Effects of the Earth's Curvature on the Dynamics of Isolated Objects. Part II: The Uniformly Translating Vortex

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ABSTRACT

Zonally propagating solutions of the primitive equations for an isolated volume of fluid are considered. In a moving stereographic projection (from the antipode of the center of mass) geometric distortion enters at $O(R^{-2})$, with R the radius of the earth, whereas planet curvature effects are $O(R^{-1})$. The imbalance between the centrifugal force and the poleward gravitational force, due to the drift c, is equilibrated by the average Coriolis force, proportional to β. The results are valid for both homogeneous and stratified cases and the lowest-order solution need not be an axisymmetric vortex. The classical β-plane approximation predicts correctly the leading order of c/β, but makes large errors in the $O(R^{-1})$ term of the vortex structure.

A method is developed to construct the correct $O(R^{-1})$ term, starting from any steady solution of the f-plane equations, as the $O(R^0)$ term. The expansion is exemplified starting with a homogeneous fluid, solid body rotating at an anticyclonic rate $-\nu f_0$, with $0 < \nu < 1$. To $O(R^{-1})$ particle orbits and isobaths belong to different families of nonconcentric circles. A water column moves faster and becomes taller the farther away it is from the equator. In order to keep its potential vorticity, the water column experiences changes of relative vorticity equal to $-(2 - \nu)/(3 - 3\nu)$ times the variations of the ambient vorticity (Coriolis parameter). The physics of this solution is compared with that of a circular and rigid disk, studied in Part I.

1. Introduction

Ball (1963) studied the shallow-water equations in the framework of the f-plane approximation (constant Coriolis parameter f and Cartesian geometry), showing that ''*the motion of the centre of gravity of a finite volume of liquid with free boundaries . . . is independent of the motion relative to the centre of gravity, and vice versa.*'' More precisely, in the absence of topography this center of mass movement is but an inertial oscillation in a circular orbit[1] and the motion relative to it satisfies the full, nonlinear, shallow-water equations. Cushman-Roisin and Nof (1985), Young (1986), and Cushman-Roisin (1987) reinterpreted the results of Ball (1963) in a ''reduced gravity'' setting in which the active volume of fluid is assumed to be floating on top of a motionless heavier liquid. As long as the ambient layer is assumed to be at rest, there is no mathematical difference between Ball's setting, a volume of fluid over the surface of the planet (which will be adopted here), or the reduced gravity ones. Maas and Zahariev (1996) further generalized these results to a three-dimensional elliptical vortex moving inside a motionless stratified fluid: in addition to the horizontal inertial oscillations, the center of mass performs vertical buoyancy oscillations.

Ball (1963) showed that conservation of the energy $\mathcal{E}$, measured in the terrestrial frame, and the vertical component of the angular momentum $\mathcal{A}$ play an important role in the motion of the fluid relative to the center of mass, and also proved that there are exact solutions of the f-plane shallow-water equations, in which the pressure (velocity) is a second (first) order polynomial of the coordinates; the time-dependent polynomial coefficients satisfy nonlinear ODE. Particular examples of these ''polynomial solutions'' were discussed by Cushman-Roisin and Nof (1985), Young (1986), and Cushman-Roisin (1987), and the general solution of the system of ODE was found by Holm (1991). It is worth recalling that, even though these are exact solutions the f-plane shallow water equations, they might be unstable, particularly the more elongated ones, to perturbations in the form of a higher degree

[1] Ball (1963) studied a more general case with a paraboloid topography. However, if this is concave and of revolution, then in the f plane there is a transformation to a rotating system which ''eliminates'' the topography while changing the value of the Coriolis parameter (Ripa 1987).

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polynomial (Cushman-Roisin 1986; Ripa 1987; Ripa and Jiménez 1988; Pavía and López 1994).

Allowing for effects of the planet's curvature changes completely Ball's scenario: the inertial oscillations are not circular but experience a secular drift, they are coupled with the internal motions, $\mathcal{A}$ is no longer an integral of motion, and polynomial exact solutions are not possible. A solid-body rotating vortex in the f-plane (the "lens"), can be shown to be stable using conservation of $\mathcal{E}$, $\mathcal{A}$, and potential vorticity (Ripa 1987, 1992); even if there were axialsymmetric solutions on the sphere (with the center off the poles), their stability could not be proved by the same method because $\mathcal{A}$ is not an integral of motion.

