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A NEW TYPE OF STABLE GENERALIZED CONVEX FUNCTIONS

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ABSTRACT. S-quasiconvex functions (Phu and An, Optimization, Vol. 38, 1996) are stable with respect to the properties: "every lower level set is convex", "each local minimizer is a global minimizer", and "each stationary point is a global minimizer" (i.e., these properties remain true if a sufficiently small linear disturbance is added to a function of this class). In this paper, we introduce a subclass of s-quasiconvex functions, namely strictly s-quasiconvex functions which guarantee the uniqueness of the minimizer. The density of the set of these functions in the set of s-quasiconvex functions and some necessary and sufficient conditions of these functions are presented.

Key words and phrases: Generalized convexity, s-quasiconvex function, Stability.

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1. INTRODUCTION

A function f is said to be stable with respect to some property (P) if there exists $\epsilon > 0$ such that $f + \xi$ fulfills (P) for all linear functions ξ satisfying $\|\xi\| < \epsilon$. It was shown in [4] that well-known kinds of generalized convex functions are often not stable with respect to the property they have to keep during the generalization, for example, quasiconvex functions (pseudoconvex functions, respectively) are not stable with respect to the property "every lower level set is convex" ("each stationary point is a global minimizer", respectively). Then the so-called s-quasiconvex functions were introduced in [4]. They are stable with respect to the properties "every lower level set is convex", "each local minimizer is a global minimizer" and "each stationary point is a global minimizer".

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Unfortunately, the uniqueness of the minimizer of *s*-quasiconvex functions does not hold while this property is included often in the sufficient conditions for the continuity of optimal solutions to parametric optimization problems (see [3]).

In this paper, we introduce strictly *s*-quasiconvex functions which guarantee the uniqueness of the minimizer. Proposition 2.3 says that under certain assumptions, we can approximate affine parts of a *s*-quasiconvex function defined on $D \subset \mathbb{R}$, by strictly convex functions to obtain a strictly *s*-quasiconvex function. Strictly *s*-quasiconvex functions are stable with respect to strict pseudoconvexity (Theorem 2.6). Finally, the necessary and sufficient conditions for a continuously differentiable function to be strictly *s*-quasiconvex are stated (Theorems 3.1 – 3.2).

From [5] and [6] the following definitions and properties are used: Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ and D be open and convex. We recall that:

f is said to be convex if, for all $x_0, x_1 \in D, \lambda \in [0,1]$,

$$f(x_\lambda) \leq (1-\lambda) f(x_0) + \lambda f(x_1), \tag{1.1}$$

where $x_\lambda = (1-\lambda)x_0 + \lambda x_1$. f is said to be strictly convex if (1.1) is a strict inequality for every distinct $x_0, x_1 \in D$.

f is said to be quasiconvex if, for all $x_0, x_1 \in D, \lambda \in [0,1]$,

$$f(x_0) \leq f(x_1) \text{ implies } f(x_\lambda) \leq f(x_1). \tag{1.2}$$

f is said to be strictly quasiconvex if the second inequality in (1.2) is strict, for every distinct $x_0, x_1 \in D, \lambda \in]0,1[$. Note that the concept "strict quasiconvexity" here is exactly the "XC" concept in [5].

A differentiable function f is said to be pseudoconvex if, for all $x_0, x_1 \in D$,

$$f(x_0) < f(x_1) \text{ implies } (x_0 - x_1)^T \nabla f(x_1) < 0, \tag{1.3}$$

where ^{T} is the matrix transposition. A differentiable function f is said to be strictly pseudoconvex if the first inequality in (1.3) is not strict, for every distinct $x_0, x_1 \in D$.

We also recall the definition of s-quasiconvex functions (“s" stands for “stable"). f is said to be s-quasiconvex if there exists $\sigma > 0$ such that

$$\frac{f(x_0) - f(x_1)}{\|x_0 - x_1\|} \leq \delta \quad \text{implies} \quad \frac{f(x_\lambda) - f(x_1)}{\|x_\lambda - x_1\|} \leq \delta \tag{1.4}$$

for $|\delta| < \sigma, x_0, x_1 \in D, x_\lambda = (1-\lambda)x_0 + \lambda x_1$ and $\lambda \in [0,1[$ ([4]).

Clearly, every convex function is s-quasiconvex and a s-quasiconvex function is quasiconvex. The following are some properties of s-quasiconvexity given in [4].

