journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 8 (2007), Issue 2, Article 51, 6 pp.

S-GEOMETRIC CONVEXITY OF A FUNCTION INVOLVING MACLAURIN'S ELEMENTARY SYMMETRIC MEAN

XIAO-MING ZHANG
ZHEJIANG HAINING TV UNIVERSITY,
HAINING, ZHEJIANG,
314400, P. R. CHINA.
zjzxm79@sohu.com

Received 23 February, 2007; accepted 27 April, 2007
Communicated by P.S. Bullen

ABSTRACT. Let $x_i > 0, i = 1, 2, \dots, n$, $x = (x_1, x_2, \dots, x_n)$, the kth elementary symmetric function of x is defined as $E_n(x,k) = \sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} x_{i_j}$ with $1 \le k \le n$, the kth elementary symmetric mean is defined as $P_n(x,k) = \left(\binom{n}{k}^{-1} E_n(x,k)\right)^{\frac{1}{k}}$, and the function f is defined as $f(x) = P_n(x, k-1) - P_n(x,k)$. The paper proves that f is a S-geometrically convex function. The result generalizes the well-known Maclaurin-Inequality.

Key words and phrases: Geometrically convex function, S-geometrically convex function, Inequality, Maclaurin-Inequality, Logarithm majorization.

2000 *Mathematics Subject Classification.* Primary 26D15.

1. INTRODUCTION

Throughout the paper we assume $\mathbb{R}^n$ be the n-dimensional Euclidean Space,

$$\mathbb{R}^n_+ = \{(x_1, x_2, \dots, x_n), x_i > 0, i = 1, 2, \dots, n\},$$

and

$$e^x = (e^{x_1}, e^{x_2}, \dots, e^{x_n}), \quad x^c = (x_1^c, x_2^c, \dots, x_n^c),$$
$$\ln x = (\ln x_1, \ln x_2, \dots, \ln x_n), x \cdot y = (x_1 y_1, x_2 y_2, \dots, x_n y_n),$$

where $c \in \mathbb{R}$, and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. And if $n \ge 2$, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n_+$. They are defined respectively by

$$A_n(x) = \frac{x_1 + x_2 + \cdots + x_n}{n}, \quad G_n(a) = \sqrt[n]{x_1 x_2 \cdots x_n}.$$

The kth elementary symmetric function of x, kth elementary symmetric mean, and function f are defined respectively as

$$E_n(x,k) = \sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} x_{i_j}, \qquad (1 \le k \le n)$$

064-07

$$P_n(x,k)=\left(\binom{n}{k}^{-1}E_n(x,k)\right)^{\frac{1}{k}},\qquad (1\le k\le n)$$
$$f(x)=P_n(x,k-1)-P_n(x,k),\qquad (2\le k\le n)$$

with $\binom{n}{k}=\frac{n!}{k!(n-k)!}$.

The following theorem is true by [1].

Theorem 1.1 (Maclaurin-Inequality)**.**

(1.1) $$A_n(x)=P_n(x,1)\ge P_n(x,2)\ge\cdots\ge P_n(x,n-1)\ge P_n(x,n)=G_n(x).$$

(1.2) $$\frac{E_n(x,n)}{E_n(x,n-1)}<\frac{E_n(x,n-1)}{E_n(x,n-2)}<\cdots<\frac{E_n(x,3)}{E_n(x,2)}<\frac{E_n(x,2)}{E_n(x,1)}<E_n(x,1).$$

References [5], [4], [2], [7], [8], [9], [10] and [6] give the definitions of n dimensional geometrically convex functions, S-geometrically convex functions and logarithm majorization, and a large number of results have been obtained. Since many functions have geometric convexity or geometric concavity, research into geometrically convex functions make sense. For a comprehensive list of recent results on geometrically convex functions, see the book [10] and the papers [5], [4], [2] [7] [8], [9] and [6] where further results are given.

The main aim of this paper is to prove the following theorem.

Theorem 1.2. *Let $n=2$, or $n\ge 3, 2\le k-1\le n-1$, and $f(x)=P_n(x,k-1)-P_n(x,k)$, then f is a $S-$geometrically convex function.*

The result generalizes the Maclaurin-Inequality.