Through the analysis of a disk dynamics, in Ripa (2000 henceforth referred to as Part I) it was shown that at low $\mathcal{E}$ there are two distinct parts of the drift velocity c: that due to the inertial oscillations c_o and that caused by the intrinsic rotation c_i, which were respectively denoted "orbital drift" and "internal drift" for simplicity (both refer to the translation of the whole solution, though, the name distinguishes the *origin* of the drift; see Table 1 in Part I). The present paper is devoted to the second effect on isolated vortices; the most difficult problem of a vortex experiencing both inertial oscillations of its center of mass and internal motions will be left for Part III of these works.

Nof (1981) and Killworth (1983) calculated the internal drift c_i of an isolated vortex in the framework of the classical β-plane approximation of the shallow-water equations, that is, using

$$x = (\lambda - \lambda_0)R\cos\theta_0, \qquad y = (\theta - \theta_0)R \qquad (1.1)$$

—where (λ_0, θ_0) are reference (longitude, latitude)—as Cartesian coordinates, but allowing for a linear variation of f with latitude. This approximation is incorrect (except in the equatorial waveguide) because the curvature corrections to a flat geometry are of the same order as the variation of the Coriolis parameter, namely, $O(R^{-1})$. However, Graef (1998) proved that the formula derived by Nof (1981) and Killworth (1983) for c_i is correct. This does not mean that the classical β plane gives the right description of all other details of the motion, as explained in Ripa (1997, hereafter referred to as R97), Part I, and this paper. More precisely, the exact arc element in the coordinates (1.1) is given by $|d\mathbf{r}| = \sqrt{\gamma^2 dx^2 + dy^2}$ with $\gamma = \sec\theta_0 \cos\theta$. Therefore, as $y \to 0$ the Coriolis parameter and metric coefficient satisfy

$$f \sim f_0 + \beta y, \qquad \gamma \sim 1 - \tau_0 y,$$

where all through these papers the symbol $\sim$ denotes "equal modulo $o(R^{-1})$," that is, results with an $O(R^{-2})$ error, and

$$f_0 = 2\Omega\sin\theta_0, \qquad \beta = 2\Omega R^{-1}\cos\theta_0,$$
$$\tau_0 = R^{-1}\tan\theta_0.$$

Note that β and τ_0 are of the same order, namely R^{-1}, except for $\theta_0 = 0$.

The inadequacy of the classical β-plane approximation is not always clearly recognized. Compare, for instance, the formulas for the ageostrophic velocity and divergence, in the quasigeostrophic scaling, given by two classical texts [Pedlosky 1979, Eqs. (13a), (13b), and (14) of section 6.3] versus [Gill 1982, Eqs. (17), (18), and (28) in section 12.2]: the former include terms proportional to τ_0 missing in the latter. In the derivation of the quasigeostrophic model, these "non-Cartesian" terms cancel out in the corresponding potential vorticity equation, which is then fortuitously described correctly by the classical β-plane approximation (Pedlosky 1979; see also R97). However, it is not unlikely that τ_0 should appear in the prognostic equation of other balance models, in a correct $O(R^{-1})$ approximation.

In the spherical coordinates (1.1), the parameters β and τ_0 appear on equal footing because both are $O(R^{-1})$. Phillips (1973) and Verkley (1990) choose other coordinates (x_p, y_p) such that the $x_p = 0$ is the meridian $\lambda = \lambda_0$, whereas $y_p = 0$ is another great circle, tangent to the zonal displacement $(\lambda_0 + d\lambda, \theta_0)$, instead of the parallel $\theta = \theta_0$. [Phillips makes a stereographic projection from the antipode $(\lambda_0 + \pi, -\theta_0)$ whereas Verkley uses spherical coordinates such that $y_p = 0$ is their equator.] With these choices of variables the classical β-plane equations are correct up to $O(R^{-1})$ in a neighborhood of $\theta = \theta_0$ and $\lambda = \lambda_0$; this is appropriate for problems in a small domain fixed to the earth (such as a sea) but not for the solutions with a secular zonal drift, such as those studied in this and companion papers, since the requirement $\lambda \approx \lambda_0$ is eventually violated. Approximations are not uniformly valid in time because the drift is along the great circle $y_p = 0$, instead of the parallel of latitude $y = 0$. (See R97 for a quantitative comparison of the predictions for a single particle made by Verkley's system, the classical β plane, and the full equations.)

