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A NEW SUBCLASS OF UNIFORMLY CONVEX FUNCTIONS AND A CORRESPONDING SUBCLASS OF STARLIKE FUNCTIONS WITH FIXED SECOND COEFFICIENT

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ABSTRACT. Making use of Linear operator theory, we define a new subclass of uniformly convex functions and a corresponding subclass of starlike functions with negative coefficients. The main object of this paper is to obtain coefficient estimates distortion bounds, closure theorems and extreme points for functions belonging to this new class. The results are generalized to families with fixed finitely many coefficients.

Key words and phrases: Univalent, Convex, Starlike, Uniformly convex, Uniformly starlike, Linear operator.

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1. INTRODUCTION

Denoted by S the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic and univalent in the unit disc $\triangle = \{z : |z| < 1\}$ and by ST and CV the subclasses of S that are respectively, starlike and convex. Goodman [2, 3] introduced and defined the following subclasses of CV and ST.

A function $f(z)$ is uniformly convex (uniformly starlike) in $\triangle$ if $f(z)$ is in CV (ST) and has the property that for every circular arc γ contained in $\triangle$, with center ξ also in $\triangle$, the arc $f(\gamma)$

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is convex (starlike) with respect to $f(\xi)$. The class of uniformly convex functions is denoted by UCV and the class of uniformly starlike functions by UST (for details see [2]). It is well known from [4, 5] that

$$f \in UCV \Leftrightarrow \left|\frac{zf''(z)}{f'(z)}\right| \leq \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}.$$

In [5], Rønning introduced a new class of starlike functions related to UCV defined as

$$f \in S_p \Leftrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| \leq \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}.$$

Note that $f(z) \in UCV \Leftrightarrow zf'(z) \in S_p$. Further, Rønning generalized the class S_p by introducing a parameter $\alpha,\ -1 \leq \alpha < 1$,

$$f \in S_p(\alpha) \Leftrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| \leq \operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\}.$$

Now we define the function $\phi(a, c; z)$ by

$$\phi(a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n, \tag{1.2}$$

for $c \neq 0, -1, -2, \dots, a \neq -1; z \in \Delta$ where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(n+\lambda)}{\Gamma(\lambda)} = \left\{\begin{array}{ll} 1; & n = 0 \\ \lambda(\lambda+1)(\lambda+2)\dots(\lambda+n-1), & n \in N = \{1, 2, \dots\} \end{array}\right\}. \tag{1.3}$$

Carlson and Shaffer [1] introduced a linear operator $L(a, c)$, by

$$\begin{aligned} L(a, c)f(z) &= \phi(a, c; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n, \quad z \in \triangle, \end{aligned} \tag{1.4}$$

where $*$ stands for the Hadamard product or convolution product of two power series

$$\varphi(z) = \sum_{n=1}^{\infty} \varphi_n z^n \ \text{ and } \ \psi(z) = \sum_{n=1}^{\infty} \psi_n z^n$$

defined by

$$(\varphi * \psi)(z) = \varphi(z) * \psi(z) = \sum_{n=1}^{\infty} \varphi_n \psi_n z^n.$$

We note that $L(a, a)f(z) = f(z)$, $L(2, 1)f(z) = zf'(z)$, $L(m+1, 1)f(z) = D^m f(z)$, where $D^m f(z)$ is the Ruscheweyh derivative of $f(z)$ defined by Ruscheweyh [6] as

$$D^m f(z) = \frac{z}{(1-z)^{m+1}} * f(z), \quad m > -1. \tag{1.5}$$

Which is equivalently,

$$D^m f(z) = \frac{z}{m!} \frac{d^m}{dz^m}\{z^{m-1} f(z)\}.$$

For $\beta \geq 0$ and $-1 \leq \alpha < 1$, we let $S(\alpha, \beta)$ denote the subclass of S consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion

$$\operatorname{Re}\left\{\frac{z(L(a, c)f(z))'}{L(a, c)f(z)} - \alpha\right\} > \beta\left|\frac{z(L(a, c)f(z))'}{L(a, c)f(z)} - 1\right|, \quad z \in \triangle. \tag{1.6}$$

We also let $TS(\alpha,\beta) = S(\alpha,\beta)\bigcap T$ where T, the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0, \quad \forall\, n \geq 2, \tag{1.7}$$

was introduced and studied by Silverman [7].