Theorem 1.1 ([4]). *Suppose $f : D \subset \mathbb{R}^n \to \mathbb{R}$.*

a) *f is s-quasiconvex iff there exists $\epsilon > 0$ such that $f + \xi$ is quasiconvex for each linear function ξ on $\mathbb{R}^n$ satisfying $\|\xi\| < \epsilon$;*
b) *f is s-quasiconvex iff there exists $\epsilon > 0$ such that $f + \xi$ is s-quasiconvex for each linear function ξ on $\mathbb{R}^n$ satisfying $\|\xi\| < \epsilon$;*
c) *A continuously differentiable function f is s-quasiconvex iff there exists $\epsilon > 0$ such that $f + \xi$ is pseudoconvex for each linear function ξ on $\mathbb{R}^n$ satisfying $\|\xi\| < \epsilon$.*

We will show that, in (1.4), both inequalities can be replaced by strict inequalities and first inequalities can be replaced by strict inequalities.

Proposition 1.2. *The following statements are equivalent:*

a) *$f : D \subset \mathbb{R}^n \to \mathbb{R}$ is s-quasiconvex;*

b) *There exists* $\sigma > 0$ *such that*

$$\frac{f(x_0)-f(x_1)}{\|x_0-x_1\|} < \delta \quad \textit{implies} \quad \frac{f(x_\lambda)-f(x_1)}{\|x_\lambda-x_1\|} < \delta \tag{1.5}$$

for $|\delta| < \sigma,\ x_0,\ x_1 \in D$ *and* $\lambda \in [0,1[\,$;

c) *There exists* $\sigma > 0$ *such that*

$$\frac{f(x_0)-f(x_1)}{\|x_0-x_1\|} < \delta \quad \textit{implies} \quad \frac{f(x_\lambda)-f(x_1)}{\|x_\lambda-x_1\|} \le \delta \tag{1.6}$$

for $|\delta| < \sigma,\ x_0,\ x_1 \in D$ *and* $\lambda \in [0,1[$.

Proof. $a) \Rightarrow b)$ Suppose that f is s-quasiconvex and $\sigma > 0$ is given in the definition of s-quasiconvex function f. Let $x_0, x_1 \in D$ and $\frac{f(x_0)-f(x_1)}{\|x_1-x_0\|} < \delta$ with $|\delta| < \sigma$. Take δ_1 such that $|\delta_1| < \sigma$ and $\frac{f(x_0)-f(x_1)}{\|x_1-x_0\|} < \delta_1 < \delta$ then $\frac{f(x_\lambda)-f(x_1)}{\|x_\lambda-x_1\|} \le \delta_1 < \delta$. Hence, (1.5) holds true.

$b) \Rightarrow c)$ It is trivial, since (1.5) implies (1.6) with the same $\sigma > 0$.

$c) \Rightarrow a)$ Suppose that f satisfies (1.6) and $\frac{f(x_0)-f(x_1)}{\|x_1-x_0\|} \le \delta$ with $|\delta| < \sigma$. Then, for each $\delta_1 \in]\delta, \sigma[\,$, we have $(f(x_0)-f(x_1))/\|x_1-x_0\| < \delta_1$. By (1.6), $\frac{f(x_\lambda)-f(x_1)}{\|x_\lambda-x_1\|} \le \delta_1$ with $\lambda \in [0,1[\,$. Hence $\frac{f(x_\lambda)-f(x_1)}{\|x_\lambda-x_1\|} \le \delta$ with $\lambda \in [0,1[\,$. Thus, f is s-quasiconvex. □

As we see from Proposition 1.2, in (1.4), replacing both inequalities by strict inequalities and replacing first inequalities by strict inequalities will not rise to new types of generalized convexity. In the following section, we replace second inequalities by strict inequalities, and in this way we shall generate a new type of generalized convexity.

2. STRICTLY s-QUASICONVEX FUNCTIONS

Let us introduce the notion of strictly s-quasiconvex functions

Definition 2.1. $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is said to be *strictly s-quasiconvex* if there exists $\sigma > 0$ such that

$$\frac{f(x_0)-f(x_1)}{\|x_0-x_1\|} \le \delta \quad \text{implies} \quad \frac{f(x_\lambda)-f(x_1)}{\|x_\lambda-x_1\|} < \delta \tag{2.1}$$

for $|\delta| < \sigma, x_0, x_1 \in D$, $x_0 \neq x_1, x_\lambda = (1-\lambda)x_0 + \lambda x_1$ and $\lambda \in]0,1[\,$.

Clearly, a strictly convex function f is strictly s-quasiconvex. Furthermore, every strictly s-quasiconvex function is s-quasiconvex and every strictly s-quasiconvex function is strictly quasiconvex.