In order to derive approximations that are uniformly valid in time, in Part I were defined *moving* coordinates (x', y') by means of a stereographic projection from $(\lambda_0 + \pi + \delta\Omega t, -\theta_0)$, where $\delta\Omega = cR^{-1}\sec\theta_0$. In the new frame a particle has a velocity $\mathbf{u}' = \mathbf{u} - \gamma c\hat{\boldsymbol{\lambda}}$ and is subject to the action of the Coriolis force $-f'\hat{\mathbf{z}} \times \mathbf{u}'$ and the imbalance between the poleward gravitational force and the equatorward centrifugal force $-\boldsymbol{\nabla}\Phi'$ (called "geoforce" in Part I), where

$$f' = \left(1 + \frac{\delta\Omega}{\Omega}\right)f,$$

$$\Phi' = \frac{1}{4}\left(1 + \frac{\delta\Omega}{2\Omega}\right)\frac{\delta\Omega}{\Omega}R^2(f^2 - f_0^2).$$

In terms of these coordinates, the arc element and the terrestrial Coriolis parameter are exactly given by $|d\mathbf{r}|$

$= \tilde{\gamma}\sqrt{dx'^2 + dy'^2}$, with $\tilde{\gamma}^{-1} = 1 + \frac{1}{4}(x'^2 + y'^2)/R^2$, and $f = (2\tilde{\gamma} - 1)f_0 + \tilde{\gamma}\beta y'$.

The transformation $(x', y') \mapsto (x, y)$ is given by in appendix A of Part I; if the interest is near $\sqrt{x'^2 + y'^2} =: r = 0$, making an expansion in r/R, it is also shown in Part I:

$$x - ct \sim x' + \tau_0 x'y', \qquad y \sim y' - \frac{1}{2}\tau_0 x'^2. \quad (1.2)$$

This yields a transformation of the horizontal velocity components, $u \sim (1 - \tau_0 y)\dot{x}$ and $v = \dot{y}$, of the form

$$u - c \sim u' + \tau_0 x'v', \qquad v \sim v' - \tau_0 x'u', \quad (1.3)$$

where $u' \sim \dot{x}'$ and $v' \sim \dot{y}'$. Furthermore, since $\delta\Omega/\Omega = O(R^{-2})$ [i.e., $c = O(R^{-1})$; see Part I],

$$f' \sim f_0 + \beta y', \qquad \Phi' \sim f_0 cy', \qquad \tilde{\gamma} \sim 1.$$

From these expressions it follows that the correct $O(R^{-1})$ equations in (x', y') coincide with the incorrect β-plane equations (i.e., without the τ_0 terms) in the coordinates $(x - ct, y)$. Consequently, as long as c is chosen so that the solution's domain remains bounded in these coordinates, the classical β-plane equations give *the right solution in the wrong frame.* Note that the circle $(x', y') = a(\cos\phi, \sin\phi)$ is not seen as a circle in $(x - ct, y)$ but as $(x - ct)^2 + y^2 \sim a^2(1 + \tau_0 a \cos\phi^2 \sin\phi)$: the apparent eccentricity is $O(R^{-1})$, namely, the same order as the difference with the f-plane solutions. This problem is not avoided using coordinates (x, y) defined with a Mercator projection, that is, $|d\mathbf{r}| = \gamma\sqrt{dx^2 + dy^2}$, since it can be shown $(x - ct)^2 + y^2 \sim a^2(1 + \tau_0 a \sin\phi)$ in this frame (see R97). Here on, (x, y) will denote the spherical coordinates (1.1); variables a and τ_0 used in R97 correspond to R and $\tau_0 R$ in this paper, whereas φ_2 from R97 should here be set equal to zero.

As done in (Ball 1963), only compact vortices are considered here, that is, bounded by a zero depth line, in the two-dimensional case, or a surface of vanishing pressure perturbation, in the three dimensional case. For nonisolated vortices, the external field plays an important role (Nof 1983; Cushman-Roisin et al. 1990; Benilov 1996; Llewellyn Smith 1997; Stern and Radko 1998); study of earth's curvature effects for these problems is beyond the scope of the present paper. The purpose of this paper is twofold: First, bulk formulas (the drift c and the average of the particles zonal velocity $\langle u\rangle$, whose difference is a consequence of the planet's curvature) are derived for the general problem in section 2, without making particular assumptions on the structure of the lowest order fields (e.g., it need not be a monopolar vortex); these results are shown to be also valid for a stratified case. Second, the structure and dynamics of a uniformly translating solution of the shallow water equations are discussed in section 3. An expansion method to find a general $O(R^{-1})$ solution is derived. The particular case when the starting, $O(R^0)$, field is a solid-body rotating vortex is described and its dynamics is compared with that of the disk, studied in Part I. Conclusions are finally given in section 4, and mathematical details are left for appendixes.

