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A GENERALIZATION OF HÖLDER AND MINKOWSKI INEQUALITIES

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ABSTRACT. In this work, we give a generalization of Hölder and Minkowski inequalities to normal sequence algebras with absolutely monotone seminorm. Our main result is Theorem 2.1 and Theorem 2.2 which state these extensions. Taking $F = \ell_1$ and $\|\cdot\|_F = \|\cdot\|_1$ in both these theorems, we obtain classical versions of these inequalities. Also, using these generalizations we construct the vector-valued sequence space $F(X, \lambda, p)$ as a paranormed space which is a most general form of the space $c_0(X, \lambda, p)$ investigated in [6].

Key words and phrases: Inequalities, Hölder, Minkowski, Sequence algebra, Vector-valued sequence space.

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1. INTRODUCTION

Hölder and Minkowski inequalities have been used in several areas of mathematics, especially in functional analysis. These inequalities have been generalized in various directions. The purpose of this paper is to give some extensions of the classical Hölder and Minkowski inequalities. We discovered that the classical versions are only a type of these extensions in ℓ_1 which is a normal sequence algebra with absolutely monotone seminorm $\|\cdot\|_1$.

We now recall some definitions and facts.

A Frechet space is a complete total paranormed space. If H is an Hausdorff space then an FH-space is a vector subspace X of H which is a Frechet space and is continuously embedded in H, that is, the topology of X is larger than the relative topology of H. Moreover if X is a normed FH-space then it is called a BH-space. An FH-space with $H = w$, the space of all complex sequences, is called an FK-space, so a BK-space is a normed FK-space. We know

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that ℓ_∞, c, c_0 and ℓ_p $(1 \leq p < \infty)$ are BK-spaces. The following relation exists among these sequence spaces:

$$\ell_p \subset c_0 \subset c \subset \ell_\infty.$$

A basis for a topological vector space X is a sequence (b_n) such that every $x \in X$ has a unique representation $x = \sum t_n b_n$. This is equivalent to the fact that $x - \sum_{n=1}^{m} t_n b_n \to 0$ $(m \to \infty)$ in the vector topology of X. For example, c_0 and ℓ_p have (e_n) as a basis (e_n is a sequence x where $x_n = 1$, $x_k = 0$ for $n \neq k$). If X has a basis (b_n) the functionals l_n, given by $l_n(x) = t_n$ when $x = \sum t_n b_n$, are linear. They are called the coordinate functionals and (b_n) is called a Schauder basis if each $l_n \in X'$, the continuous dual of X. A basis of a Frechet space must be a Schauder basis [7]. An FK-space X is said to have AK, or be an AK-space, if $X \supset \phi$ (the space of all finite sequences) and (e_n) is a basis for X, i.e. for each x, $x^{[n]} \to x$, where $x^{[n]}$, the nth section of x is $\sum_{k=1}^{n} x_k e_k$; otherwise expressed, $x = \sum x_k e_k$ for all $x \in X$ [8]. The spaces c_0 and ℓ_p are AK-spaces but c and ℓ_∞ are not. We say that a sequence space F is an AK-BK space if it is both a BK and an AK-space.

An algebra A over a field K is a vector space A over K such that for each ordered pair of elements $x, y \in A$ a unique product $xy \in A$ is defined with the properties

(1) $(xy)z = x(yz)$

(2a) $x(y+z) = xy + xz$

(2b) $(x+y)z = xz + yz$

(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$

for all $x, y, z \in A$ and scalars α [4].

If $K = \mathbb{R}$ (real field) or $\mathbb{C}$ (complex field) then A is said to be a real or complex algebra, respectively.

Let F be a sequence space and x, y be arbitrary members of F. F is called a sequence algebra if it is closed under the multiplication defined by $xy = (x_i y_i), i \geq 1$, and is called normal or solid if $y \in F$ whenever $|y_i| \leq |x_i|$, for some $x \in F$. If F is both a normal and sequence algebra then it is called a normal sequence algebra. For example, c is a sequence algebra but not normal. w, ℓ_∞, c_0 and ℓ_p $(0 < p < \infty)$ are normal sequence algebras.

A paranorm p on a normal sequence space F is said to be absolutely monotone if $p(x) \leq p(y)$ for $x, y \in F$ with $|x_i| \leq |y_i|$ for each i [3].