Theorem 2.1. *A function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is strictly s-quasiconvex iff there exists $\epsilon > 0$ such that $f + \xi$ is strictly quasiconvex for each linear function ξ on $\mathbb{R}^n$ satisfying $\|\xi\| < \epsilon$.*

Proof. (a) Necessity: Assume that f is strictly s-quasiconvex. Choose $\epsilon = \sigma$ and suppose ξ is a linear function satisfying $\|\xi\| < \epsilon$, where σ is given in Definition 2.1. Then

$$\frac{f(x_0)-f(x_1)}{\|x_1-x_0\|} \le \xi\left(\frac{x_1-x_0}{\|x_1-x_0\|}\right) = \xi\left(\frac{x_1-x_\lambda}{\|x_1-x_\lambda\|}\right),$$

for every distinct $x_0, x_1 \in D$ satisfying $f(x_0) + \xi(x_0) \le f(x_1) + \xi(x_1)$ and for all $\lambda \in]0,1[$. Since

$$\left|\xi\left(\frac{x_1-x_\lambda}{\|x_1-x_\lambda\|}\right)\right| \le \|\xi\| < \epsilon = \sigma,$$

and since f is strictly s-quasiconvex, we have

$$\frac{f(x_\lambda)-f(x_1)}{\|x_\lambda - x_1\|} < \xi\left(\frac{x_1 - x_\lambda}{\|x_1 - x_\lambda\|}\right).$$

Therefore, $f(x_\lambda)+\xi(x_\lambda) < f(x_1)+\xi(x_1)$, i.e., $f+\xi$ is strictly quasiconvex.

(b) Sufficiency: Suppose that there exists $\epsilon > 0$ such that $f+\xi$ is strictly quasiconvex for each linear function ξ on $\mathbb{R}^n$ satisfying $\|\xi\| < \epsilon$. Choose $\sigma = \epsilon$ and suppose that $x_0, x_1 \in D$ satisfy $\frac{f(x_0)-f(x_1)}{\|x_1-x_0\|} \le \delta$ with $|\delta| < \epsilon$. By the Hahn-Banach theorem, there exists a linear function ξ satisfying $\|\xi\| = \delta$ and $\xi\left(\frac{x_1-x_0}{\|x_1-x_0\|}\right) = \delta$. Then,

$$\frac{f(x_0)-f(x_1)}{\|x_1 - x_0\|} \le \xi\left(\frac{x_1 - x_0}{\|x_1 - x_0\|}\right).$$

Hence, $f(x_0)+\xi(x_0) \le f(x_1)+\xi(x_1)$. Since $f+\xi$ is strictly quasiconvex, we have $f(x_\lambda) + \xi(x_\lambda) < f(x_1)+\xi(x_1)$ for all $\lambda \in]0,1[$. It follows that

$$\frac{f(x_\lambda)-f(x_1)}{\|x_\lambda - x_1\|} < \xi\left(\frac{x_1 - x_\lambda}{\|x_1 - x_\lambda\|}\right) = \xi\left(\frac{x_1 - x_0}{\|x_1 - x_0\|}\right) = \delta$$

for all $\lambda \in]0,1[$. □

We now consider the density of the set of strictly s-quasiconvex functions in the set of s-quasiconvex functions.

Proposition 2.2. *If a s-quasiconvex $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is not strictly s-quasiconvex then it is affine on a certain interval in D.*

Proof. Suppose that f is not strictly s-quasiconvex. Since f is s-quasiconvex, there exists $\epsilon > 0$ such that $f+\xi$ is quasiconvex for each linear function ξ on $\mathbb{R}^n$ satisfying $\|\xi\| < \epsilon$ (Theorem 1.1). On the other hand, in view of Theorem 2.1, $f+\xi$ is not strictly quasiconvex for some linear function ξ on $\mathbb{R}^n$ satisfying $\|\xi\| < \epsilon$. Since $f+\xi$ is quasiconvex, we conclude that $f+\xi$ is constant on a certain interval. Hence, f is affine on this interval. □

Proposition 2.3. *Suppose that $f :]a,b[\subset \mathbb{R} \to \mathbb{R}$ is s-quasiconvex and let $\epsilon > 0$. If it is affine only on a finite number of intervals $[a_i, b_i] \subset]a,b[$, $(i = 1,2,\dots,k)$ then there exist strictly convex functions g_i defined on $[a_i,b_i]$ $(i=1,2,\dots,k)$ such that*

$$h(x) = \begin{cases} g_i(x) & \text{if } x \in [a_i,b_i]\ (i=1,2,\dots,k), \\ f(x) & \text{if } x \in]a,b[\setminus \cup_{i=1,2,\dots,k} [a_i,b_i] \end{cases}$$

is strictly s-quasiconvex and $\|f-h\| := \sup_{x\in]a,b[} |f(x)-h(x)| < \epsilon$.

Proof. Assume without loss of generality that f is affine only on $[a_1,b_1]$. By Theorem 1.1 (a), there exists $\epsilon_0 > 0$ such that $f+\xi$ is quasiconvex for each linear function ξ on $\mathbb{R}$ satisfying $\|\xi\| < \epsilon_0$. Assume without loss of generality that $f(a_1) \le f(b_1)$.