2. RELATIVE DEFINITION AND A LEMMA

Lemma 2.1 ([3])**.** *Let $H\subseteq\mathbb{R}^n$ be a symmetric convex set with a nonempty interior, $\phi: H\to\mathbb{R}$ be continuously differentiable on the interior of H and continuous on H. Necessary and sufficient conditions for ϕ to be S-convex(concave) on H are that ϕ is symmetric on H, and*

$$(x_1-x_2)\left(\frac{\partial\phi}{\partial x_1}-\frac{\partial\phi}{\partial x_2}\right)\ge(\le)\,0,$$

for all x in the interior of H.

Definition 2.1 ([9], [10, p. 89], [6])**.** Let $x\in\mathbb{R}_+^n$, $y\in\mathbb{R}_+^n$, $\left(x_{[1]},x_{[2]},\dots,x_{[n]}\right)$ and $\left(y_{[1]},y_{[2]},\dots,y_{[n]}\right)$ be the decreasing queues of $(x_1,x_2,\dots,x_n)$ and $(y_1,y_2,\dots,y_n)$ respectively. We say $(x_1,x_2,\dots,x_n)$ logarithm majorizes $(y_1,y_2,\dots,y_n)$, denote $\ln x\succ\ln y$ if

$$\begin{cases}\prod_{i=1}^{k}x_i\ge\prod_{i=1}^{k}y_i, & k=1,2,\dots,n-1,\\ \prod_{i=1}^{n}x_i=\prod_{i=1}^{n}y_i.\end{cases}$$

Remark 2.2. If x logarithm majorizes y, then

$$\begin{cases}\ln\left(\prod_{i=1}^{k}x_i\right)\ge\ln\left(\prod_{i=1}^{k}y_i\right), & k=1,2,\dots,n-1,\\ \ln\left(\prod_{i=1}^{n}x_i\right)=\ln\left(\prod_{i=1}^{n}y_i\right).\end{cases}$$

$$\begin{cases} \sum_{i=1}^{k} \ln x_i \geq \sum_{i=1}^{k} \ln y_i, \quad k = 1, 2, \dots, n-1, \\ \sum_{i=1}^{n} \ln x_i = \sum_{i=1}^{n} \ln y_i. \end{cases}$$

So we can denote $\ln x \succ \ln y$ in the Definition 2.1.

Lemma 2.3 ([10, p. 97])**.** $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$ *logarithm majorizes*

$$\bar{G}(x) = (G_n(x), G_n(x), \dots, G_n(x)).$$

Definition 2.2 ([7])**.** Let $E \subseteq \mathbb{R}_+^n$, then E is said to be a logarithm convex set, if for any $x, y \in E$, $\alpha, \beta > 0, \alpha + \beta = 1$, it have $x^\alpha y^\beta \in E$.

Remark 2.4. Let $E \subseteq \mathbb{R}_+^n$, $\ln E = \{\ln x | x \in E\}$. Then $x, y \in E$ if only if $\ln x, \ln y \in \ln E$, and $x^\alpha y^\beta \in E$ if only if $\alpha \ln x + \beta \ln y \in \ln E$, so E is a logarithm convex set if and only if $\ln E$ is a convex set.

Definition 2.3 ([10, p. 107])**.** Let $E \subseteq \mathbb{R}_+^n$, $f : E \to [0, +\infty)$. Then f is called an $S-$geometrically convex function, if for any $x, y \in E \subseteq \mathbb{R}_+^n$, $\ln x \succ \ln y$, we have

$$f(x) \geq f(y). \tag{2.1}$$

And f is called an $S-$geometrically concave function, if the inequality (2.1) is reversed.