The norm $\|x\|_\infty = \sup |x_k|$ which makes the spaces ℓ_∞, c, c_0 a BK-space, is absolutely monotone. For $p \geq 1$, the norm $\|x\| = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}$ over ℓ_p is absolutely monotone. Also, for $0 < p < 1$, the p-norm $\|x\|_p = \sum_{k=1}^{\infty} |x_k|^p$ over ℓ_p is absolutely monotone.

An Orlicz function is a function $M : [0, \infty) \longrightarrow [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. We say that the Orlicz function M satisfies the $\acute{\Delta}$-condition if there exist positive constants a and u such that $M(xy) \leq aM(x)M(y)$ $(x, y \geq u)$. By means of M, Lindenstrauss and Tzafriri [2] constructed the sequence space

$$\ell_M = \left\{ x \in w : \sum M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}$$

with the norm $\|x\|_M = \inf \left\{ \rho > 0 : \sum M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$. This norm is absolutely monotone and ℓ_M is normal since M is non-decreasing. Also if M satisfies the $\acute{\Delta}$-condition then ℓ_M is a sequence algebra.

Now we give a useful inequality from classical analysis.

Lemma 1.1. *Let f be a function such that $f''(x) \geq 0$ for $x > 0$. Then for $0 < a < x < b$*

$$\begin{aligned}\frac{f(x)-f(a)}{x-a} &= \frac{1}{x-a}\int_a^x f'(t)dt \\ &\leq f'(x) \\ &\leq \frac{1}{b-x}\int_x^b f'(t)dt \\ &= \frac{f(b)-f(x)}{b-x}.\end{aligned}$$

Hence

$$f(x) \leq \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)$$

[7].

Apply this to the function $f(x) = -\ln x$ with $\theta = (b-x)/(b-a)$. Then for all a, b positive numbers and $0 \leq \theta \leq 1$, we have

(1.1) $$a^\theta b^{1-\theta} \leq a\theta + (1-\theta)b.$$

Next, we give a lemma associated with the theorems in Section 2.

Lemma 1.2.

a) *Let F be a normal sequence algebra, $u = (u_n) \in F$ and $p \geq 1$. Then $u^p = (u_n^p) \in F$.*
b) *If F is a normal sequence space, $\|\cdot\|_F$ is an absolutely monotone seminorm on F and $u = (u_n) \in F$ then $|u| = (|u_n|) \in F$ and $\||u|\|_F = \|u\|_F$.*

Proof. **a)** We define two sequences $a = (a_n)$ and $b = (b_n)$ such that

$$a_n = \begin{cases} u_n & \text{if } |u_n| \geq 1 \\ 0 & \text{if } |u_n| < 1 \end{cases} \quad \text{and} \quad b_n = \begin{cases} 0 & \text{if } |u_n| \geq 1 \\ u_n & \text{if } |u_n| < 1 \end{cases}.$$

So $u_n = a_n + b_n$ and $u_n^p = a_n^p + b_n^p$. Obviously, $a, b \in F$. Since $p < [p] + 1$, we have

$$|a_n|^p \leq |a_n|^{[p]+1},$$

where $[p]$ denotes the integer part of p. Since F is a sequence algebra, the sequence $a^{[p]+1}$ is a member of F by induction, and so $a^p \in F$. Furthermore, since F is normal and $|b_n|^p \leq |b_n|$, we have $b^p \in F$. Hence $u^p \in F$.

b) It is a direct consequence of normality and absolute monotonicity. □

2. GENERALIZATIONS

Our main results are the following theorems which state the extensions of Hölder and Minkowski inequalities. Taking $F = \ell_1$ and $\|\cdot\|_F = \|\cdot\|_1$ in both Theorem 2.1 and Theorem 2.2, we get classical versions of these inequalities. Moreover, if we change the choices of F and $\|\cdot\|_F$ then we can obtain many different inequalities corresponding to these generalizations. Therefore, the following results are quite productive.