First, consider the case $f(a_1) < f(b_1)$. Choose $g_1(x) := \alpha x^2 + \beta x + \gamma$, $(\alpha,\beta,\gamma \in \mathbb{R}, \alpha > 0)$ such that

$$\begin{aligned} g_1(a_1) = f(a_1),\ \ g_1(b_1) &= f(b_1) \\ 0 &< g_1'(a_1) \\ \epsilon &> \sup_{x\in[a_1,b_1]} |f(x) - g_1(x)|. \end{aligned}$$

We are now in a position to show that the sum of the function

$$h(x) = \begin{cases} g_1(x) & \text{if } x \in [a_1, b_1], \\ f(x) & \text{if } x \in]a, b[\setminus [a_1, b_1] \end{cases}$$

and ξ is quasiconvex for each linear function ξ satisfying $\|\xi\| < \min\{\epsilon_0, g_1'(a_1)\}$. Suppose that $\xi(x) = -ax$, $a > 0$. Since $a < g_1'(a_1)$ and g_1 is strictly convex on $[a_1, b_1]$, $f(a_1) + a(x - a_1) < f(a_1) + g_1'(a_1)(x - a_1) < g_1(x)$ for every $x \in]a_1, b_1]$. It follows that $g_1(a_1) - aa_1 = f(a_1) - aa_1 < g_1(x) - ax$. Hence,

$$g_1(a_1) + \xi(a_1) < g_1(x) + \xi(x). \tag{2.2}$$

for every $x \in]a_1, b_1]$. Let $x_0, x_1 \in]a, b[\subset \mathbb{R}$ and $\lambda \in]0, 1[$.

We now consider the case $x_0 \in]-\infty, a_1[\cap]a, b[$ and $x_1 \in [a_1, b_1]$. If $x_\lambda \in [a_1, x_1]$ then, by quasiconvexity of $g_1 + \xi$ and by (2.2) (with $x = x_1$),

$$\begin{aligned} h(x_\lambda) + \xi(x_\lambda) &= g_1(x_\lambda) + \xi(x_\lambda) \\ &\le \max\{g_1(a_1) + \xi(a_1), g_1(x_1) + \xi(x_1)\} \\ &= g_1(x_1) + \xi(x_1) = h(x_1) + \xi(x_1). \end{aligned}$$

If $x_\lambda \in [x_0, a_1[$ then, by quasiconvexity of $f + \xi$ and by (2.2) (with $x = x_1$),

$$\begin{aligned} h(x_\lambda) + \xi(x_\lambda) &= f(x_\lambda) + \xi(x_\lambda) \\ &\le \max\{f(x_0) + \xi(x_0), f(a_1) + \xi(a_1)\} \\ &\le \max\{f(x_0) + \xi(x_0), g_1(x_1) + \xi(x_1)\} \\ &= \max\{h(x_0) + \xi(x_0), h(x_1) + \xi(x_1)\}. \end{aligned}$$

Similarly, if either $x_0 \in]-\infty, a_1[\cap]a, b[$ and $x_1 \in]b_1, +\infty] \cap]a, b[$ or $x_0 \in [a_1, b_1]$ and $x_1 \in]b_1, +\infty[\cap]a, b[$, we have

$$h(x_\lambda) + \xi(x_\lambda) \le \max\{h(x_0) + \xi(x_0), h(x_1) + \xi(x_1)\}$$

for all $x_\lambda \in [x_0, x_1]$. It implies that $h + \xi$ is quasiconvex for each linear function ξ satisfying $\|\xi\| < \min\{\epsilon_0, \epsilon_1\}$. By Theorem 1.1 (a), h is s-quasiconvex.

On the other hand, since f is not affine on any interval contained in $D \setminus [a_1, b_1]$ and g_1 is strictly convex, h is not affine on any intervals. By Proposition 2.2, h is strictly s-quasiconvex. Since $\sup_{x \in [a_1, b_1]} |f(x) - g_1(x)| < \epsilon$, we conclude that $\|f - h\| < \epsilon$.

Finally, we consider the case $f(a_1) = f(b_1)$. By Theorem 1.1 (b), there exists $\epsilon_0 > 0$ such that $f + \xi$ is s-quasiconvex for each linear function ξ on $\mathbb{R}$ satisfying $\|\xi\| < \min\{\epsilon/2, \epsilon_0\}$. Set $\bar{f} = f + \xi$ where ξ is a linear function on $\mathbb{R}$ satisfying $\|\xi\| < \min\{\epsilon/2, \epsilon_0\}$ and $\xi(a_1) < \xi(b_1)$. Then $\bar{f}$ is s-quasiconvex, affine on $[a_1, b_1]$ and $\bar{f}(a_1) < \bar{f}(b_1)$. Applying the above case, there exists a strictly s-quasiconvex function h such that $\|\bar{f} - h\| < \epsilon/2$. It follows that

$$\|f - h\| = \|f + \xi - h + \xi\| \le \|\bar{f} - h\| + \|\xi\| < \epsilon.$$

□

From Proposition 2.3, we have the following.

