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A RECENT NOTE ON THE ABSOLUTE RIESZ SUMMABILITY FACTORS

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ABSTRACT. The purpose of this note is to present a theorem having conditions of new type and to weaken some assumptions given in two previous papers simultaneously.

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1. INTRODUCTION

Recently there have been a number of papers written dealing with absolute summability factors of infinite series, see e.g. [3] – [9]. Among others in [6] we also proved a theorem of this type improving a result of H. Bor [3]. Very recently H. Bor and L. Debnath [5] enhanced a theorem of S.M. Mazhar [9] considering a quasi β-power increasing sequence $\{X_n\}$ for some $0 < \beta < 1$ instead of the case $\beta = 0$.

The purpose of this note is to moderate the conditions of the theorems of Bor-Debnath and ours.

To recall these theorems we need some definitions.

A positive sequence $\mathbf{a} := \{a_n\}$ is said to be *quasi* β*- power increasing* if there exists a constant $K = K(\beta, \mathbf{a}) \geq 1$ such that

$$K\, n^\beta\, a_n \geq m^\beta\, a_m \tag{1.1}$$

holds for all $n \geq m$. If (1.1) stays with $\beta = 0$ then $\mathbf{a}$ is simply called a *quasi increasing* sequence. In [6] we showed that this latter class is equivalent to the class of *almost increasing* sequences.

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A series $\sum a_n$ with partial sums s_n is said to be summable $|\overline{N}, p_n|_k,\ k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty,$$

where $\{p_n\}$ is a sequence of positive numbers such that

$$P_n := \sum_{\nu=0}^{n} p_\nu \to \infty$$

and

$$t_n := \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu\, s_\nu.$$

First we recall the theorem of Bor and Debnath.

Theorem 1.1. *Let* $\mathbf{X} := \{X_n\}$ *be a quasi* β*-power increasing sequence for some* $0 < \beta < 1$, *and* $\lambda := \{\lambda_n\}$ *be a real sequence. If the conditions*

(1.2) $$\sum_{n=1}^{m} \frac{1}{n} P_n = O(P_m),$$

(1.3) $$\lambda_n\, X_n = O(1),$$

(1.4) $$\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m),$$

(1.5) $$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m),$$

and

(1.6) $$\sum_{n=1}^{\infty} n\, X_n |\Delta^2 \lambda_n| < \infty, \quad (\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1})$$

are satisfied, then the series $\sum a_n \lambda_n$ *is summable* $|\overline{N}, p_n|_k,\ k \geq 1.$

In my view, the proof of Theorem 1.1 has a little gap, but the assertion is true.

Our mentioned theorem [7] reads as follows.

Theorem 1.2. *If* $\mathbf{X}$ *is a quasi increasing sequence and the conditions (1.4), (1.5),*

(1.7) $$\sum_{n=1}^{\infty} \frac{1}{n} |\lambda_n| < \infty,$$

(1.8) $$\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty$$

and

(1.9) $$\sum_{n=1}^{\infty} n\, X_n |\Delta |\Delta \lambda_n|| < \infty$$

are satisfied, then the series $\sum a_n \lambda_n$ *is summable* $|\overline{N}, p_n|_k,\ k \geq 1.$

2. RESULT

Now we prove the following theorem.

Theorem 2.1. *If the sequence* $\mathbf{X}$ *is quasi* β*-power increasing for some* $0 \le \beta < 1$, λ *satisfies the conditions*

$$\sum_{n=1}^{m} \lambda_n = o(m) \tag{2.1}$$

and

$$\sum_{n=1}^{m} |\Delta\,\lambda_n| = o(m), \tag{2.2}$$

furthermore (1.4), (1.5) and

$$\sum_{n=1}^{\infty} n\,X_n(\beta)|\Delta|\Delta\,\lambda_n|| < \infty \tag{2.3}$$

hold, where $X_n(\beta) := \max(n^\beta\,X_n, \log n)$, *then the series* $\sum a_n\,\lambda_n$ *is summable* $|\overline{N}, p_n|_k$, $k \ge 1$.

Remark 2.2. It seems to be worth comparing the assumptions of these theorems.

By Lemma 3.3 it is clear that $(1.7) \Rightarrow (2.1)$, furthermore if $\mathbf{X}$ is quasi increasing then $(1.8) \Rightarrow (2.2)$. It is true that (2.3) in the case $\beta = 0$ claims a little bit more than (1.9) does, but only if $X_n < K\ \log n$. However, in general, $X_n \ge K\ \log n$ holds, see (1.4) and (1.5). In the latter case, Theorem 2.1 under weaker conditions provides the same conclusion as Theorem 1.2.

If we analyze the proofs of Theorem 1.1 and Theorem 1.2, it is easy to see that condition (1.2) replaces (1.7), (1.3) and (1.6) jointly imply (1.8), finally (1.9) requires less than (1.6). Thus we can say that the conditions of Theorem 2.1 also claim less than that of Theorem 1.1.

3. LEMMAS

Later on we shall use the notation $L \ll R$ if there exists a positive constant K such that $L \le K\,R$ holds.

To avoid needless repetition we collect the relevant partial results proved in [3] into a lemma.

In [3] the following inequality is verified implicitly.

