. *Let E, F be Banach spaces, $U \subset E$ a domain, $K \subset U \times F$ such that $A = \Pr_1 K$ is at most countable and let $h : U \times F \to F$ be continuous on $U \times F$, differentiable on $(U \times F) \backslash K$ such that*

$$\ell\left(\frac{\partial h}{\partial y}(x,y)\right) > 0 \text{ on } (U \times F) \backslash K$$

and there exists $\omega : [0,\infty) \to [0,\infty)$ continuous such that

$$\left\| \frac{\partial h}{\partial x}(x,y) \right\| \Big/ \ell\left(\frac{\partial h}{\partial y}(x,y)\right) \leq \omega(\|y\|)$$

for every $(x,y) \in (U \times F) \backslash K$. Suppose that $\varphi : U \to F$ is continuous on U, differentiable on $U \backslash A$, $\varphi(a) = b$ and

$$h(x,\varphi(x)) = h(a,b)$$

for every $x \in U$.

Then, if $\alpha \in \partial U$ and $C \subset U$ is convex such that $\alpha \in \overline{C}$, either

$$\lim_{\substack{x\to\alpha \\ x\in C}} \|\varphi(x)\| = \ell \in F,$$

or

$$\lim_{\substack{x\to\alpha \\ x\in C}} \|\varphi(x)\| = \infty.$$

Also, if $\omega > 0$ on $(1,\infty)$ and

$$\int_1^\infty \frac{ds}{\omega(s)} = \infty,$$

then

$$\lim_{\substack{x\to\alpha \\ x\in C}} \varphi(x) = \ell \in F.$$

Proof. We see that if $x \in U \setminus A$, then $(x,\varphi(x)) \in (U \times F) \setminus K$, hence h is differentiable in $(x,\varphi(x))$ and we have

$$\frac{\partial h}{\partial x}(x,\varphi(x)) + \frac{\partial h}{\partial y}(x,\varphi(x)) \cdot \varphi'(x) = 0$$

and we see that

$$\|\varphi'(x)\| \cdot \ell\left(\frac{\partial h}{\partial y}(x,\varphi(x))\right) \leq \left\| \frac{\partial h}{\partial y}(x,\varphi(x))(\varphi'(x)) \right\| = \left\| \frac{\partial h}{\partial x}(x,\varphi(x)) \right\|.$$

It results that

$$\|\varphi'(x)\| \leq \left\| \frac{\partial h}{\partial x}(x,\varphi(x)) \right\| \Big/ \ell\left(\frac{\partial h}{\partial y}(x,\varphi(x))\right) \leq \omega(\|\varphi(x)\|)$$

for every $x \in U \setminus A$ and we now apply Theorem 5. □

Remark 8. If E is an infinite dimensional Banach space and

$$K = \bigcup_{p=1}^{\infty} K_p \text{ with } K_p \subset E$$

are compact sets for every $p \in \mathbb{N}$ and $y \in E$, then the set

$$M(K,y) = \{w \in E |\text{ there exists } t > 0 \text{ and } x \in K \text{ such that } w = tx\}$$

is also a countable union of compact sets and hence a "thin" set. Keeping the notations from Theorem 5, we see that the basic inequality

$$\|\varphi(z_1)-\varphi(z_2)\| \le \sup_{z\in[z_1,z_2]} \omega(\|\varphi(z)\|) \qquad \text{if } [z_1,z_2]\subset U \tag{3}$$

is also valid for K a countable union of compact sets and φ as in Theorem 5.

If $\dim E = n$ and $K\subset E$ has a σ-finite $(n-1)$-dimensional measure (i.e. $K=\bigcup_{p=1}^{\infty} K_p$, with $m_{n-1}(K_p)<\infty$ for every $p\in\mathbb{N}$, where m_q is the q-Hausdorff measure from $\mathbb{R}^n$), a theorem of Gross shows that if $H\subset E$ is a hyperplane and $P:E\to H$ is the projection on H, then $P^{-1}(z)\cap K$ is at most countable with the possible exception of a set $B\subset H$, with $m_{n-1}(B)=0$. Applying as in Theorem 5 the theorem of Denjoi and Bourbaki on each interval from $P^{-1}(z)\cap K$ for every $z\in H\setminus B$ and using a natural limiting process, we see that if $\dim E=n$ and $K\subset E$ has a σ-finite $(n-1)$-dimensional measure, then the inequality (3) also holds. It is easy see now that Theorem 5 and Theorem 7 hold if the set K, respectively the set $A=\mathrm{Pr}_1 K$ are chosen to be a countable union of compact sets if $\dim E=\infty$ and having σ-finite $(n-1)$-dimensional measure if $\dim E=n$.