Lemma 2.5 ([10, p. 108])**.** *Let $E \subseteq \mathbb{R}_+^n$ be a symmetric logarithm convex set with a nonempty interior, $f : E \to [0, +\infty)$ be symmetric continuously differentiable on the interior of E and continuous on E. Then f is a S-geometrically convex function, if the following inequality*

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2}\right) \geq 0 \tag{2.2}$$

holds for all x in the interior of E. And f a is S-geometrically concave function, if the inequality (2.2) *is reversed.*

Proof. Let $\ln E = \{\ln x | x \in E\}$, then $\ln E$ is a symmetric convex set and has a nonempty interior. Again, let $\varepsilon > 0$, $g(x) = f(x) + \varepsilon$ with $x \in E$, and $h(y) = \ln g(e^y)$ with $y \in \ln E = \{\ln x | x \in E\}$, then $g : E \to (0, +\infty)$, $h : \ln E \to (-\infty, +\infty)$. Further let $x = e^y$,

$$\begin{aligned} \frac{\partial h}{\partial y_1} &= \frac{\partial (\ln g(e^y))}{\partial y_1} \\ &= \frac{1}{g(e^y)} \cdot \frac{\partial (g(e^y))}{\partial y_1} \\ &= \frac{1}{g(x)} \cdot \frac{\partial (g(x))}{\partial x_1} \cdot e^{y_1} \\ &= \frac{x_1}{g(x)} \cdot \frac{\partial g}{\partial x_1}. \end{aligned}$$

Similarly,

$$\frac{\partial h}{\partial y_2} = \frac{x_2}{g(x)} \cdot \frac{\partial g}{\partial x_2}.$$

According to inequality 2.2,

$$(y_1-y_2)\left(\frac{\partial h}{\partial y_1}-\frac{\partial h}{\partial y_2}\right)=\frac{(\ln x_1-\ln x_2)}{g(x)}\left(x_1\frac{\partial g}{\partial x_1}-x_2\frac{\partial g}{\partial x_2}\right)$$
$$=\frac{(\ln x_1-\ln x_2)}{g(x)}\left(x_1\frac{\partial f}{\partial x_1}-x_2\frac{\partial f}{\partial x_2}\right)\geq 0.$$

Then by Lemma 2.1, we know that h is a S-convex function. For any $u,v\in E$ with $\ln u \succ \ln v$, we have

$$h(\ln u)\geq h(\ln v)\,,\quad \ln g\left(e^{\ln u}\right)\geq \ln g\left(e^{\ln v}\right),$$

and

$$g(u)\geq g(v)\,,\quad f(u)\geq f(v)\,.$$

So f is a S- geometrically convex function.

If the inequality (2.2) is reversed, we similarly have that f is a S-geometrically concave function.

The proof of Lemma 2.5 is completed. □

3. THE PROOF OF THEOREM 1.2

Proof of Theorem 1.2. If $n=2$, then $k=2$.

$$f(x)=P_n(x,k-1)-P_n(x,k)=\frac{x_1+x_2}{2}-\sqrt{x_1x_2},$$

$$\frac{\partial f}{\partial x_1}=\frac{1}{2}-\frac{1}{2}\sqrt{\frac{x_2}{x_1}},\quad x_1\frac{\partial f}{\partial x_1}=\frac{1}{2}x_1-\frac{1}{2}\sqrt{x_1x_2},$$

$$(\ln x_1-\ln x_2)\left(x_1\frac{\partial f}{\partial x_1}-x_2\frac{\partial f}{\partial x_2}\right)=(\ln x_1-\ln x_2)\left(\frac{x_1-x_2}{2}\right)\geq 0.$$

According to Lemma 2.5, if $n=2$, Theorem 1.2 is true.

If $n\geq 3$, $k\geq 3$, Letting $\bar{x}=(x_3,x_4,\dots,x_n)$, $E_{n-2}(\bar{x},0)=1$, we have

$$f(x)=P_n(x,k-1)-P_n(x,k)$$
$$=\left(\binom{n}{k-1}^{-1}E_n(x,k-1)\right)^{\frac{1}{k-1}}-\left(\binom{n}{k}^{-1}E_n(x,k)\right)^{\frac{1}{k}},$$