2. General equations

First consider an homogeneous fluid, henceforth referred to as "the vortex," in a compact volume bounded by the earth's radii R and $R + h(\mathbf{x}, t)$, where the horizontal position $\mathbf{x}$ is expressed in any coordinates on the sphere. Each water column moves with a horizontal velocity $\mathbf{u}(\mathbf{x}, t)$. It is important to write down the evolution equations in a coordinate-free form, namely

$$\left.\begin{aligned} \partial_t h + \nabla\cdot(h\mathbf{u}) &= 0\\ \partial_t\mathbf{u} + (f + \xi)\hat{\mathbf{z}}\times\mathbf{u} + \nabla b &= 0\end{aligned}\right\} \qquad \mathbf{x}\in D(t),$$

$$h(\mathbf{x}, t) = 0 \qquad \mathbf{x}\in\partial D(t),$$

where

$$\xi = \hat{\mathbf{z}}\cdot\nabla\times\mathbf{u}, \qquad b = gh + \frac{1}{2}\mathbf{u}^2$$

are the vertical relative vorticity and Bernoulli head, respectively. The total volume is conserved

$$\frac{d}{dt}\iint_D h\, dS = 0.$$

Ball (1963) derived, in the case of Cartesian geometry, the theorem

$$\frac{d\langle\mu\rangle}{dt} = \left\langle\frac{D\mu}{Dt}\right\rangle \quad (2.1)$$

for any $\mu(\mathbf{x}, t)$, where $\langle\mu\rangle := (\iint_D h\mu\, dS)/(\iint_D h\, dS)$ and $D/Dt = \partial_t + \mathbf{u}\cdot\nabla$. It is easy to show that it is also valid on the sphere (in any coordinates and frame).[2] [If the domain is not limited by $h = 0$ and the far-field asymptotic condition is of the form $h \to h_\infty$, then the angle brackets are not an average, but rather denote $\langle\mu\rangle := (\iint_D h\mu\, dS)/(\iint_D (h - h_\infty)\, dS)$.]

Secondly, the equations of motion for a stratified isolated volume of fluid are

$$\left.\begin{aligned} \partial_z w + \nabla\cdot\mathbf{u} &= 0\\ \partial_t\mathbf{u} + (f + \xi)\hat{\mathbf{z}}\times\mathbf{u} + \nabla b &= 0\\ D\vartheta/Dt = 0, \qquad \partial_z p &= \vartheta\end{aligned}\right\} \qquad (\mathbf{x}, z)\in D_3(t),$$

$$p(\mathbf{x}, z, t) = 0 \qquad (\mathbf{x}, z)\in\partial D_3(t),$$

where $D/Dt = \partial_t + \mathbf{u}\cdot\nabla + w\partial_z$, $\vartheta(\mathbf{x}, z, t)$ is the buoyancy field, $b = p + \frac{1}{2}\mathbf{u}^2$, and $p(\mathbf{x}, z, t)$ is the kinematic pressure deviation from the atmospheric pressure, in Ball's volume of fluid setting, or from the reference

[2] Since $(d/dt)\iint_D \mu h\, dS = \iint_D \partial_t(\mu h)\, dS + \oint_{\partial D}\mu h\mathbf{u}\cdot\hat{\mathbf{n}}\, dl = \iint_D [\partial_t(\mu h) + \nabla\cdot(\mu h\mathbf{u})]\, dS = \iint_D h\, D\mu/Dt\, dS$.

pressure profile of the surrounding fluid, in a "reduced gravity" setting like that of Maas and Zahariev (1996). It is easy to see that Ball's theorem (2.1) is also valid in three dimensions, where $\langle \cdots \rangle$ represents a volume average in the domain $D_3(t)$.

Different orthogonal coordinates (x_1, x_2) can be used on the sphere. If the arc element takes the form

$$|d\mathbf{r}| = \sqrt{\gamma_1^2 dx_1^2 + \gamma_2^2 dx_2^2},$$

then the area element and the differential operators are

$$dS = \gamma_1\gamma_2\, dx_1\, dx_2,$$
$$\boldsymbol{\nabla} b = (\gamma_1^{-1}\partial_{x_1} b)\hat{\mathbf{x}}_1 + (\gamma_2^{-1}\partial_{x_2} b)\hat{\mathbf{x}}_2,$$
$$\boldsymbol{\nabla}\cdot\mathbf{u} = \gamma_1^{-1}\gamma_2^{-1}(\partial_{x_1}(\gamma_2 u_1) + \partial_{x_2}(\gamma_1 u_2)),$$
$$\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}\times\mathbf{u} = \gamma_1^{-1}\gamma_2^{-1}(\partial_{x_1}(\gamma_2 u_2) - \partial_{x_2}(\gamma_1 u_1)).$$