The main object of this paper is to obtain necessary and sufficient conditions for the functions $f(z) \in TS(\alpha,\beta)$. Furthermore we obtain extreme points, distortion bounds and closure properties for $f(z) \in TS(\alpha,\beta)$ by fixing the second coefficient.

2. The Class $S(\alpha,\beta)$

In this section we obtain necessary and sufficient conditions for functions $f(z)$ in the classes $TS(\alpha,\beta)$.

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in $S(\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}|a_n| \leq 1-\alpha, \tag{2.1}$$

$-1 \leq \alpha < 1,\ \beta \geq 0.$

Proof. It suffices to show that

$$\beta\left|\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1\right| - \operatorname{Re}\left\{\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1\right\} \leq 1-\alpha.$$

We have

$$\begin{aligned}\beta\left|\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1\right| &- \operatorname{Re}\left\{\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1\right\}\\ &\leq (1+\beta)\left|\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1\right|\\ &\leq \frac{(1+\beta)\sum_{n=2}^{\infty}(n-1)\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|}{1-\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|}.\end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}|a_n| \leq 1-\alpha,$$

and hence the proof is complete. □

Theorem 2.2. *A necessary and sufficient condition for $f(z)$ of the form (1.7) to be in the class $TS(\alpha,\beta)$, $-1 \leq \alpha < 1,\ \beta \geq 0$ is that*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}\, a_n \leq 1-\alpha. \tag{2.2}$$

Proof. In view of Theorem 2.1, we need only to prove the necessity. If $f(z) \in TS(\alpha,\beta)$ and z is real then

$$\frac{1-\sum_{n=2}^{\infty} n\frac{(a)_{n-1}}{(c)_{n-1}}\, a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}\, a_n z^{n-1}} - \alpha \geq \beta\left|\frac{\sum_{n=2}^{\infty}(n-1)\frac{(a)_{n-1}}{(c)_{n-1}}\, a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}\, a_n z^{n-1}}\right|.$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}\,a_n \le 1-\alpha,\ -1\le\alpha<1,\ \beta\ge 0.$$

□

Corollary 2.3. *Let the function $f(z)$ defined by (1.7) be in the class $TS(\alpha,\beta)$. Then*

$$a_n \le \frac{(1-\alpha)(c)_{n-1}}{[n(1+\beta)-(\alpha+\beta)](a)_{n-1}},\ \ n\ge 2,\ -1\le\alpha\le 1,\ \ \beta\ge 0.$$

Remark 2.4. In view of Theorem 2.2, we can see that if $f(z)$ is of the form (1.7) and is in the class $TS(\alpha,\beta)$ then

$$a_2 = \frac{(1-\alpha)(c)}{(2+\beta-\alpha)(a)}. \tag{2.3}$$

By fixing the second coefficient, we introduce a new subclass $TS_b(\alpha,\beta)$ of $TS(\alpha,\beta)$ and obtain the following theorems.

Let $TS_b(\alpha,\beta)$ denote the class of functions $f(z)$ in $TS(\alpha,\beta)$ and be of the form

$$f(z) = z - \frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)}z^2 - \sum_{n=3}^{\infty}a_n z^n\ \ (a_n\ge 0),\ 0\le b\le 1. \tag{2.4}$$

Theorem 2.5. *Let function $f(z)$ be defined by (2.4). Then $f(z)\in TS_b(\alpha,\beta)$ if and only if*

$$\sum_{n=3}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}\,a_n \le (1-b)(1-\alpha), \tag{2.5}$$

$-1\le\alpha<1$, $\beta\ge 0$.

Proof. Substituting

$$a_2 = \frac{b(1-\alpha)}{(2+\beta-\alpha)}\frac{(c)}{(a)},\ \ 0\le b\le 1.$$

in (2.2) and simple computation leads to the desired result. □

Corollary 2.6. *Let the function $f(z)$ defined by (2.4) be in the class $TS_b(\alpha,\beta)$. Then*

$$a_n \le \frac{(1-b)(1-\alpha)(c)_{n-1}}{[n(1+\beta)-(\alpha+\beta)](a)_{n-1}},\ \ n\ge 3,\ -1\le\alpha\le 1,\ \beta\ge 0. \tag{2.6}$$

Theorem 2.7. *The class $TS_b(\alpha,\beta)$ is closed under convex linear combination.*

Proof. Let the function $f(z)$ be defined by (2.4) and $g(z)$ defined by

$$g(z) = z - \frac{b(1-\alpha)}{(2+\beta-\alpha)}\frac{(c)}{(a)}z^2 - \sum_{n=3}^{\infty}d_n z^n, \tag{2.7}$$

where $d_n\ge 0$ and $0\le b\le 1$.