Lemma 3.1. *Let* T_n *denote the* n*-th* $(\overline{N}, p_n)$ *mean of the series* $\sum\, a_n\,\lambda_n$. *If* $\{X_n\}$ *is a sequence of positive numbers, and* $\lambda_n \to 0$, *plus (1.7) and (1.5) hold, then*

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k \ll |\lambda_m| X_m + \sum_{n=1}^{m} |\Delta\,\lambda_n| X_n + \sum_{n=1}^{m} |t_n|^k |\Delta\,\lambda_n|.$$

Lemma 3.2 ([7]). *Let* $\{\gamma_n\}$ *be a sequence of real numbers and denote*

$$\Gamma_n := \sum_{k=1}^{n} \gamma_k \qquad \text{and} \qquad R_n := \sum_{k=n}^{\infty} |\Delta\,\gamma_k|.$$

If $\Gamma_n = o(n)$ *then there exists a natural number* $\mathbb{N}$ *such that*

$$|\gamma_n| \le 2R_n$$

for all $n \ge \mathbb{N}$.

Lemma 3.3 ([1, 2.2.2., p. 72]). *If $\{\mu_n\}$ is a positive, monotone increasing and tending to infinity sequence, then the convergence of the series $\sum a_n \mu_n^{-1}$ implies the estimate*

$$\sum_{i=1}^{n} a_i = o(\mu_n).$$

4. PROOF OF THEOREM 2.1

In order to use Lemma 3.1 we first have to show that its conditions follow from the assumptions of Theorem 2.1. Thus we must show that

$$\lambda_n \to 0. \tag{4.1}$$

By Lemma 3.2, condition (2.1) implies that

$$|\lambda_n| \le 2 \sum_{k=n}^{\infty} |\Delta \lambda_k|,$$

and by (2.2)

$$|\Delta \lambda_n| \le 2 \sum_{k=n}^{\infty} |\Delta |\Delta \lambda_k||, \tag{4.2}$$

whence

$$|\lambda_n| \ll \sum_{k=n}^{\infty} n |\Delta |\Delta \lambda_n|| \tag{4.3}$$

holds. Thus (2.3) and (4.3) clearly prove (4.1).

Next we verify (1.7). In view of (4.3) and (2.3)

$$\sum_{n=1}^{\infty} \frac{1}{n} |\lambda_n| \ll \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=n}^{\infty} k |\Delta |\Delta \lambda_k|| \ll \sum_{k=1}^{\infty} k |\Delta |\Delta \lambda_k|| \log k < \infty,$$

that is, (1.7) is satisfied.

In the following steps we show that

$$|\lambda_n| X_n \ll 1, \tag{4.4}$$

$$\sum_{n=1}^{\infty} |\Delta \lambda_n| X_n \ll 1 \tag{4.5}$$

and

$$\sum_{n=1}^{\infty} |t_n|^k |\Delta \lambda_k| \ll 1. \tag{4.6}$$

Utilizing the quasi monotonicity of $\{n^\beta X_n\}$, (2.3) and (4.3) we get that

$$|\lambda_n| X_n \le n^\beta |\lambda_n| X_n \ll \sum_{k=n}^{\infty} k^\beta |X_k| k |\Delta |\Delta \lambda_k|| \ll \sum_{k=n}^{\infty} k\, X_k(\beta) |\Delta |\Delta \lambda_k|| < \infty. \tag{4.7}$$

Similar arguments give that

$$\begin{aligned}\sum_{n=1}^{\infty} |\Delta\,\lambda_n| X_n &\ll \sum_{n=1}^{\infty} X_n \sum_{k=n}^{\infty} |\Delta|\Delta\,\lambda_k|| \\ &= \sum_{k=1}^{\infty} |\Delta|\Delta\,\lambda_k|| \sum_{n=1}^{k} n^{\beta} X_n\, n^{-\beta} \\ &\ll \sum_{k=1}^{\infty} k^{\beta} X_k |\Delta|\Delta\,\lambda_k|| \sum_{n=1}^{k} n^{-\beta} \\ &\ll \sum_{k=1}^{\infty} k\, X_k |\Delta|\Delta\,\lambda_k|| < \infty. \end{aligned} \tag{4.8}$$

Finally to verify (4.6) we apply Abel transformation as follows:

$$\begin{aligned}\sum_{n=1}^{m} |t_n|^k |\Delta\,\lambda_n| &\ll \sum_{n=1}^{m-1} |\Delta(n|\Delta\,\lambda_n|)| \sum_{i=1}^{n} \frac{1}{i} |t_i|^k + m|\Delta\,\lambda_m| \sum_{n=1}^{m} \frac{1}{n} |t_n|^k \\ &\ll \sum_{n=1}^{m-1} n|\Delta|\Delta\,\lambda_n|| X_n + \sum_{n=1}^{m-1} |\Delta\,\lambda_{n+1}| X_{n+1} + m|\Delta\,\lambda_m| X_m. \end{aligned} \tag{4.9}$$

Here the first term is bounded by (2.3), the second one by (4.5), and the third term by (2.3) and (4.2), namely

$$m|\Delta\,\lambda_m| X_m \ll m\, X_m \sum_{n=m}^{\infty} |\Delta|\Delta\,\lambda_n|| \ll \sum_{n=m}^{\infty} n\, X_n |\Delta|\Delta\,\lambda_n|| < \infty. \tag{4.10}$$

Herewith (4.6) is also verified.

Consequently Lemma 3.1 exhibits that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k < \infty,$$

and this completes the proof of our theorem.

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