$$\frac{\partial f}{\partial x_1}=\frac{1}{k-1}\cdot\binom{n}{k-1}^{-\frac{1}{k-1}}\cdot(E_n(x,k-1))^{\frac{1}{k-1}-1}\sum_{2\leq i_1<\dots<i_j\leq n}\prod_{j=1}^{k-2}x_{i_j}$$
$$-\frac{1}{k}\cdot\binom{n}{k}^{-\frac{1}{k}}\cdot(E_n(x,k))^{\frac{1}{k}-1}\sum_{2\leq i_1<\dots<i_j\leq n}\prod_{j=1}^{k-1}x_{i_j},$$

$$x_1\frac{\partial f}{\partial x_1}=\frac{1}{k-1}\cdot\binom{n}{k-1}^{-\frac{1}{k-1}}$$
$$\cdot(E_n(x,k-1))^{\frac{1}{k-1}-1}\left[x_1E_{n-2}(\bar{x},k-2)+x_1x_2E_{n-2}(\bar{x},k-3)\right]$$
$$-\frac{1}{k}\cdot\binom{n}{k}^{-\frac{1}{k}}\cdot(E_n(x,k))^{\frac{1}{k}-1}\left[x_1E_{n-2}(\bar{x},k-1)+x_1x_2E_{n-2}(\bar{x},k-2)\right],$$

and

$$x_2 \frac{\partial f}{\partial x_2} = \frac{1}{k-1} \cdot \binom{n}{k-1}^{-\frac{1}{k-1}} \cdot \left(E_n\left(x, k-1\right)\right)^{\frac{1}{k-1}-1} \left[x_2 E_{n-2}\left(\bar{x}, k-2\right) + x_1 x_2 E_{n-2}\left(\bar{x}, k-3\right)\right] - \frac{1}{k} \cdot \binom{n}{k}^{-\frac{1}{k}} \cdot \left(E_n\left(x, k\right)\right)^{\frac{1}{k}-1} \left[x_2 E_{n-2}\left(\bar{x}, k-1\right) + x_1 x_2 E_{n-2}\left(\bar{x}, k-2\right)\right].$$

So

$$(3.1)\quad (\ln x_1 - \ln x_2)\left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2}\right) = (\ln x_1 - \ln x_2) \cdot \frac{1}{k-1} \cdot \binom{n}{k-1}^{-\frac{1}{k-1}} \cdot \left(E_n\left(x, k-1\right)\right)^{\frac{1}{k-1}-1} \cdot (x_1 - x_2) \cdot E_{n-2}\left(\bar{x}, k-2\right) - (\ln x_1 - \ln x_2) \cdot \frac{1}{k} \cdot \binom{n}{k}^{-\frac{1}{k}} \cdot \left(E_n\left(x, k\right)\right)^{\frac{1}{k}-1} \cdot (x_1 - x_2) \cdot E_{n-2}\left(\bar{x}, k-1\right).$$

On the other hand, by (1.2), we deduce

$$(x_1 + x_2) E_{n-2}\left(\bar{x}, k-1\right) \cdot E_{n-2}\left(\bar{x}, k-2\right) + x_1 x_2 \left[k E_{n-2}^2\left(\bar{x}, k-2\right) - (k-1) E_{n-2}\left(\bar{x}, k-3\right) \cdot E_{n-2}\left(\bar{x}, k-1\right)\right] \geq 0,$$

so

$$k \cdot \left[(x_1 + x_2) E_{n-2}\left(\bar{x}, k-1\right) + x_1 x_2 E_{n-2}\left(\bar{x}, k-2\right)\right] \cdot E_{n-2}\left(\bar{x}, k-2\right) - (k-1) \cdot \left[(x_1 + x_2) E_{n-2}\left(\bar{x}, k-2\right) + x_1 x_2 E_{n-2}\left(\bar{x}, k-3\right)\right] \cdot E_{n-2}\left(\bar{x}, k-1\right) \geq 0,$$

$$k \cdot E_n\left(x, k\right) \cdot E_{n-2}\left(\bar{x}, k-2\right) - (k-1) \cdot E_n\left(x, k-1\right) \cdot E_{n-2}\left(\bar{x}, k-1\right) \geq 0.$$