Assuming that $f(z)$ and $g(z)$ are in the class $TS_b(\alpha,\beta)$, it is sufficient to prove that the function $H(z)$ defined by

$$H(z) = \lambda f(z) + (1-\lambda)g(z),\ \ (0\le\lambda\le 1) \tag{2.8}$$

is also in the class $TS_b(\alpha,\beta)$.
Since

$$H(z) = z - \frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)}z^2 - \sum_{n=3}^{\infty}\{\lambda a_n + (1-\lambda)d_n\}z^n, \tag{2.9}$$

$a_n \geq 0,\ d_n \geq 0,\ 0 \leq b \leq 1$, we observe that

$$\sum_{n=3}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}(\lambda a_n+(1-\lambda)d_n) \leq (1-b)(1-\alpha) \tag{2.10}$$

which is, in view of Theorem 2.5, again, implies that $H(z) \in TS_b(\alpha, \beta)$ which completes the proof of the theorem. □

Theorem 2.8. *Let the functions*

$$f_j(z) = z - \frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)}z^2 - \sum_{n=3}^{\infty} a_{n,\,j} z^n,\ a_{n,j} \geq 0 \tag{2.11}$$

be in the class $TS_b(\alpha, \beta)$ for every j $(j = 1, 2, \ldots, m)$. Then the function $F(z)$ defined by

$$F(z) = \sum_{j=1}^{m} \mu_j f_j(z), \tag{2.12}$$

is also in the class $TS_b(\alpha, \beta)$, where

$$\sum_{j=1}^{m} \mu_j = 1. \tag{2.13}$$

Proof. Combining the definitions (2.11) and (2.12), further by (2.13) we have

$$F(z) = z - \frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)}z^2 - \sum_{n=3}^{\infty}\left(\sum_{j=1}^{m} \mu_j a_{n,j}\right) z^n. \tag{2.14}$$

Since $f_j(z) \in TS_b(\alpha, \beta)$ for every $j = 1, 2, \ldots, m$, Theorem 2.5 yields

$$\sum_{n=3}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}\, a_{n,j} \leq (1-b)(1-\alpha), \tag{2.15}$$

for $j = 1, 2, \ldots, m$. Thus we obtain

$$\begin{aligned} &\sum_{n=3}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}\left(\sum_{j=1}^{m} \mu_j a_{n,j}\right) \\ &\qquad = \sum_{j=1}^{m}\mu_j\left(\sum_{n=3}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}a_{n,j}\right) \\ &\qquad \leq (1-b)(1-\alpha) \end{aligned}$$

in view of Theorem 2.5. So, $F(z) \in TS_b(\alpha, \beta)$. □

Theorem 2.9. *Let*

$$f_2(z) = z - \frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)}z^2 \tag{2.16}$$

and

$$f_n(z) = z - \frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)}z^2 - \frac{(1-b)(1-\alpha)(c)_{n-1}}{[n(1+\beta)-(\alpha+\beta)](a)_{n-1}}z^n \tag{2.17}$$

for $n = 3, 4, \ldots$. Then $f(z)$ is in the class $TS_b(\alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z), \tag{2.18}$$

where $\lambda_n \geq 0$ and $\sum_{n=2}^{\infty} \lambda_n = 1$.