Theorem 2.1. *Let F be a sequence algebra and $\|\cdot\|_F$ be an absolutely monotone seminorm on F. Suppose $u = (u_n), v = (v_n) \in F$. Then*

$$\|uv\|_F \leq \|u^p\|_F^{1/p}\|v^q\|_F^{1/p},$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Assume that $x_n = |u_n|^p$ and $y_n = |v_n|^q$. It is immediate from Lemma 1.2(a) that $x = (x_n)$ and $y = (y_n)$ are members of F. Let $M = \|x\|_F$ and $N = \|y\|_F$. Then it follows from inequality (1.1) that for each n,

$$\left(\frac{x_n}{M}\right)^\theta \left(\frac{y_n}{N}\right)^{1-\theta} \leq \theta \frac{x_n}{M} + (1-\theta)\frac{y_n}{N}$$

as $0 \leq \theta \leq 1$. Because $\|\cdot\|_F$ is an absolutely monotone seminorm we write

$$\left\| \left(\left(\frac{x_n}{M}\right)^\theta \left(\frac{y_n}{N}\right)^{1-\theta} \right) \right\|_F \leq \left\| \left(\theta \frac{x_n}{M} + (1-\theta)\frac{y_n}{N} \right) \right\|_F.$$

Hence

$$\frac{1}{M^\theta N^{1-\theta}} \left\| \left(x_n^\theta y_n^{1-\theta} \right) \right\|_F \leq 1,$$

so that

$$\left\| \left(x_n^\theta y_n^{1-\theta} \right) \right\|_F \leq \|(x_n)\|_F^\theta \|(y_n)\|_F^{1-\theta}.$$

Setting $\theta = 1/p$, we get

$$\left\| \left(x_n^{1/p} y_n^{1/q} \right) \right\|_F \leq \|(x_n)\|_F^{1/p} \|(y_n)\|_F^{1/q},$$

and putting $x_n = |u_n|^p$ and $y_n = |v_n|^q$, we obtain

$$\|(|u_n v_n|)\|_F \leq \|(|u_n|^p)\|_F^{1/p} \|(|v_n|^q)\|_F^{1/q}.$$

So, it follows from Lemma 1.2(b) that

$$\|uv\|_F \leq \|u^p\|_F^{1/p} \|v^q\|_F^{1/q}.$$

□

Theorem 2.2. *Let F be a normal sequence algebra and $\|\cdot\|_F$ be an absolutely monotone seminorm on F. Then for every $u = (u_n), v = (v_n) \in F$ and $p \geq 1$,*

$$\|(u+v)^p\|_F^{1/p} \leq \|u^p\|_F^{1/p} + \|v^p\|_F^{1/p},$$

where $(u+v)^p = ((u_n + v_n)^p)$.

Proof. For $p = 1$, it is obvious.

Let $p > 1$. Proceeding with the manner of the proof in the classical version, we write

$$(u+v)^p = u\,(u+v)^{p-1} + v\,(u+v)^{p-1}.$$

It follows from Theorem 2.1 that

$$\begin{aligned} \|(u+v)^p\|_F &\leq \|u^p\|_F^{1/p} \left\| (u+v)^{(p-1)q} \right\|_F^{1/q} + \|v^p\|_F^{1/p} \left\| (u+v)^{(p-1)q} \right\|_F^{1/q} \\ &= \left(\|u^p\|_F^{1/p} + \|v^p\|_F^{1/p} \right) \left\| (u+v)^{(p-1)q} \right\|_F^{1/q}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Hence, dividing the first and last terms by $\left\| (u+v)^{(p-1)q} \right\|_F^{1/q} = \|(u+v)^p\|_F^{1/q}$, we obtain the inequality. □

Example 2.1. Taking $F = \ell_\infty$ and $\|\cdot\|_F = \|\cdot\|_\infty$ in both Theorem 2.1 and Theorem 2.2, we obtain the inequalities

$$\sup_n |u_n v_n| \leq \left\{ \sup_n |u_n|^p \right\}^{\frac{1}{p}} \cdot \left\{ \sup_n |v_n|^q \right\}^{\frac{1}{q}}$$

and

$$\left\{\sup_n |u_n+v_n|^p\right\}^{\frac{1}{p}} \leq \left\{\sup_n |u_n|^p\right\}^{\frac{1}{p}} + \left\{\sup_n |v_n|^p\right\}^{\frac{1}{p}},$$

where $u, v \in \ell_\infty$ and $\frac{1}{p}+\frac{1}{q}=1$ as $p>1$. Hence, in fact, these elementary inequalities are extended Hölder and Minkowski inequalities respectively.