The following theorem is the main theorem of the paper and it gives some cases when the implicit function is globally defined or some estimates of the maximal balls on which some implicit function is defined.

We say that a domain D from a Banach space is starlike with respect to the point $a\in D$ if $[a,x]\subset D$ for every $x\in D$, and if D is a domain in the Banach space E and $a\in D$. We set

$$D_a=\{x\in D|\,[a,x]\subset D\}.$$

Theorem 9. *Let E,F be Banach spaces, $\dim F<\infty, D\subset E$ a domain, $K\subset D\times F$ at most countable, and $A=\mathrm{Pr}_1 K$.Also, let $h:D\times F\to F$ be continuous on $D\times F$ and differentiable on $(D\times F)\setminus K$ such that*

$$\ell\left(\frac{\partial h}{\partial y}(x,y)\right)>0 \text{ on } (D\times F)\setminus K.$$

In addition, there exists $\omega:[0,\infty)\to[0,\infty)$ continuous such that $\omega>0$ on $(0,\infty)$ and

$$\left\|\frac{\partial h}{\partial x}(x,y)\right\| \Big/ \ell\left(\frac{\partial h}{\partial y}(x,y)\right)\le \omega(\|y\|)$$

for every $(x,y)\in(D\times F)\setminus K$. Then, if $(a,b)\in D\times F$ and

$$Q_{a,b}=D_a\cap B\left(a,\int_{\|b\|}^{\infty}\frac{ds}{\omega(s)}\right),$$

there exists a unique continuous map $\varphi:Q_{a,b}\to F$, differentiable on $Q_{a,b}\setminus A$ such that $h(x,\varphi(x))=h(a,b)$ for every $x\in Q_{a,b}$. If D is starlike with respect to a and

$$\int_1^{\infty}\frac{ds}{\omega(s)}=\infty,$$

then $Q_{a,b}=D$ and $\varphi:D\to F$ is globally defined on D.

Proof. Let $z\in Q_{a,b}$ and let $B=\{x\in[a,z]|$ there exists an open, convex domain $D_x\subset Q_{a,b}$ such that $[a,x]\subset D_x\}$ and a continuous implicit function $\varphi_x:D_x\to F$, differentiable on $D_x\setminus A$ such that $\varphi_x(a)=b$ and $h(u,\varphi_x(u))=h(a,b)$ for every $u\in D_x$. We see that B is open, and from Theorem 1, $B \neq \emptyset$. We show that B is a closed set. Let $x_p \in B,\ x_p \to x$, and we can suppose that $x_p \in [a, x)$ for every $p \in \mathbb{N}$, and let

$$Q = \bigcup_{p=1}^{\infty} D_{x_p}.$$

We define $\Psi : Q \to F$ by $\Psi(u) = \varphi_{x_p}(u)$ for $u \in D_{x_p}$ and $p \in \mathbb{N}$ and the definition is correct. Indeed, if $p, q \in \mathbb{N}, p \neq q$ let $U_{p_q} = D_{x_p} \cap D_{x_q}$. Then U_{pq} is a nonempty, open and convex set, hence it is a nonempty domain. If

$$V_{pq} = \left\{u \in U_{pq} | \varphi_{x_q}(u) = \varphi_{x_p}(u)\right\},$$

we see that $a \in V_{pq}$, hence $V_{pq} \neq \emptyset$ and we see that V_{pq} is a closed set in U_{pq}. Using the property of the local unicity of the implicit function from Theorem 1, we obtain that V_{pq} is also an open set. Since U_{pq} is a domain, it results that $U_{pq} = V_{pq}$ and hence that Ψ is correctly defined. We also see immediately that $\Psi(a) = b$, and Ψ is continuous on Q and differentiable on $Q \setminus A$.