Proof. We suppose that $f(z)$ can be expressed in the form (2.18). Then we have

$$\begin{aligned} f(z) &= z - \frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)} z^2 - \sum_{n=3}^{\infty} \lambda_n \frac{(1-b)(1-\alpha)(c)_{n-1}}{[n(1+\beta)-(\alpha+\beta)](a)_{n-1}} z^n \\ &= z - \sum_{n=2}^{\infty} A_n z^n, \end{aligned} \tag{2.19}$$

where

$$A_2 = \frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)} \tag{2.20}$$

and

$$A_n = \frac{\lambda_n (1-b)(1-\alpha)(c)_{n-1}}{[n(1+\beta)-(\alpha+\beta)](a)_{n-1}}, \quad n = 3, 4, \ldots. \tag{2.21}$$

Therefore

$$\begin{aligned} \sum_{n=2}^{\infty} [n(1+\beta)-(\alpha+\beta)] \frac{(a)_{n-1}}{(c)_{n-1}} A_n &= b(1-\alpha) + \sum_{n=3}^{\infty} \lambda_n (1-b)(1-\alpha) \\ &= (1-\alpha)[b + (1-\lambda_2)(1-b)] \\ &\leq (1-\alpha), \end{aligned} \tag{2.22}$$

it follows from Theorem 2.2 and Theorem 2.5 that $f(z)$ is in the class $TS_b(\alpha, \beta)$. Conversely, we suppose that $f(z)$ defined by (2.4) is in the class $TS_b(\alpha, \beta)$. Then by using (2.6), we get

$$a_n \leq \frac{(1-b)(1-\alpha)(c)_{n-1}}{[n(1+\beta)-(\alpha+\beta)](a)_{n-1}}, \quad (n \geq 3). \tag{2.23}$$

Setting

$$\lambda_n = \frac{[n(1+\beta)-(\alpha+\beta)](a)_{n-1}}{(1-b)(1-\alpha)(c)_{n-1}} a_n, \quad (n \geq 3) \tag{2.24}$$

and

$$\lambda_2 = 1 - \sum_{n=3}^{\infty} \lambda_n, \tag{2.25}$$

we have (2.18). This completes the proof of Theorem 2.9. □

Corollary 2.10. *The extreme points of the class $TS_b(\alpha, \beta)$ are functions $f_n(z)$, $n \geq 2$ given by Theorem 2.9.*

3. DISTORTION THEOREMS

In order to obtain distortion bounds for the function $f \in TS_b(\alpha, \beta)$, we first prove the following lemmas.

Lemma 3.1. *Let the function* $f_3(z)$ *be defined by*

$$f_3(z) = z - \frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)}z^2 - \frac{(1-b)(1-\alpha)(c)_2}{(3+2\beta-\alpha)(a)_2}z^3 \,. \tag{3.1}$$

Then, for $0 \le r < 1$ *and* $0 \le b \le 1$,

$$|f_3(re^{i\theta})| \ge r - \frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)}r^2 - \frac{(1-b)(1-\alpha)(c)_2}{(3+2\beta-\alpha)(a)_2}r^3 \tag{3.2}$$

with equality for $\theta = 0$. *For either* $0 \le b < b_0$ *and* $0 \le r \le r_0$ *or* $b_0 \le b \le 1$,

$$|f_3(re^{i\theta})| \le r + \frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)}r^2 - \frac{(1-b)(1-\alpha)(c)_2}{(3+2\beta-\alpha)(a)_2}r^3 \tag{3.3}$$

with equality for $\theta = \pi$, *where*

$$\begin{aligned} b_0 = {} & \frac{1}{2(1-\alpha)(c)(c)_2} \\ & \times \{-[(3+2\beta-\alpha)(a)_2(c) + 4(2+\beta-\alpha)(a)(c)_2 - (1-\alpha)(c)(c)_2] \\ & + [((3+2\beta-\alpha)(a)_2(c) + 4(2+\beta-\alpha)(a)(c)_2 - (1-\alpha)(c)(c)_2^2 \\ & \qquad + 16(2+\beta-\alpha)(1-\alpha)(a)(c)(c)_2^2]^{1/2}\} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} r_0 = {} & \frac{1}{b(1-b)(1-\alpha)(c)_2}\{-2(1-b)(2+\beta-\alpha)(a)(c+1) \\ & + [4(1-b)^2(2+\beta-\alpha)^2(a)^2(c+1)^2 \\ & \qquad + b^2(1-b)(3+2\beta-\alpha)(1-\alpha)(a)_2(c)_2]^{1/2}\}. \end{aligned} \tag{3.5}$$