Example 2.2. Now put $F=\ell_M$ and $\|\cdot\|_F=\|\cdot\|_M$ in Theorem 2.1 and Theorem 2.2, where M satisfies the Δ'-condition. In this case, we write the inequalities

$$\inf\left\{\rho>0: \sum M\left(\frac{|x_k y_k|}{\rho}\right)\leq 1\right\} \leq \left\{\inf\left\{\rho>0: \sum M\left(\frac{|x_k|^p}{\rho}\right)\leq 1\right\}\right\}^{\frac{1}{p}} \cdot \left\{\inf\left\{\rho>0: \sum M\left(\frac{|y_k|^q}{\rho}\right)\leq 1\right\}\right\}^{\frac{1}{q}}$$

and

$$\left\{\inf\left\{\rho>0: \sum M\left(\frac{|x_k+y_k|^p}{\rho}\right)\leq 1\right\}\right\}^{\frac{1}{p}} \leq \left\{\inf\left\{\rho>0: \sum M\left(\frac{|x_k|^p}{\rho}\right)\leq 1\right\}\right\}^{\frac{1}{p}} + \left\{\inf\left\{\rho>0: \sum M\left(\frac{|y_k|^p}{\rho}\right)\leq 1\right\}\right\}^{\frac{1}{p}}$$

as Hölder and Minkowski inequalities respectively.

3. An Application

Now let us introduce the class $F(X,\lambda,p)$ of vector-valued sequence spaces which includes the space $c_0(X,\lambda,p)$ investigated in [6] with some linear topological properties. Theorem 2.2 makes it possible to improve some topological properties of the space $F(X,\lambda,p)$.

Let F be an AK-BK normal sequence algebra such that the norm $\|\cdot\|_F$ of F is absolutely monotone and X be a seminormed space. Also suppose that $\lambda=(\lambda_k)$ is a non-zero complex sequence and $p=(p_k)$ is a sequence of strictly positive real numbers. Define the vector-valued sequence class

$$F(X,\lambda,p)=\left\{x\in s(X):\ \left([q(\lambda_k x_k)]^{p_k}\right)\in F\right\},$$

where q is the seminorm of X and $s(X)$ is the most general X-termed sequence space. $F(X,\lambda,p)$ becomes a linear space under natural co-ordinatewise vector operations if and only if $p\in\ell_\infty$ (see Lascarides [1]). Taking $F=c_0$ and X as a Banach space we get the space $c_0(X,\lambda,p)$ in [6].

Lemma 3.1. *Let $0<t_k\leq 1$. If a_k and b_k are complex numbers then we have*

$$|a_k+b_k|^{t_k}\leq |a_k|^{t_k}+|b_k|^{t_k}$$

[5, p.5].

Lemma 3.2. *Let (X,q) be a seminormed space, and F a normal AK-BK space with an absolutely monotone norm $\|\cdot\|_F$. Suppose $p=(p_k)$ is a bounded sequence of positive real numbers. Then the map*

$$\widetilde{x}_n:[0,\infty)\to[0,\infty)\,;\ \widetilde{x}_n(u)=\left\|\sum_{k=1}^{n}\left[uq(\lambda_k x_k)\right]^{p_k}e_k\right\|_F$$

defined by means of $x=(x_k)\in F(X,\lambda,p)$ and a positive integer n, is continuous, where (e_k) is a unit vector basis of F.

Proof. Since the norm function is continuous it is sufficient to show that the mappings defined by

$$g_k:[0,\infty)\to F,\ g_k(u)=\left[uq_k(\lambda_k x_k)\right]^{p_k}e_k$$

are continuous. Let $u_i\to 0$ $(i\to\infty)$, then

$$g_k(u_i)\to(0,0,\dots)\quad(i\to\infty)$$

for each k. Hence, each g_k is sequential continuous (it is equivalent to continuity here). □

Theorem 3.3. *Define the function $g:F(X,\lambda,p)\longrightarrow\mathbb{R}$ by*

$$g(x)=\left\|\left(\left[q(\lambda_k x_k)\right]^{p_k}\right)\right\|_F^{1/M},$$

where $M=\max(1,\sup p_n)$. Then g is a paranorm on $F(X,\lambda,p)$.