For instance, for the rescaled spherical coordinates (1.1) it is $Dx/Dt = u/\gamma$ and $Dy/Dt = \upsilon$, and the shallow water equations take the form

$$\frac{Dh}{Dt} + \frac{h}{\gamma}\frac{\partial u}{\partial x} + \frac{h}{\gamma}\frac{\partial(\gamma\upsilon)}{\partial y} = 0,$$
$$\frac{Du}{Dt} - (f + \tau u)\upsilon + \frac{g}{\gamma}\frac{\partial h}{\partial x} = 0,$$
$$\frac{D\upsilon}{Dt} + (f + \tau u)u + g\frac{\partial h}{\partial y} = 0,$$

with $\tau := -\gamma^{-1}\, d\gamma/dy = R^{-1}\tan\theta$. (Note that $\tau \sim \tau_0$ as $y/R \to 0$.) On the other hand, in the moving stereographic coordinates derived in Part I and described in the introduction, the shallow-water equations are, exactly,

$$\partial_t' h + \boldsymbol{\nabla}'\cdot(h\mathbf{u}') = 0,$$
$$\partial_t'\mathbf{u}' + (f' + \xi')\hat{\mathbf{z}}\times\mathbf{u}' + \boldsymbol{\nabla}' b' = 0, \qquad (2.2)$$

where ∂_t' is the operator for the time derivative at fixed $x' = (x', y')$ and

$$\xi' = \hat{\mathbf{z}}\cdot\boldsymbol{\nabla}'\times\mathbf{u}', \qquad b' = gh + \frac{1}{2}\mathbf{u}'^2 + \Phi'.$$

Notice that the potential vorticity is the same in both systems $q = (f + \xi)/h \equiv (f' + \xi')/h$. Since $\gamma_1 = \gamma_2 = \tilde{\gamma}$, the system (2.2) for $(h, u', \upsilon')(x', y', t)$ can be explicitly written as

$$\partial_t' h + \tilde{\gamma}^{-2}\boldsymbol{\nabla}_0'\cdot(\tilde{\gamma} h\mathbf{u}') = 0,$$
$$(\partial_t' + \tilde{\gamma}^{-1}\mathbf{u}'\cdot\boldsymbol{\nabla}_0')\mathbf{u}' + f'\hat{\mathbf{z}}\times\mathbf{u}' + \tilde{\gamma}^{-1}\boldsymbol{\nabla}_0'(gh + \Phi') = 0, \qquad (2.3)$$

where $\boldsymbol{\nabla}_0'$ is the nabla operator, as if (x', y') were Cartesian coordinates.

Bulk properties for uniform translation

Assume a steadily propagating solution. In spherical coordinates, all dynamical fields are functions of $(x - ct, y)$ or $(x - ct, y, z)$ for some constant c (which is an internal drift c_i, since inertial oscillations are not included in this calculation). Consequently, $d\langle x\rangle/dt = c$ and $d\langle\upsilon\rangle/dt = 0$. Moreover, from $dS = \gamma\, dx\, dy$ and $d\gamma/dy = -\gamma\tau$, it follows $\langle\partial p/\partial y\rangle = \frac{1}{2}\langle p\tau\rangle$, and finally $\langle\upsilon\rangle = 0$ implies $\langle u\rangle = -f_0^{-1}\langle(f - f_0)u + u^2\tau + \frac{1}{2}p\tau\rangle$ and $c = \langle\gamma^{-1}u\rangle$, exactly. (The center of mass zonal velocity U is equal to c due to the lack, here, of center of mass oscillations.) In an expansion in inverse powers of R, to lowest order it is

$$\langle u\rangle \sim -f_0^{-1}\left\langle \beta y u_0 + u_0^2\tau_0 + \frac{1}{2}p_0\tau_0\right\rangle,$$
$$c \sim \langle u\rangle + \tau_0\langle y u_0\rangle.$$

These results are quite general. They apply to both homogeneous and stratified isolated volumes of fluid. They might even be valid for a nonisolated, uniformly translating solution, as long as the $O(R^{-2})$ terms and the remainder in the integration by parts of $\langle\partial p/\partial y\rangle$ can be neglected. Notice that to calculate the lowest order contribution to the bulk quantities, namely $O(R^{-1})$, it is not necessary to study the motion of the center of mass [as done, for instance, by Killworth (1983) in the classical β plane], and it is enough to know the $O(R^0)$ solution, which need not be axisymmetric.