Proof. We employ the technique as used by Silverman and Silvia [8]. Since

$$\begin{aligned} \frac{\partial |f_3(re^{i\theta})|^2}{\partial\theta} = 2(1-\alpha)r^3\sin\theta \Bigg\{ & \frac{b(c)}{(2+\beta-\alpha)(a)} + \frac{4(1-b)(c)_2}{(3+2\beta-\alpha)(a)_2} r\cos\theta \\ & - \frac{b(1-b)(1-\alpha)(c)(c)_2}{(2+\beta-\alpha)(3+2\beta-\alpha)(a)(a)_2}r^2 \Bigg\} \end{aligned} \tag{3.6}$$

we can see that

$$\frac{\partial |f_3(re^{i\theta})|^2}{\partial\theta} = 0 \tag{3.7}$$

for $\theta_1 = 0$, $\theta_2 = \pi$, and

$$\theta_3 = \cos^{-1}\left(\frac{1}{r}\frac{b[(1-b)(1-\alpha)(c)_2r^2 - (3+2\beta-\alpha)(a)_2]}{4(1-b)(2+\beta-\alpha)(a)(c+1)}\right) \tag{3.8}$$

since θ_3 is a valid root only when $-1 \le \cos\theta_3 \le 1$. Hence we have a third root if and only if $r_0 \le r < 1$ and $0 \le b \le b_0$. Thus the results of the theorem follow from comparing the extremal values $|f_3(re^{i\theta_k})|$, $k = 1, 2, 3$ on the appropriate intervals. □

Lemma 3.2. *Let the functions* $f_n(z)$ *be defined by (2.17) and* $n \ge 4$. *Then*

$$|f_n(re^{i\theta})| \le |f_4(-r)|. \tag{3.9}$$

Proof. Since

$$f_n(z) = z - \frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)} z^2 - \frac{(1-b)(1-\alpha)(c)_{n-1}}{[n(1+\beta)-(\alpha+\beta)](a)_{n-1}} z^n$$

and $\frac{r^n}{n}$ is a decreasing function of n, we have

$$\begin{aligned} |f_n(re^{i\theta}| &\leq r + \frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)} r^2 - \frac{(1-b)(1-\alpha)(c)_3}{[4+3\beta-\alpha](a)_3} r^4 \\ &= -f_4(-r), \end{aligned}$$

which shows (3.9). □

Theorem 3.3. *Let the function $f(z)$ defined by (2.4) belong to the class $TS_b(\alpha,\beta)$, then for $0 \leq r < 1$,*

$$|f(re^{i\theta})| \geq r - \frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)} r^2 - \frac{(1-b)(1-\alpha)(c)_2}{[3+2\beta-\alpha](a)_2} r^3 \tag{3.10}$$

with equality for $f_3(z)$ at $z = r$, and

$$|f(re^{i\theta})| \leq \max\left\{ \max_\theta |f_3(re^{i\theta})|, -f_4(-r) \right\}, \tag{3.11}$$

where $\max_\theta |f_3(re^{i\theta})|$ is given by Lemma 3.1.

Proof. The proof of Theorem 3.3 is obtained by comparing the bounds of Lemma 3.1 and Lemma 3.2. □

Remark 3.4. Taking $b = 1$ in Theorem 3.3 we obtain the following result.

Corollary 3.5. *Let the function $f(z)$ defined by (1.7) be in the class $TS(\alpha,\beta)$. Then for $|z| = r < 1$, we have*

$$r - \frac{(1-\alpha)(c)}{(2+\beta-\alpha)(a)} r^2 \leq |f(z)| \leq r + \frac{(1-\alpha)(c)}{(2+\beta-\alpha)(a)} r^2. \tag{3.12}$$

Lemma 3.6. *Let the function $f_3(z)$ be defined by (3.1). Then, for $0 \leq r < 1$, and $0 \leq b \leq 1$,*

$$|f_3'(re^{i\theta})| \geq 1 - \frac{2b(1-\alpha)(c)}{(2+\beta-\alpha)(a)} r - \frac{3(1-b)(1-\alpha)(c)_2}{(3+2\beta-\alpha)(a)_2} r^2 \tag{3.13}$$

with equality for $\theta = 0$. For either $0 \leq b < b_1$ and $0 \leq r \leq r_1$ or $b_1 \leq b \leq 1$,

$$|f_3'(re^{i\theta})| \leq 1 + \frac{2b(1-\alpha)(c)}{(2+\beta-\alpha)(a)} r - \frac{3(1-b)(1-\alpha)(c)_2}{(3+2\beta-\alpha)(a)_2} r^2 \tag{3.14}$$