Proof. It is obvious that $g(\theta)=0$ and $g(-x)=g(x)$. From the absolute monotonicity of $\|\cdot\|_F$, Lemma 3.1 and Theorem 2.2, we get

$$\begin{aligned}g(x+y)&=\left\|\left(\left(\left[q(\lambda_k x_k+\lambda_k y_k)\right]^{p_k/M}\right)^M\right)\right\|_F^{1/M}\\&\le\left\|\left(\left(\left[q(\lambda_k x_k)\right]^{p_k/M}+\left[q(\lambda_k y_k)\right]^{p_k/M}\right)^M\right)\right\|_F^{1/M}\\&\le\left\|\left(\left[q(\lambda_k x_k)\right]^{p_k}\right)\right\|_F^{1/M}+\left\|\left(\left[q(\lambda_k y_k)\right]^{p_k}\right)\right\|_F^{1/M}\\&=g(x)+g(y)\end{aligned}$$

for $x,y\in F(X,\lambda,p)$.

To show the continuity of scalar multiplication assume that (μ^n) is a sequence of scalars such that $|\mu^n-\mu|\to 0$ $(n\to\infty)$ and $g(x^n-x)\to 0$ $(n\to\infty)$ for an arbitrary sequence $(x^n)\subset F(X,\lambda,p)$. We shall show that

$$g(\mu^n x^n-\mu x)\to 0\,(n\to\infty)\,.$$

Say $\tau_n=|\mu^n-\mu|$ and we get

$$\begin{aligned}g(\mu^n x^n-\mu x)&=\left\|\left(\left[q\left(\lambda_k\left(\mu^n x_k^n-\mu x_k\right)\right)\right]^{p_k}\right)\right\|_F^{1/M}\\&=\left\|\left(\left[q\left(\lambda_k\left(\mu^n x_k^n-\mu^n x_k+\mu^n x_k-\mu x_k\right)\right)\right]^{p_k}\right)\right\|_F^{1/M}\\&\le\left\|\left(\left[|\mu^n|\,q\left(\lambda_k\left(x_k^n-x_k\right)\right)\right)+\tau_n q(\lambda_k x_k)\right]^{p_k}\right)\right\|_F^{1/M}\\&\le\left\|\left(\left\{\left[A(k,n)\right]^{p_k/M}+\left[B(k,n)\right]^{p_k/M}\right\}^M\right)\right\|_F^{1/M},\end{aligned}$$

where $A(k,n) = Rq(\lambda_k(x_k^n - x_k))$, $B(k,n) = \tau_n q(\lambda_k x_k)$ and $R = \max\{1, \sup|\mu^n|\}$. Again by Theorem 2.2 we can write

$$\begin{aligned} g(\mu^n x^n - \mu x) &\le \|(A(k,n))\|_F^{1/M} + \|(B(k,n))\|_F^{1/M} \\ &\le R\left\|\left(\left[\frac{A}{R}\right]^{p_k}\right)\right\|_F^{1/M} + \|(B(k,n))\|_F^{1/M} \\ &= Rg(x^n - x) + \|(B(k,n))\|_F^{1/M}. \end{aligned}$$

Since $g(x^n - x) \to 0\,(n \to \infty)$ we must show that $\|(B(k,n))\|_F^{1/M} \to 0\ (n \to \infty)$. We can find a positive integer n_0 such that $0 \le \tau_n \le 1$ for $n \ge n_0$. Say $t_k = [q(\lambda_k x_k)]^{p_k}$. Since $t = (t_k) \in F$ and F is an AK-space, we get

$$\left\|t - \sum_{k=1}^{m} t_k e_k\right\|_F = \left\|\sum_{k=m+1}^{\infty} [q(\lambda_k x_k)]^{p_k} e_k\right\|_F \to 0 \quad (m \to \infty),$$

where (e_k) is a unit vector basis of F. Therefore, for every $\varepsilon > 0$ there exists a positive integer m_0 such that

$$\left\|\sum_{k=m_0+1}^{\infty} [q(\lambda_k x_k)]^{p_k} e_k\right\|_F^{\frac{1}{M}} < \frac{\varepsilon}{2}.$$