Again, according to (1.1), the following inequality holds.

$$k \cdot P_n\left(x, k-1\right) \cdot E_n\left(x, k\right) \cdot E_{n-2}\left(\bar{x}, k-2\right) - (k-1) \cdot P_n\left(x, k\right) \cdot E_n\left(x, k-1\right) \cdot E_{n-2}\left(\bar{x}, k-1\right) \geq 0,$$

then

$$\frac{1}{k-1} \cdot \binom{n}{k-1}^{-\frac{1}{k-1}} \cdot \left(E_n\left(x, k-1\right)\right)^{\frac{1}{k-1}-1} \cdot E_{n-2}\left(\bar{x}, k-2\right) - \frac{1}{k} \cdot \binom{n}{k}^{-\frac{1}{k}} \cdot \left(E_n\left(x, k\right)\right)^{\frac{1}{k}-1} \cdot E_{n-2}\left(\bar{x}, k-1\right) \geq 0.$$

Finally by (3.1), we can state that

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2}\right) \geq 0,$$

Thus Theorem 1.2 holds by Lemma 2.5. □

Remark 3.1. If $n \geq 3, k = 2$, we know that f is neither a S-geometrically convex function nor a S-geometrically concave function.

Remark 3.2. If $n = 2$, or $n \geq 3, 2 \leq k-1 < k \leq n$, and x logarithm majorizes y, according to Definition 2.3, we can state that

$$P_n\left(x, k-1\right) - P_n\left(x, k\right) \geq P_n\left(y, k-1\right) - P_n\left(y, k\right).$$

By Lemma 2.3 and

$$P_n\left(\bar{G}(x), k-1\right) - P_n\left(\bar{G}(x), k\right) = 0,$$

we know that Theorem 1.2 generalizes the Maclaurin-Inequality.

REFERENCES

[1] J.-C. KUANG, *Applied Inequalities*, Jinan: Shandong Science and Techology Press, 3rd. Ed., 2004. (Chinese)

[2] S.J. LI, An important generating function inequality, *Journal of Fuzhou Teachers College* (Natural Science Edition), **18**(1) (1999), 37–40. (Chinese)

[3] A.W. MARSHALL AND I. OLKIN, *Inequalities: Theory of Majorization and its Applications*, London, Academic Press, 1979, 54–57.

[4] J. MATKOWSKI, L^p-like paranorms, Selected Topics in Functional Equations and Iteration Theory, Proceedings of the Austrian-Polish Seminar, *Graz Math. Ber.*, **316** (1992), 103–138.

[5] P. MONTEL, Sur les functions convéxes et les fonctions sousharmoniques, *Journal de Math.*, **7**(9) (1928), 29–60.

[6] C.P. NICULESCU, Convexity according to the geometric mean, *Mathematical Inequalities & Applications*, **3**(2) (2000), 155–167.

[7] X.M. ZHANG, An inequality of the Hadamard type for the geometrical convex functions, *Mathematics in Practice and Theory*, **34**(9) (2004), 171–176. (Chinese)

[8] X.M. ZHANG, Some theorems on geometric convex functions and its applictions, *Journal of Capital Normal University*, **25**(2) (2004), 11–13. (Chinese)

[9] D.H. YANG, About inequality of geometric convex functions, *Hebei University Learned Journal* (Natural Science Edition), **22**(4) (2002), 325–328. (Chinese)

[10] X.M. ZHANG, *Geometrically Convex Functions*, Hefei: An'hui University Press, 2004. (Chinese)

[11] X.M. ZHANG AND Y.D. WU, Geometrically convex functions and solution of a question, *RGMIA Res. Rep. Coll.*, **7**(4) (2004), Art. 11. [ONLINE: `http://rgmia.vu.edu.au/v7n4.html`].

[12] N.G. ZHENG AND X.M. ZHANG, Some monotonicity questions for geometrically convex functions, *RGMIA Res. Rep. Coll.*, **7**(3) (2004), Art. 20. [ONLINE: `http://rgmia.vu.edu.au/v7n3.html`].