Let $q : [0, 1] \to Q$ be defined by

$$q(t) = (1 - t)\, a + tx \text{ for } t \in [0, 1].$$

Then

$$D^+ q(t) = \|x - a\| < \int_{||b|}^{\infty} \frac{ds}{\omega(s)}.$$

Let

$$m = \inf_{t\in[0,1)} \|\Psi(q(t))\|, \quad M = \sup_{t\in[0,1)} \|\Psi(q(t))\|.$$

As in Theorem 7, we see that

$$\|\Psi'(u)\| \leq \omega(\|\Psi(u)\|) \text{ for every } u \in Q \setminus A,$$

hence

$$\|\Psi(z_1) - \Psi(z_2)\| \leq \sup_{z\in[z_1,z_2]} \omega(\|\Psi(z)\|)$$

if $[z_1, z_2] \subset Q$. This implies that relations (1) and (2) from Theorem 5 also hold and we see that

$$\int_m^M \frac{ds}{\omega(s)} \leq \|x - a\|.$$

Then

$$\int_{\|b\|}^M \frac{ds}{\omega(s)} \leq \int_m^M \frac{ds}{\omega(s)} \leq \|x - a\|$$

and this implies that $M < \infty$, hence

$$\ell = \sup_{t\in[0,M]} \omega(t) < \infty.$$

Using (1) and Theorem 5, we see that

$$\|\Psi(q(d)) - \Psi(q(c))\| \leq \ell \cdot \|x - a\| \cdot |d - c| \text{ for every } 0 \leq c < d < 1$$

and this implies that

$$\lim_{\substack{u\to x \\ u\in \operatorname{Im} q}} \Psi(u) = w \in F.$$

Using Theorem 1, we can find $r, \delta > 0$ and a unique continuous implicit function $\Psi_x : B(x, r) \to B(w, \delta)$, differentiable on $B(x, r) \setminus A$ such that

$$\Psi_x(x) = w \text{ and } h(u, \Psi_x(u)) = h(a, b)$$

for every $u \in B(x,r)$. Let $0 < \varepsilon < r$ and $p_\varepsilon \in \mathbb{N}$ be such that

$$\|x_p - x\| < \varepsilon \text{ and } \|\Psi(x_p) - w\| < \delta$$

for $p \geq p_\varepsilon$ and let $p \geq p_\varepsilon$ be fixed. Since

$$\varphi_{x_p}(x_p) = \Psi(x_p) \in B(w,\delta),$$

we see from Remark 3 that $\varphi_{x_p}(x_p) = \Psi_x(x_p)$ and hence the set

$$U = \left\{u \in D_{x_p} \cap B(x,\varepsilon) \,|\varphi_{x_p}(u) = \Psi_x(u)\right\}$$

is nonempty. We also see that $D_{x_p} \cap B(x,\varepsilon)$ is an open, nonempty, convex set, hence it is a domain and U is an open, closed and nonempty subset of $D_{x_p} \cap B(x,\varepsilon)$, and this implies that

$$U = D_{x_p} \cap B(x,\varepsilon).$$

Let $U_0 = D_{x_p} \cup B(x,\varepsilon)$. We can now correctly define $\Phi : U_0 \to F$ by

$$\Phi(u) = \varphi_{x_p}(u) \text{ if } u \in D_{x_p}$$

and

$$\Phi(u) = \Psi_x(u) \text{ if } u \in B(x,\varepsilon)$$

and we see that Φ is continuous on U_0, differentiable on $U_0 \setminus A$, $\Phi(a) = b$ and $h(u,\Phi(u)) = h(a,b)$ for every $u \in U_0$. It results that $x \in B$, hence B is also a closed set and since $[a,z]$ is a connected set, we see that $B = [a,z]$.