For $n \ge n_0$ write $[(\tau_n q(\lambda_k x_k))]^{p_k} \le [f(q(\lambda_k x_k))]^{p_k}$ for each k. On the other hand, we can write

$$\left\|\sum_{k=m_0+1}^{\infty} [\tau_n q(\lambda_k x_k)]^{p_k} e_k\right\|_F^{\frac{1}{M}} \le \left\|\sum_{k=m_0+1}^{\infty} [q(\lambda_k x_k)]^{p_k} e_k\right\|_F^{\frac{1}{M}} < \frac{\varepsilon}{2}.$$

Now, from Lemma 3.2, the function

$$\widetilde{x}_{m_0}(u) = \left\|\sum_{k=1}^{m_0} [(uq(\lambda_k x_k))]^{p_k} e_k\right\|_F$$

is continuous. Hence, there exists a $\delta\ (0 < \delta < 1)$ such that

$$\widetilde{x}_{m_0}(u) \le \left(\frac{\varepsilon}{2}\right)^M,$$

for $0 < u < \delta$. Also we can find a number Δ such that $\tau_n < \delta$ for $n > \Delta$. So for $n > \Delta$ we have

$$(\widetilde{x}_{m_0}(\tau_n))^{1/M} = \left\|\sum_{k=1}^{m_0} [\tau_n q(\lambda_k x_k)]^{p_k} e_k\right\|_F < \frac{\varepsilon}{2},$$

and eventually we get

$$\begin{aligned}
\|([\tau_n q(\lambda_k x_k)]^{p_k})\|_F^{1/M} &= \left\|\sum_{k=1}^{\infty}[\tau_n q(\lambda_k x_k)]^{p_k} e_k\right\|_F^{\frac{1}{M}} \\
&= \left\|\sum_{k=1}^{m_0}[\tau_n q(\lambda_k x_k)]^{p_k} e_k + \sum_{k=m_0+1}^{\infty}[\tau_n q(\lambda_k x_k)]^{p_k} e_k\right\|_F^{\frac{1}{M}} \\
&\le \left\|\sum_{k=1}^{m_0}[\tau_n q(\lambda_k x_k)]^{p_k} e_k\right\|_F^{\frac{1}{M}} + \left\|\sum_{k=m_0+1}^{\infty}[\tau_n q(\lambda_k x_k)]^{p_k} e_k\right\|_F^{\frac{1}{M}} \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This shows that $\|(B(k,n))\|_F^{1/M} \to 0\ (n\to\infty)$. □

Theorem 3.4. *Let (X,q) be a complete seminormed space. Then $F(X,\lambda,p)$ is complete with the paranorm g. If X is a Banach space then $F(X,\lambda,p)$ is an FK-space, in particular, an AK-space.*

Proof. Let (x^n) be a Cauchy sequence in $F(X,\lambda,p)$. Therefore

$$g(x^n - x^m) = \|([q(\lambda_k(x_k^n - x_k^m))]^{p_k})\|_F^{1/M} \to 0 \quad (m,n\to\infty),$$

also, since F is an FK-space, for each k

$$[q(\lambda_k(x_k^n - x_k^m))]^{p_k} \to 0 \quad (m,n\to\infty)$$

and so $|\lambda_k|\, q(x_k^n - x_k^m) \to 0\ (m,n\to\infty)$. Because of the completeness of X, there exists an $x_k \in X$ such that $q(x_k^n - x_k) \to 0\ (n\to\infty)$ for each k. Define the sequence $x=(x_k)$ with these points. Now we can determine a sequence $\eta_k \in c_0\ (0<\eta_k^n\le 1)$ such that

$$[|\lambda_k|\, q(x_k^n - x_k)]^{p_k} \le \eta_k^n [q(\lambda_k x_k^n)]^{p_k} \tag{3.1}$$

since $q(x_k^n - x_k)\to 0$. On the other hand,

$$[q(\lambda_k x_k)]^{p_k} \le D\left\{[q(\lambda_k(x_k^n - x_k))]^{p_k} + [q(\lambda_k x_k^n)]^{p_k}\right\},$$

where $D=\max\left(1,2^{H-1}\right)$; $H=\sup p_k$. From (3.1) we have

$$\begin{aligned}
[q(\lambda_k x_k)]^{p_k} &\le D(1+\eta_k^n)[q(\lambda_k x_k^n)]^{p_k} \\
&\le 2D[q(\lambda_k x_k^n)]^{p_k}.
\end{aligned}$$