with equality for $\theta = \pi$, where

$$\begin{aligned} b_1 = {} & \frac{1}{6(1-\alpha)(c)(c_2)} \\ & \times \{-[(3+2\beta-\alpha)(a)_2(c) + 6(2+\beta-\alpha)(a)(c)_2 - 3(1-\alpha)(c)(c)_2] \\ & + \{((3+2\beta-\alpha)(a)_2(c) + 6(2+\beta-\alpha)(a)(c)_2 - 3(1-\alpha)(c)(c)_2)^2 \\ & \qquad + 72(2+\beta-\alpha)(1-\alpha)(a)(c)(c_2^2)\}^{1/2}\} \end{aligned} \tag{3.15}$$

and

$$r_1 = \frac{1}{3b(1-b)(1-\alpha)(c_2)}\{-3(1-b)(2+\beta-\alpha)(a)(c+1) + [8(1-b)^2(2+\beta-\alpha)^2(a)^2(c+1)^2 + 3b^2(1-b)(3+2\beta-\alpha)(1-\alpha)(a)_2(c)_2]^{1/2}\}. \tag{3.16}$$

Proof. The proof of Lemma 3.6 is much akin to the proof of Lemma 3.1. □

Theorem 3.7. *Let the function $f(z)$ defined by (2.4) belong to the class $TS_b(\alpha,\beta)$, then for* $0 \le r < 1$,

$$|f'(re^{i\theta})| \ge 1 - \frac{2b(1-\alpha)(c)}{(2+\beta-\alpha)(a)}r - \frac{3(1-b)(1-\alpha)(c)_2}{[3+2\beta-\alpha](a)_2}r^2 \tag{3.17}$$

with equality for $f_3'(z)$ at $z = r$, and

$$|f'(re^{i\theta})| \le \max\left\{\max_\theta |f_3'(re^{i\theta})|, -f_4'(-r)\right\}, \tag{3.18}$$

where $\max_\theta |f_3'(re^{i\theta})|$ is given by Lemma 3.6.

Remark 3.8. Putting $b = 1$ in Theorem 3.7 we obtain the following result.

Corollary 3.9. *Let the function $f(z)$ defined by (1.2) be in the class $TS(\alpha,\beta)$. Then for $|z| = r < 1$, we have*

$$1 - \frac{2(1-\alpha)(c)}{(2+\beta-\alpha)(a)}r \le |f'(z)| \le 1 + \frac{2(1-\alpha)(c)}{(2+\beta-\alpha)(a)}r. \tag{3.19}$$

4. THE CLASS $TS_{b_n,k}(\alpha,\beta)$

Instead of fixing just the second coefficient, we can fix finitely many coefficients. Let $TS_{b_n,k}(\alpha,\beta)$ denote the class of functions in $TS_b(\alpha,\beta)$ of the form

$$f(z) = z - \sum_{n=2}^{k} \frac{b_n(1-\alpha)(c)_{n-1}}{[n(1+\beta)-(\alpha+\beta)](a)_{n-1}}z^n - \sum_{n=k+1}^{\infty} a_n z^n, \tag{4.1}$$

where $0 \le \sum_{n=2}^{k} b_n = b \le 1$. Note that $TS_{b_2,2}(\alpha,\beta) = TS_b(\alpha,\beta)$.

Theorem 4.1. *The extreme points of the class $TS_{b_n,k}(\alpha,\beta)$ are*

$$f_k(z) = z - \sum_{n=2}^{k} \frac{b_n(1-\alpha)(c)_{n-1}}{[n(1+\beta)-(\alpha+\beta)](a)_{n-1}}z^n$$

and

$$f_n(z) = z - \sum_{n=2}^{k} \frac{b_n(1-\alpha)(c)_{n-1}}{[n(1+\beta)-(\alpha+\beta)](a)_{n-1}}z^n - \sum_{n=k+1}^{\infty} \frac{(1-b)(1-\alpha)(c)_{n-1}}{[n(1+\beta)-(\alpha+\beta)](a)_{n-1}}z^n$$

The details of the proof are omitted, since the characterization of the extreme points enables us to solve the standard extremal problems in the same manner as was done for $TS_b(\alpha,\beta)$.

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