We have therefore proved that for every $z \in Q_{a,b}$ there exists a convex domain D_z such that $[a,z] \subset D_z$ and a unique continuous implicit function $\varphi_z : D_z \to F$, differentiable on $D_z \setminus A$ such that

$$\varphi_z(a) = b,\ h(u,\varphi_z(u)) = h(a,b) \text{ for every } u \in D_z.$$

We now define $\varphi : Q_{a,b} \to F$ by $\varphi(x) = \varphi_z(x)$ for $x \in D_z$ and we see, as before, that the definition is correct, that $\varphi(a) = b, \varphi$ is continuous on $Q_{a,b}$, differentiable on $Q_{a,b} \setminus A$ and $h(x,\varphi(x)) = h(a,b)$ for every $x \in Q_{a,b}$. □

Remark 10. The result from Theorem 9 extends a global implicit function theorem from [3]. The result from [3] also involves an inequality containing

$$\left\|\frac{\partial h}{\partial x}(x,y)\right\| \quad \text{and} \quad \ell\left(\frac{\partial h}{\partial y}(x,y)\right)$$

and it says that if E,F are Banach spaces, $h : E \times F \to F$ is a C^1 map such that $\frac{\partial h}{\partial y}(x,y) \in \mathrm{Isom}(F,F)$ for every $(x,y) \in E \times F$ and there exists $\omega : [0,\infty) \to (0,\infty)$ continuous such that

$$\left(1 + \left\|\frac{\partial h}{\partial x}(x,y)\right\|\right) \Big/ \ell\left(\frac{\partial h}{\partial y}(x,y)\right) \leq \omega\left(\max\left(\|x\|,\|y\|\right)\right)$$

for every $(x,y) \in E \times F$, then, for $(x_{0,}y_0) \in E \times F$, $z_0 = h(x_0,y_0)$ and

$$r = \int_{\max(\|x_0\|,\|y_0\|)}^{\infty} \frac{ds}{1+\omega(s)},$$

there exists a C^1 map $\varphi : B(x_0,r) \times B(z_0,r) \to F$ such that $h(x,\varphi(x,z)) = z$ for every $(x,z) \in B(x_0,r) \times B(z_0,r)$. The main advantage of our new global implicit function theorems is that these theorems hold even if the map h is defined on a proper subset of $E \times F$, namely, on a set $D \times F \subset E \times F$, where $D \subset E$ is an open starlike domain.

Example 1. *A known global inversion theorem of Hadamard, Lévy and John says that if* E,F *are Banach spaces,* $f:E\to F$ *is a* C^1 *map such that* $f'(x)\in\operatorname{Isom}(E,F)$ *for every* $x\in E$ *and there exists* $\omega:[0,\infty)\to(0,\infty)$ *continuous such that*

$$\int_1^\infty \frac{ds}{\omega(s)}=\infty \quad \textit{and} \quad \left\|f'(x)^{-1}\right\|\le\omega(\|x\|)$$

for every $x\in E$*, then it results that* $f:E\to F$ *is a* C^1 *diffeomorphism (see* [11], [14], [12], [3], [7]*). If* $E=F=\mathbb{R}^n$ *or if* $\dim E=\infty$ *and* $f=I-T$ *with* T *compact, we can drop the continuity of the derivative on* E *and we can impose the essential condition "*$f'(x)\in\operatorname{Isom}(F,F)$*" with the possible exception of a "thin" set (see* [4], [5],[6]*) and we will still obtain that* $f:E\to F$ *is a homeomorphism.*

Now, let E,F be Banach spaces, $D\subset E$ a domain, $a\in D$, $b\in F$, $g:D\to F$ be differentiable on D, $f:F\to F$ be differentiable on F such that $f'(y)\in\operatorname{Isom}(F,F)$ for every $y\in F$ and there exists $\omega:[0,\infty)\to(0,\infty)$ continuous such that

$$\left\|f'(y)^{-1}\right\|\le\omega(\|y\|) \text{ for every } y\in F.$$

Let $h:D\times F\to F$ be defined by $h(x,y)=f(y)-g(x)$ for $x\in D$, $y\in F$, $r_0=d(a,\partial D)$ and suppose that

$$M_r=\sup_{x\in B(a,r)}\|g'(x)\|<\infty \text{ for every } 0<r\le r_0.$$

Then

$$\left\|\frac{\partial h}{\partial x}(x,y)\right\| \Big/ \ell\left(\frac{\partial h}{\partial y}(x,y)\right)\le\|g'(x)\|\cdot\left\|f'(y)^{-1}\right\|\le M_r\cdot\omega(\|y\|)$$

if $(x,y)\in B(a,r)\times F$ and $0<r\le r_0$ and let

$$\delta_r=\min\left\{r,\frac{1}{M_r}\cdot\int_{\|b\|}^\infty\frac{ds}{\omega(s)}\right\} \text{ for } 0<r\le r_0.$$