So we get $x\in F(X,\lambda,p)$. Now, for each $\varepsilon>0$ there exist $n_0(\varepsilon)$ such that

$$[g(x^n - x^m)]^M < \varepsilon^M \quad \text{for } n,m>n_0.$$

Also, we may write from the AK-property of F that

$$\begin{aligned}
\left\|\sum_{k=1}^{m_0}[q(\lambda_k(x_k^n - x_k^m))]^{p_k} e_k\right\|_F &\le \left\|\sum_{k=1}^{\infty}[q(x_k^n - x_k^m)]^{p_k} e_k\right\|_F \\
&= \left[g(x^n - x^m)^M\right].
\end{aligned}$$

Letting $m \to \infty$ we have

$$\left\| \sum_{k=1}^{m_0} \left[q \left(\lambda_k \left(x_k^n - x_k^m \right) \right) \right]^{p_k} e_k \right\|_F \to \left\| \sum_{k=1}^{m_0} \left[q \left(\lambda_k \left(x_k^n - x_k \right) \right) \right]^{p_k} e_k \right\|_F < \varepsilon^M \quad \text{for } n > n_0.$$

Since (e_k) is a Schauder basis for F,

$$\left\| \sum_{k=1}^{m_0} \left[q \left(\lambda_k \left(x_k^n - x_k \right) \right) \right]^{p_k} e_k \right\|_F \to \left\| \left(\left[q \left(\lambda_k \left(x_k^n - x_k \right) \right) \right]^{p_k} \right) \right\|_F < \varepsilon^M \quad \text{as } m_0 \to \infty.$$

Then we get $g(x^n - x) < \varepsilon$ for $n > n_0$ so $g(x^n - x) \to 0$ $(n \to \infty)$.

For the rest of the theorem; we can say immediately that $F(X, \lambda, p)$ is a Frechet space, because X is a Banach space. Also, the projections

$$\hat{P}_k \colon F(X, \lambda, p) \longrightarrow X; \qquad \hat{P}_k(x) = x_k$$

are continuous since $P_k = |\lambda_k| \left(q \circ \hat{P}_k \right)$ for each k. Where P_k's are coordinate mappings on F and they are continuous since F is an FK-space.

Let $x^{[n]}$ be the nth section of an element x of $F(X, \lambda, p)$. We must prove that $x^{[n]} \to x$ in $F(X, \lambda, p)$ for each $x \in F(X, \lambda, p)$. Indeed,

$$\begin{aligned} g\left(x - x^{[n]} \right) &= g\left(0, 0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots \right) \\ &= \left\| \sum_{k=n+1}^{\infty} \left[q \left(\lambda_k x_k \right) \right]^{p_k} e_k \right\|_F \to 0 \end{aligned}$$

since F is an AK-space. Hence $F(X, \lambda, p)$ is an AK-space. □

REFERENCES

[1] C.G. LASCARIDES, A study of certain sequence spaces of Maddox and a generalization of a theorem of Iyer, *Pacific J. Math.*, **38** (1971), 487–500.

[2] J. LINDENSTRAUSS AND L. TZAFRIRI On Orlicz sequence space, *Israel J. Math.*, **10** (1971), 379–390.

[3] P.K. KAMTHAN AND M. GUPTA, *Sequence Spaces and Series*, Marcel Dekker Inc., New York and Basel, 1981.

[4] E. KREYSZIG, *Introductory Functional Analysis with Applications*, John Wiley & Sons, New York, 1978.

[5] S. NANDA AND B. CHOUDHARY, *Functional Analysis with Applications*, John Wiley & Sons, New York, 1989.

[6] J. K. SRIVASTAVA AND B. K. SRIVASTAVA, Generalized Sequence Space $c_0(X, \lambda, p)$, *Indian J. of Pure Appl. Math.*, **27**(1) (1996), 73–84.

[7] A. WILANSKY, *Modern Methods in Topological Vector Spaces*, Mac-Graw Hill, New York, 1978.

[8] A. WILANSKY, *Summability Through Functional Analysis*, Mathematics Studies 85, North-Holland, Amsterdam, 1984.