Corollary 2.4. *The set of strictly s-quasiconvex functions defined on $D =]a, b[\subset \mathbb{R}$ is dense in the set ω of s-quasiconvex functions, which are affine only on a finite number of intervals in $]a, b[$.*

We do not know whether the conclusion of Corollary 2.4 holds for the case $D \subset \mathbb{R}^n$, $n > 1$. Note that the uniqueness of the minimizer of strictly s-quasiconvex functions follows directly from the uniqueness of the minimizer of strictly quasiconvex functions.

We now consider continuously differentiable functions.

Lemma 2.5. *If $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is strictly pseudoconvex then it is strictly quasiconvex.*

Proof. Suppose that f is strictly pseudoconvex. Let $x_0, x_1 \in D$, $x_0 \neq x_1$ be such that $f(x_0) \leq f(x_1)$. We want to show that $f(x_\lambda) < f(x_1)$ for all $x_\lambda \in]x_0, x_1[$. Assume the contrary that there exists $x_{\bar\lambda} \in]x_0, x_1[$ such that

$$f(x_{\bar\lambda}) \geq f(x_1) \geq f(x_0).$$

By the strictly pseudoconvexity of f, we have

$$(x_1 - x_{\bar\lambda})^T \nabla f(x_{\bar\lambda}) < 0 \text{ and } (x_0 - x_{\bar\lambda})^T \nabla f(x_{\bar\lambda}) < 0. \tag{2.3}$$

Set $s{:} = (x_1 - x_{\bar\lambda})/\|x_1 - x_{\bar\lambda}\|$ then $-s{:} = (x_0 - x_{\bar\lambda})/\|x_0 - x_{\bar\lambda}\|$. It follows from (2.3) that $s^T \nabla f(x_{\bar\lambda}) < 0$ and $-s^T \nabla f(x_{\bar\lambda}) < 0$, a contradiction. □

Theorem 2.6. *Suppose that $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Then, f is strictly s-quasiconvex iff there exists $\epsilon > 0$ such that $f + \xi$ is strictly pseudoconvex for each linear function ξ on $\mathbb{R}^n$ satisfying $\|\xi\| < \epsilon$.*

Proof. (a) Necessity: Assume that f is strictly s-quasiconvex. Then, it is s-quasiconvex. By Theorem 1.1, there exists $\epsilon_1 > 0$ such that $f + \xi$ is pseudoconvex for each linear function ξ on $\mathbb{R}^n$ satisfying $\|\xi\| < \epsilon_1$. On the either hand, by Theorem 2.1, there exists $\epsilon_2 > 0$ such that $f + \xi$ is strictly quasiconvex for each linear function ξ on $\mathbb{R}^n$ satisfying $\|\xi\| < \epsilon_2$. Therefore, $f + \xi$ is pseudoconvex and strictly quasiconvex for each linear function ξ on $\mathbb{R}^n$ satisfying $\|\xi\| < \epsilon{:} = \min\{\epsilon_1, \epsilon_2\}$. Thus $f + \xi$ is pseudoconvex and XC (see [5]). By Theorem 1 [5], $f + \xi$ is strictly pseudoconvex for each linear function ξ on $\mathbb{R}^n$ satisfying $\|\xi\| < \epsilon$.

(b) Sufficiency: Suppose that there exists $\epsilon > 0$ such that $f + \xi$ is strictly pseudoconvex for each linear function ξ on $\mathbb{R}^n$ satisfying $\|\xi\| < \epsilon$. By Lemma 2.5, $f + \xi$ is strictly quasiconvex. According to Theorem 2.1, f is strictly s-quasiconvex. □

3. NECESSARY AND SUFFICIENT CONDITIONS FOR STRICTLY s-QUASICONVEX FUNCTIONS

Our next objective is to give necessary and sufficient conditions for a continuously differentiable function to be strictly s-quasiconvex.

Theorem 3.1. *Suppose that $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Then, f is strictly s-quasiconvex iff there exists $\sigma > 0$ such that*

$$\frac{f(x_0) - f(x_1)}{\|x_0 - x_1\|} \leq \delta \quad \textit{implies} \quad \frac{(x_0 - x_1)^T}{\|x_0 - x_1\|} \nabla f(x_1) < \delta \tag{3.1}$$

for all $|\delta| < \sigma, x_0, x_1 \in D$.