Using Theorem 9, we see that there exists a unique differentiable map $\varphi:B(a,\delta_r)\to F$ such that $\varphi(a)=b$ and $h(x,\varphi(x))=h(a,b)$ for every $x\in B(a,\delta_r)$, i.e. $f(\varphi(x))=g(x)+h(a,b)$ for every $x\in B(a,r)$ and every $0<r\le r_0$.

If

$$r_0\cdot M_{r_0}<\int_{\|b\|}^\infty\frac{ds}{\omega(s)},$$

then φ is defined on $B(a,r_0)$. Additionally, if $D=B(a,r_0)$, then φ is globally defined on D and $f\circ\varphi=g+h(a,b)$ on D. In the special case $D=E$, $g(x)=x$ for every $x\in E$,

$$\int_1^\infty\frac{ds}{\omega(s)}=\infty \quad\text{and}\quad b=f(a),$$

then $f(\varphi(x))=x$ for every $x\in E$, and φ is defined on E and is the inverse of f and it results that $f:F\to F$ is a homeomorphism. In this way we obtain an alternative proof of the Hadamard-Lévy-John theorem.

Remark 11. The global implicit function problem

$$h(t,x)=h(a,b),\quad x(a)=b$$

considered before has two basic properties:

(1) It satisfies the differential inequality $\|\varphi'(x)\|\le\omega(\|\varphi(x)\|)$.
(2) It has the property of the local existence and local unicity of the solutions around each point (t_0,x_0).

This shows that by considering some other conditions of local existence and local unicity of the implicit function instead of the conditions from Theorem 1, we can produce corresponding global implicit function results. Using the conditions of local existence and local unicity from Theorem 11 of [8], we obtain the following corresponding version of Theorem 9.

Theorem 12. *Let E, F be Banach spaces, $\dim E = \infty$, $\dim F < \infty$, $D \subset E$ a domain, $K \subset D \times F$,*

$$K = \bigcup_{p=1}^{\infty} K_p$$

with K_p compact sets for every $p \in \mathbb{N}$, $A = \Pr_1 K$, $h : D \times F \to F$ continuous on $D \times F$ and differentiable on $(D \times F) \backslash K$ such that

$$\ell\left(\frac{\partial h}{\partial y}(x, y)\right) > 0 \text{ on } (D \times F) \backslash K,$$

and there exists $\omega : [0, \infty) \to [0, \infty)$ continuous such that $\omega > 0$ on $(0, \infty)$ and

$$\left\|\frac{\partial h}{\partial x}(x, y)\right\| \Big/ \ \ell\left(\frac{\partial h}{\partial y}(x, y)\right) \leq \omega(\|y\|)$$

for every $(x, y) \in (D \times F) \backslash K$. Suppose that the map $y \to h(x, y)$ is a light map on F for every $x \in D$. Then, if $a, b \in D \times F$ and

$$Q_{a,b} = D_a \cap B\left(a, \int_{\|b\|}^{\infty} \frac{ds}{\omega(s)}\right),$$

there exists a unique continuous implicit function $\varphi : Q_{a,b} \to F$, differentiable on $Q_{a,b} \backslash A$ such that $\varphi(a) = b$ and $h(x, \varphi(x)) = h(a, b)$ for every $x \in Q_{a,b}$ and if D is starlike with respect to a and

$$\int_{1}^{\infty} \frac{ds}{\omega(s)} = \infty,$$

then $Q_{a,b} = D$ and $\varphi : D \to F$ is globally defined on D.

Remark 13. The condition "the map $y \to h(x, y)$ is a light map on F for every $x \in D$" is satisfied if $\frac{\partial h}{\partial y}$ exists on $D \times F$ and

$$\ell\left(\frac{\partial h}{\partial y}(x, y)\right) > 0 \text{ for every } (x, y) \in D \times F.$$

Using the conditions of local existence and local unicity from Theorem 7 of [8], we obtain the following global implicit function theorem.

