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A NOTE ON INTEGRAL INEQUALITIES INVOLVING THE PRODUCT OF TWO FUNCTIONS

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ABSTRACT. In this note, we establish new integral inequalities involving two functions and their derivatives. The discrete analogues of the main results are also given.

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1. INTRODUCTION

Inequalities have proved to be one of the most powerful and far-reaching tools for the development of many branches of mathematics. The monographs [1] – [3] contain an extensive number of surveys of inequalities up to the year of their publications. In the last few decades, much significant development in the classical and new inequalities, particularly in analysis has been witnessed. The aim of the present note is to establish new integral inequalities, providing approximation formulae which can be used to estimate the deviation of the product of two functions. The discrete versions of the main results are also given.

2. STATEMENT OF RESULTS

Our main results are given in the following theorem.

Theorem 2.1. *Let* $f, g \in C^1([a,b], \mathbb{R})$, $[a,b] \subset \mathbb{R}$, $a < b$. *Then*

$$\left| f(x)\, g(x) - \frac{1}{2}\left[g(x)\, F + f(x)\, G \right] \right| \leq \frac{1}{4}\left[|g(x)| \int_a^b |f'(t)|\, dt + |f(x)| \int_a^b |g'(t)|\, dt \right], \tag{2.1}$$

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and

$$|f(x)g(x) - [g(x)F + f(x)G] + FG| \le \frac{1}{4}\left(\int_a^b |f'(t)|\,dt\right)\left(\int_a^b |g'(t)|\,dt\right), \tag{2.2}$$

for all $x \in [a,b]$, where

$$F = \frac{f(a)+f(b)}{2}, \qquad G = \frac{g(a)+g(b)}{2}. \tag{2.3}$$

The constant $\frac{1}{4}$ in (2.1) and (2.2) is sharp.

Remark 2.2. If we take $g(x) = 1$ and hence $g'(x) = 0$ in (2.1), then by simple calculation we get the inequality

$$|f(x) - F| \le \frac{1}{2}\int_a^b |f'(t)|\,dt, \tag{2.4}$$

which is established in [5, p.28]. We believe that the inequality established in (2.2) is new to the literature.

The discrete versions of the inequalities in Theorem 2.1 are embodied in the following theorem.

Theorem 2.3. *Let $\{u_i\}, \{v_i\}$ for $i = 0,1,2,\ldots,n$, $n \in \mathbb{N}$ be sequences of real numbers. Then*

$$\left|u_i v_i - \frac{1}{2}[v_i U + u_i V]\right| \le \frac{1}{4}\left[|v_i|\sum_{j=0}^{n-1}|\Delta u_j| + |u_i|\sum_{j=0}^{n-1}|\Delta v_j|\right], \tag{2.5}$$

and

$$|u_i v_i - [v_i U + u_i V] + UV| \le \frac{1}{4}\left(\sum_{j=0}^{n-1}|\Delta u_j|\right)\left(\sum_{j=0}^{n-1}|\Delta v_j|\right), \tag{2.6}$$

for $i = 0,1,2,\ldots,n$, where

$$U = \frac{u_0 + u_n}{2}, V = \frac{v_0 + v_n}{2}, \tag{2.7}$$

and Δ is the forward difference operator. The constant $\frac{1}{4}$ in (2.5) and (2.6) is sharp.

3. PROOF OF THEOREM 2.1

From the hypotheses of Theorem 2.1 we have the following identities (see [5], [6, p. 267]):

$$f(x) - F = \frac{1}{2}\left[\int_a^x f'(t)\,dt - \int_x^b f'(t)\,dt\right], \tag{3.1}$$

$$g(x) - G = \frac{1}{2}\left[\int_a^x g'(t)\,dt - \int_x^b g'(t)\,dt\right]. \tag{3.2}$$

Multiplying both sides of (3.1) and (3.2) by $g(x)$ and $f(x)$ respectively, adding the resulting identities and rewriting we have

$$\begin{aligned} f(x)g(x) &- \frac{1}{2}[g(x)F + f(x)G] \\ &= \frac{1}{4}\left[g(x)\left[\int_a^x f'(t)\,dt - \int_x^b f'(t)\,dt\right] + f(x)\left[\int_a^x g'(t)\,dt - \int_x^b g'(t)\,dt\right]\right]. \end{aligned} \tag{3.3}$$

From (3.3) and using the properties of modulus we have

$$\left|f(x)g(x) - \frac{1}{2}[g(x)F + f(x)G]\right| \le \frac{1}{4}\left[|g(x)|\int_a^b |f'(t)|\,dt + |f(x)|\int_a^b |g'(t)|\,dt\right].$$

This is the required inequality in (2.1).

Multiplying the left sides and right sides of (3.1) and (3.2) we get

$$\begin{aligned} f(x)g(x) - [g(x)F + f(x)G] + FG \\ = \frac{1}{4}\left[\int_a^x f'(t)\,dt - \int_x^b f'(t)\,dt\right]\left[\int_a^x g'(t)\,dt - \int_x^b g'(t)\,dt\right]. \end{aligned} \tag{3.4}$$

From (3.4) and using the properties of modulus we have

$$|f(x)g(x) - [g(x)F + f(x)G] + FG| \le \frac{1}{4}\left[\int_a^b |f'(t)|\,dt\right]\left[\int_a^b |g'(t)|\,dt\right].$$

This proves the inequality in (2.2).