Proof. (a) Necessity: Assume that f is strictly s-quasiconvex. Then, by Theorem 2.6, there exists $\epsilon > 0$ such that $f + \xi$ is strictly pseudoconvex for each linear function ξ on $\mathbb{R}^n$ satisfying $\|\xi\| < \epsilon$. Set $\sigma{:} = \epsilon$. Suppose that $x_0, x_1 \in D$, and $\frac{f(x_0) - f(x_1)}{\|x_1 - x_0\|} \leq \delta$ for $|\delta| < \epsilon$. Choose a linear function ξ such that $\|\xi\| = \delta$ and $\xi\left((x_1 - x_0)/\|x_1 - x_0\|\right) = \delta$. Then, $f(x_0) + \xi(x_0) \leq f(x_1) + \xi(x_1)$. Since $f + \xi$ is strictly pseudoconvex,

$$\frac{(x_0 - x_1)^T}{\|x_0 - x_1\|} \nabla (f + \xi)(x_1) < 0.$$

Clearly, ξ can be expressed in the form $\xi(x) = x^T a$, with some $a \in \mathbb{R}^n$. Hence,

$$\begin{aligned} 0 &> \frac{(x_0 - x_1)^T}{\|x_1 - x_0\|} \nabla (f + \xi)(x_1) \\ &= \frac{(x_0 - x_1)^T}{\|x_1 - x_0\|} \nabla f(x_1) + \frac{(x_0 - x_1)^T}{\|x_1 - x_0\|} \nabla \xi(x_1) \\ &= \frac{(x_0 - x_1)^T}{\|x_1 - x_0\|} \nabla f(x_1) + \frac{(x_0 - x_1)^T}{\|x_1 - x_0\|} a \\ &= \frac{(x_0 - x_1)^T}{\|x_1 - x_0\|} \nabla f(x_1) + \xi\left(\frac{x_0 - x_1}{\|x_1 - x_0\|}\right). \end{aligned}$$

Thus,

$$\frac{(x_0 - x_1)^T}{\|x_0 - x_1\|} \nabla f(x_1) < \xi\left(\frac{x_1 - x_0}{\|x_1 - x_0\|}\right) = \delta.$$

Therefore, (3.1) holds true.

(b) Sufficiency: Suppose that there exists $\sigma > 0$ satisfying (3.1). We prove that f is strictly s-quasiconvex. Suppose that $\frac{f(x_0) - f(x_1)}{\|x_1 - x_0\|} \le \delta$ with $|\delta| < \sigma$. Choose a linear function ξ such that $\|\xi\| = \delta$ and $\xi\left(\frac{x_1 - x_0}{\|x_1 - x_0\|}\right) = \delta$. Then,

$$f(x_0) + \xi(x_0) \le f(x_1) + \xi(x_1). \tag{3.2}$$

Consider the differentiable function $\phi : [0, 1] \to \mathbb{R}$ defined as follows

$$\phi(\lambda) : = (f + \xi)(x_\lambda) = (f + \xi)((1 - \lambda)x_0 + \lambda x_1).$$

We are now in a position to show that $\phi(\lambda) < \phi(1)$ for all $\lambda \in]0, 1[$

Assume the contrary that $\phi(\lambda) \ge \phi(1)$, for some $\lambda \in]0, 1[$. Then, there exists $\lambda_0 \in [\lambda, 1[$, such that

$$\phi(\lambda_0) \ge \phi(1), \phi'(\lambda_0) = (x_1 - x_0)^T \nabla (f + \xi)(x_{\lambda_0}) \le 0,$$

where $x_{\lambda_0} = (1 - \lambda_0)x_0 + \lambda_0 x_1$. This yields

$$f(x_1) + \xi(x_1) = \phi(1) \le \phi(\lambda_0) = f(x_{\lambda_0}) + \xi(x_{\lambda_0}). \tag{3.3}$$

By (3.2) and (3.3), $f(x_0) + \xi(x_0) \le f(x_{\lambda_0}) + \xi(x_{\lambda_0})$. Hence,

$$\frac{f(x_0) - f(x_{\lambda_0})}{\|x_{\lambda_0} - x_0\|} \le \xi\left(\frac{x_{\lambda_0} - x_0}{\|x_{\lambda_0} - x_0\|}\right) = \xi\left(\frac{x_1 - x_0}{\|x_1 - x_0\|}\right) = \delta.$$

It follows from (3.1) that

$$\frac{(x_0 - x_{\lambda_0})^T}{\|x_{\lambda_0} - x_0\|} \nabla f(x_{\lambda_0}) < \delta = \xi\left(\frac{x_{\lambda_0} - x_0}{\|x_{\lambda_0} - x_0\|}\right).$$