Theorem 14. *Let $n \geq 2,\ D \subset \mathbb{R}^n$ be a domain, $h : D \times \mathbb{R}^m \to \mathbb{R}^m$ be differentiable and let $K \subset D \times \mathbb{R}^m$,*

$$K = \bigcup_{p=1}^{\infty} K_p$$

with K_p closed sets such that $m_{n-2}(\Pr_1 K_p) = 0$ for every $p \in \mathbb{N},\ A = \Pr_1 K,$ such that $\frac{\partial h}{\partial y}(x, y) \in \operatorname{Isom}(\mathbb{R}^m, \mathbb{R}^m)$ for every $(x, y) \in (D \times \mathbb{R}^m) \backslash K$ and the map $y \to h(x, y)$ is a light map on $\mathbb{R}^m$ for every $x \in D$. Suppose that there exists $\omega : [0, \infty) \to [0, \infty)$ continuous such that $\omega > 0$ on $(0, \infty)$ and

$$\left\|\frac{\partial h}{\partial x}(x, y)\right\| \Big/ \ \ell\left(\frac{\partial h}{\partial y}(x, y)\right) \leq \omega(\|y\|) \text{ for every } (x, y) \in (D \times F) \backslash K.$$

Then, if $(a, b) \in D \times F$ *and*

$$Q_{a,b} = D_a \cap B\left(a, \int_{\|b\|}^{\infty} \frac{ds}{\omega(s)}\right),$$

there exists a unique continuous implicit function $\varphi : Q_{a,b} \to F$*, differentiable on* $Q_{a,b} \backslash A$ *such that* $\varphi(a) = b$ *and* $h(x, \varphi(x)) = h(a, b)$ *for every* $x \in Q_{a,b}$*, and if* D *is starlike with respect to* a *and*

$$\int_1^{\infty} \frac{ds}{\omega(s)} = \infty,$$

then $Q_{a,b} = D$ *and* $\varphi : D \to F$ *is globally defined on* D*.*

Proof. We see that $m_{m+n-2}(K_p) = 0$ for every $p \in \mathbb{N}$ and A has σ-finite $(n-1)$-dimensional measure. We now apply the local implicit function theorem from Theorem 7 of [8], Remark 8 and the preceding arguments. □

Using the classical implicit function theorem, we obtain the following global implicit function theorem

Theorem 15. *Let* E*,* F *be Banach spaces,* $D \subset E$ *a domain,* $h : D \times F \to F$ *be continuous such that* $\frac{\partial h}{\partial y}$ *exists on* $D \times F$*, it is continuous on* $D \times F$ *and*

$$\ell\left(\frac{\partial h}{\partial y}(x, y)\right) > 0 \text{ for every } (x, y) \in D \times F.$$

Also, let $K \subset D \times F$ *be such that* $A = \mathrm{Pr}_1 K$ *is a countable union of compact sets if* $\dim E = \infty$ *and has* σ*-finite* $(n-1)$*-dimensional measure if* $\dim E = n$*. Additionally, suppose that* h *is differentiable on* $(D \times F) \backslash K$ *and there exists* $\omega : [0, \infty) \to [0, \infty)$ *continuous such that* $\omega > 0$ *on* $(0, \infty)$ *and*

$$\left\|\frac{\partial h}{\partial x}(x, y)\right\| \Big/ \ \ell\left(\frac{\partial h}{\partial y}(x, y)\right) \leq \omega(\|y\|)$$

for every $(x, y) \in (D \times F) \backslash K$*. Then, if* $(a, b) \in D \times F$ *and*

$$Q_{a,b} = D_a \cap B\left(a, \int_{\|b\|}^{\infty} \frac{ds}{\omega(s)}\right),$$

there exists a unique continuous implicit function $\varphi : Q_{a.b} \to F$*, differentiable on* $Q_{a,b} \backslash A$ *such that* $\varphi(a) = b$ *and* $h(x, \varphi(x)) = h(a, b)$ *for every* $x \in Q_{a,b}$*. If* D *is starlike with respect to* a *and*

$$\int_1^{\infty} \frac{ds}{\omega(s)} = \infty,$$

then $Q_{a,b} = D$ *and* $\varphi : D \to F$ *is globally defined.*

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