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A NOTE ON CERTAIN INEQUALITIES FOR THE GAMMA FUNCTION

JÓZSEF SÁNDOR

DEPARTMENT OF MATHEMATICS AND INFORMATICS
BABEŞ-BOLYAI UNIVERSITY
STR. KOGĂLNICEANU 1
400084 CLUJ-NAPOCA, ROMANIA
jjsandor@hotmail.com

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ABSTRACT. We obtain a new proof of a generalization of a double inequality on the Euler gamma function, obtained by C. Alsina and M. S. Tomás [1].

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1. INTRODUCTION

The Euler Gamma function Γ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

By using a geometrical method, recently C. Alsina and M. S. Tomás [1] have proved the following double inequality:

Theorem 1.1. *For all $x \in [0, 1]$, and all nonnegative integers n one has*

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1. \tag{1.1}$$

While the interesting method of [1] is geometrical, we will show in what follows that, by certain simple analytical arguments it can be proved that (1.1) holds true for all real numbers n, and all $x \in [0, 1]$. In fact, this will be a consequence of a monotonicity property.

Let $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ $(x > 0)$ be the "digamma function". For properties of this function, as well as inequalities, or representation theorems, see e.g. [2], [4], [5], [7]. See also [3] and [6] for a survey of results on the gamma and related functions.

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2. MAIN RESULTS

Our method is based on the following auxiliary result:

Lemma 2.1. *For all $x > 0$ one has the series representation*

$$\psi(x) = -\gamma + (x-1)\sum_{k=0}^{\infty} \frac{1}{(k+1)(x+k)}. \tag{2.1}$$

This is well-known. For proofs, see e.g. [4], [7].

Lemma 2.2. *For all $x > 0$, and all $a \geq 1$ one has*

$$\psi(1+ax) \geq \psi(1+x). \tag{2.2}$$

Proof. By (2.1) we can write $\psi(1+ax) \geq \psi(1+x)$ iff

$$-\gamma + ax\sum_{k=0}^{\infty} \frac{1}{(k+1)(1+ax+k)} \geq -\gamma + x\sum_{k=0}^{\infty} \frac{1}{(k+1)(1+x+k)}.$$

Now, remark that

$$\frac{a}{(k+1)(1+ax+k)} - \frac{1}{(k+1)(1+x+k)} = \frac{a-1}{(1+x+k)(1+ax+k)} \geq 0$$

by $a \geq 1$, $x > 0$, $k \geq 0$. Thus inequality (2.2) is proved. There is equality only for $a = 1$. □

We notice that (2.2) trivially holds true for $x = 0$ for all a.

Theorem 2.3. *For all $a \geq 1$, the function*

$$f(x) = \frac{\Gamma(1+x)^a}{\Gamma(1+ax)}$$

is a decreasing function of $x \geq 0$.

Proof. Let

$$g(x) = \log f(x) = a \log \Gamma(1+x) - \log \Gamma(1+ax).$$

Since

$$g'(x) = a[\psi(1+x) - \psi(1+ax)],$$

by Lemma 2.2 we get $g'(x) \leq 0$, so g is decreasing. This implies the required monotonicity of f. □

Corollary 2.4. *For all $a \geq 1$ and all $x \in [0,1]$ one has*

$$\frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(x+1)^a}{\Gamma(ax+1)} \leq 1. \tag{2.3}$$

Proof. For $x \in (0,1]$, by Theorem 2.3, $f(1) \leq f(x) \leq f(0)$, which by $\Gamma(1) = \Gamma(2) = 1$ implies (2.3). For $a = n \geq 1$ integer, this yields relation (1.1). □

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