Then, ξ can be expressed in the form $\xi(x) = x^T a$, with some $a \in \mathbb{R}^n$. Therefore,

$$\begin{aligned} 0 &> \frac{(x_0 - x_{\lambda_0})^T}{\|x_{\lambda_0} - x_0\|} \nabla f(x_{\lambda_0}) + \xi\left(\frac{x_0 - x_{\lambda_0}}{\|x_{\lambda_0} - x_0\|}\right) \\ &= \frac{(x_0 - x_{\lambda_0})^T}{\|x_{\lambda_0} - x_0\|} \nabla f(x_{\lambda_0}) + \frac{(x_0 - x_{\lambda_0})^T}{\|x_{\lambda_0} - x_0\|} a \\ &= \frac{(x_0 - x_{\lambda_0})^T}{\|x_{\lambda_0} - x_0\|} (\nabla f(x_{\lambda_0}) + a) \\ &= \frac{(x_0 - x_{\lambda_0})^T}{\|x_{\lambda_0} - x_0\|} \nabla (f + \xi)(x_{\lambda_0}). \end{aligned}$$

Hence $(x_0 - x_{\lambda_0})^T \nabla (f+\xi)(x_{\lambda_0}) < 0$ which yields $(x_1 - x_0)^T \nabla (f+\xi)(x_{\lambda_0}) > 0$. Thus, $\phi'(x_{\lambda_0}) > 0$, a contradiction. Therefore, $\phi(\lambda) < \phi(1)$ for all $\lambda \in]0,1[$. It follows that $f(x_\lambda) + \xi(x_\lambda) < f(x_1) + \xi(x_1)$. Hence,

$$\frac{f(x_\lambda) - f(x_1)}{\|x_\lambda - x_1\|} < \xi\left(\frac{x_1 - x_\lambda}{\|x_\lambda - x_1\|}\right) = \xi\left(\frac{x_1 - x_0}{\|x_1 - x_0\|}\right) = \delta,$$

i.e., f is strictly s-quasiconvex. □

Theorem 3.2. *A continuously differentiable function f on $D \subset \mathbb{R}^n$ is strictly s-quasiconvex iff there exists $\alpha > 0$ such that f is strictly convex on every segment $[x_0, x_1]$ satisfying*

$$\left|\frac{(x_1 - x_0)^T}{\|x_1 - x_0\|} \nabla f(x_\lambda)\right| < \alpha \quad \textit{for all} \quad x_\lambda \in [x_0, x_1]. \tag{3.4}$$

Proof. (a) Necessity: Assume that f is strictly s-quasiconvex. Choose $\alpha = \sigma$, where σ is given in Definition 2.1. Let $[x_0, x_1] \in D$ satisfy (3.4). We have to show that f is strictly convex on $[x_0, x_1]$. Take $y_0, y_1 \in [x_0, x_1]$, $\lambda \in [0,1]$. By the mean-value theorem, there exists $\bar{y} \in [y_0, y_1]$ such that

$$\left|\frac{f(y_1) - f(y_0)}{\|y_1 - y_0\|}\right| = \left|\frac{(y_1 - y_0)^T}{\|y_1 - y_0\|} \nabla f(\bar{y})\right| < \alpha = \sigma.$$

Therefore, by Definition 2.1,

$$\frac{f(y_1) - f(y_0)}{\|y_1 - y_0\|} < \frac{f(y_\lambda) - f(y_1)}{\|y_\lambda - y_1\|}$$

for all $y_\lambda \in [y_0, y_1]$. It follows that $f(y_\lambda) < (1-\lambda) f(y_0) + \lambda f(y_1)$. Hence, f is strictly convex on $[x_0, x_1]$.

(b) Sufficiency: Assume that there is an $\alpha > 0$ such that f is strictly convex on every segment $[x_0, x_1]$ satisfying (3.4). Choose $\sigma = \alpha$. We have to show that for $|\delta| < \sigma$, $x_0, x_1 \in D$, $x_\lambda = (1-\lambda)x_0 + \lambda x_1$ and $\lambda \in]0,1[$, (2.1) is satisfied. Assume the contrary that

$$\frac{f(x_0) - f(x_1)}{\|x_0 - x_1\|} \leq \delta \quad \text{but} \quad \frac{f(x_\lambda) - f(x_1)}{\|x_\lambda - x_1\|} \geq \delta. \tag{3.5}$$

In analogy to the proof of Theorem 2.2 [4], we consider the function

$$g(t) := f\left(x_1 + t\frac{x_0 - x_1}{\|x_0 - x_1\|}\right) - \delta t, \quad 0 \leq t \leq \|x_1 - x_0\|.$$