To prove the sharpness of the constant $\frac{1}{4}$ in (2.1) and (2.2), assume that the inequalities (2.1) and (2.2) hold with constants $c > 0$ and $k > 0$ respectively. That is,

$$\begin{aligned} \left|f(x)g(x) - \frac{1}{2}[|g(x)|F + |f(x)|G]\right| \\ \le c\left[|g(x)|\int_a^b |f'(t)|\,dt + |f(x)|\int_a^b |g'(t)|\,dt\right], \end{aligned} \tag{3.5}$$

and

$$|f(x)g(x) - [|g(x)|F + |f(x)|G] + FG| \le k\left(\int_a^b |f'(t)|\,dt\right)\left(\int_a^b |g'(t)|\,dt\right), \tag{3.6}$$

for $x \in [a,b]$. In (3.5) and (3.6), choose $f(x) = g(x) = x$ and hence $f'(x) = g'(x) = 1$, $F = G = \frac{a+b}{2}$. Then by simple computation, we get

$$\left|x - \frac{1}{2}(a+b)\right| \le 2c(b-a), \tag{3.7}$$

and

$$\left|x(x-(a+b)) + \left(\frac{a+b}{2}\right)^2\right| \le k(b-a)^2. \tag{3.8}$$

By taking $x = b$, from (3.7) we observe that $c \ge \frac{1}{4}$ and from (3.8) it is easy to observe that $k \ge \frac{1}{4}$, which proves the sharpness of the constants in (2.1) and (2.2). The proof is complete.

4. PROOF OF THEOREM 2.3

From the hypotheses of Theorem 2.3 we have the following identities (see [5], [6, p. 352]):

$$u_i - U = \frac{1}{2}\left[\sum_{j=0}^{i-1}\Delta u_j - \sum_{j=i}^{n-1}\Delta u_j\right] \tag{4.1}$$

and

$$v_i - V = \frac{1}{2}\left[\sum_{j=0}^{i-1}\Delta v_j - \sum_{j=i}^{n-1}\Delta v_j\right]. \tag{4.2}$$

Multiplying both sides of (4.1) and (4.2) by v_i and u_i $(i = 0, 1, 2, \ldots, n)$ respectively, adding the resulting identities and rewriting we get

$$u_i v_i - \frac{1}{2}\left[v_i U + u_i V\right] = \frac{1}{4}\left[v_i\left[\sum_{j=0}^{i-1}\Delta u_j - \sum_{j=i}^{n-1}\Delta u_j\right] + u_i\left[\sum_{j=0}^{i-1}\Delta v_j - \sum_{j=i}^{n-1}\Delta v_j\right]\right]. \tag{4.3}$$

Multiplying the left sides and right sides of (4.1) and (4.2) we have

$$u_i v_i - \left[v_i U + u_i V\right] + UV = \frac{1}{4}\left[\sum_{j=0}^{i-1}\Delta u_j - \sum_{j=i}^{n-1}\Delta u_j\right]\left[\sum_{j=0}^{i-1}\Delta v_j - \sum_{j=i}^{n-1}\Delta v_j\right]. \tag{4.4}$$

From (4.3) and (4.4) and following the proof of Theorem 2.1, we get the desired inequalities in (2.5) and (2.6).

Assume that the inequalities (2.5) and (2.6) hold with constants $\alpha > 0$ and $\beta > 0$ respectively. Taking $\{u_i\} = \{v_i\} = \{i\}$ for $i = 0, 1, 2, \ldots, n$ and $U = V = \frac{n}{2}$ and following similar arguments to those used in the last part of the proof of Theorem 2.1, it is easy to observe that $\alpha \geq \frac{1}{4}$ and $\beta \geq \frac{1}{4}$ and hence the constants in (2.5) and (2.6) are sharp. The proof is complete.

Remark 4.1. Dividing both sides of (3.3) and (3.4) by $(b-a)$, then integrating both sides with respect to x over $[a, b]$ and closely looking at the proof of Theorem 2.1 we get

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{2(b-a)}\left[F\int_a^b g(x)\,dx + G\int_a^b f(x)\,dx\right]\right| \\ &\qquad \leq \frac{1}{4(b-a)}\left[\left(\int_a^b |g(x)|\,dx\right)\left(\int_a^b |f'(x)|\,dx\right)\right. \\ &\qquad\qquad \left. + \left(\int_a^b |f(x)|\,dx\right)\left(\int_a^b |g'(x)|\,dx\right)\right], \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)}\left[F\int_a^b g(x)\,dx + G\int_a^b f(x)\,dx - FG\right]\right| \\ &\qquad \leq \frac{1}{4}\left(\int_a^b |f'(x)|\,dx\right)\left(\int_a^b |g'(x)|\,dx\right). \end{aligned} \tag{4.6}$$

We note that the inequalities (4.5) and (4.6) are similar to those of the well known inequalities due to Grüss and Čebyšev, see [3, 4].

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