Since g is continuous, the set $A := \operatorname{argmax}_{0 \leq t \leq \|x_0 - x_1\|} g(t)$ is nonempty and closed. Moreover, (3.5) implies that

$$g(\|x_0 - x_1\|) \leq g(0) \leq g(\|x_\lambda - x_1\|).$$

If either 0 or $\|x_0 - x_1\|$ belongs to A so does $\|x_\lambda - x_1\|$. This implies that $A \cap]0, \|x_0 - x_1\|[\neq \emptyset$. Take $z \in A \cap]0, \|x_0 - x_1\|[$. Then $g'(z) = 0$. It follows that

$$\left| \frac{(x_1 - x_0)^T}{\|x_1 - x_0\|} \nabla f \left(x_1 + z \frac{x_1 - x_0}{\|x_0 - x_1\|} \right) \right| = |\delta| < \sigma = \alpha.$$

Since ∇f is continuous and $z \in]0, \|x_0 - x_1\|[$, there exists $\omega > 0$ such that

$$\left| \frac{(x_1 - x_0)^T}{\|x_1 - x_0\|} \nabla f \left(x_1 + t \frac{x_1 - x_0}{\|x_0 - x_1\|} \right) \right| < \alpha$$

holds true for $t \in [z - \omega, z + \omega] \subset]0, \|x_0 - x_1\|[$. This implies by our assumption that g is strictly convex on $[z - \omega, z + \omega]$. Since $g'(z) = 0$, we conclude that z is a minimizer of g on $[z - \omega, z + \omega]$. It follows from $z \in A$ that g is constant on $[z - \omega, z + \omega]$, in contradiction with the strict convexity of g. This completes our proof. □

The following corollary is a direct result of Theorem 3.2.

Corollary 3.3. *A continuously differentiable function f on $]a, b[\subset \mathbb{R}$ is strictly s-quasiconvex iff there exists $\alpha > 0$ such that f is strictly convex on the level set*

$$L(|f'|, \alpha) := \{x \in]a, b[: |f'(x)| < \alpha\}.$$

Example 3.1. The functions

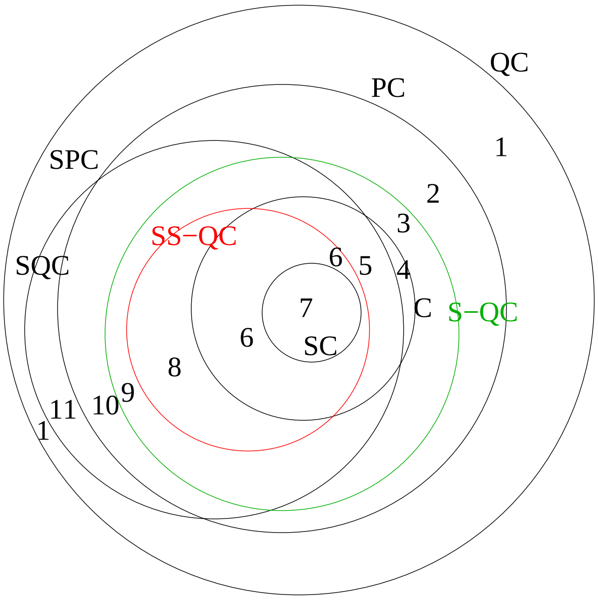
$$\begin{aligned} f_1(x) &= \sqrt{|x|}, \ x \in [-1, 1], \\ f_2(x) &= -\cos x, \ x \in [-2, 2], \\ f_3(x) &= \ln x, \ x \in [1, 2] \end{aligned}$$

given in [4] are not only s-quasiconvex but also strictly s-quasiconvex. Since a strictly s-quasiconvex function is strictly quasiconvex, a convex function which is constant on some interval is not strictly s-quasiconvex.

4. CONCLUDING REMARKS

Based on the results in the above sections and [4] – [5], Fig. 4 gives a complete description of the relations existing between strict s-quasiconvexity (SS-QC), s-quasiconvexity (S-QC), strict quasiconvexity (SQC), quasiconvexity (QC), strict pseudoconvexity (SPC), pseudoconvexity (PC), strict convexity (SC), and convexity (C) of continuously differentiable functions. This figure consists of 11 disjoint regions, numbered from 1 to 11. Here all abbreviations refer to circular regions, apart from SPC which refers to the intersection of the circles defined by PC and SQC. QC refers to the entire interior of the largest circle, S-QC refers to the union of the regions 3-9, and SS-QC refers to the union of the regions 6-8.

In [1], we introduced the notion of s-quasimonotone maps which are stable with respect to their characterizations. In analogy to this paper, we can generate a new type of generalized monotonicity, namely strict s-quasimonotonicity and show that in the case of a differentiable map, strict s-quasimonotonicity of the gradient is equivalent to strict s-quasiconvexity of the underlying function. This will be a subject of another paper. Also, an application of this trend in the theory of general economic equilibrium was presented in [2].



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