It is much simpler to calculate bulk formulas for homogeneous or stratified uniformly propagating vortices in stereographic coordinates, for which the solution is steady, and therefore $\langle\mathbf{u}'\rangle$ is constant (in fact, c must be chosen so that $\langle\mathbf{u}'\rangle = 0$). Using Ball's theorem (2.1) with $d\langle\mathbf{u}'\rangle/dt = \mathbf{0}$ yields, exactly,

$$\langle f'\mathbf{u}'\rangle = \hat{\mathbf{z}}\times\langle\boldsymbol{\nabla}'(p + \Phi')\rangle.$$

Now $\langle\boldsymbol{\nabla}' p\rangle$ is proportional to $\iint(\frac{1}{2}\boldsymbol{\nabla}' h^2)\tilde{\gamma}^2\, dx'\, dy'$ in the homogeneous case or to $\iiint(\boldsymbol{\nabla}' p)\tilde{\gamma}^2\, dx'\, dy'\, dz$ in the stratified case; integrating by parts it is found $\langle\boldsymbol{\nabla}' p\rangle = O(R^{-2})$ in both cases because $\tilde{\gamma} = 1 + O(R^{-2})$ and $p = 0$ in the boundary. Using $\Phi' \sim f_0 c y'$, it is then found $\beta\langle y'\mathbf{u}'\rangle \sim -f_0 c\hat{\mathbf{x}}'$; that is, $\beta\langle y'\upsilon'\rangle = O(R^{-2})$ and

$$c \sim -\beta f_0^{-1}\langle y' u'\rangle. \qquad (2.4a)$$

Even though $\langle u'\rangle = 0$, from (1.3) it follows

$$\langle u\rangle \sim c + \tau_0\langle x'\upsilon'\rangle \qquad (2.4b)$$

and $\tau_0\langle x' u'\rangle = O(R^{-2})$. These results can be cast in a form more similar to those obtained for the disk, in Part I, as follows. Writing the horizontal velocity field in terms of the rotation velocity $\omega(x', y', z)$ and radial velocity $u_r'(x', y', z)$

$$u' = -\omega y' + \frac{x'}{r}u_r', \qquad \upsilon' = \omega x' + \frac{y'}{r}u_r',$$

from $d\langle x'y'\rangle/dt = 0$ it follows $\langle\tilde{\gamma}^{-1}(y'u' + x'\upsilon')\rangle = 0$, and therefore $\langle y'u' + x'\upsilon'\rangle \sim 0$. Consequently, since $\langle y'u'\rangle - \langle x'\upsilon'\rangle = \frac{1}{2}\langle\omega r^2\rangle$, it is finally found $\langle y'u'\rangle \sim -\langle x'\upsilon'\rangle \sim -\frac{1}{2}\langle\omega r^2\rangle$, which implies

$$c \sim \frac{1}{2}\beta f_0^{-1}\langle \omega r^2 \rangle, \qquad \langle u \rangle \sim \frac{1}{2}(\beta f_0^{-1} + \tau_0)\langle \omega r^2 \rangle, \quad (2.5)$$

which are exactly the formulas obtained for the uniformly propagating disk (see Table 1 in Part I and recall that $\beta f_0^{-1} + \tau_0 = \beta f_0^{-1} \sec^2\theta_0$). Notice that the only assumption made is the existence of a uniformly propagating solution (an approximate example of which is given in section 3): it is not necessary to make any hypothesis on the shape of this solution (e.g., the lowest order field need not be axisymmetric).

3. Structure of the purely translating solutions

For simplicity, a homogeneous vortex will be considered here. In order to find the structure of a vortex in pure precession, $\partial_t' = 0$ is made in the exact evolution equations (2.2). The first one is satisfied defining a transport function, $h\mathbf{u}' = \hat{\mathbf{z}} \times \nabla'\psi$, which can then be used to write the potential vorticity as

$$q = \frac{f' + \nabla'(h^{-1}\nabla'\psi)}{h}.$$

The second equation in (2.2) then implies $b' = B(q)$ and $\psi = \Psi(q)$, where the functions $B(q)$ and $\Psi(q)$ could be multivalued and are related by

$$\frac{dB}{dq} = q\frac{d\Psi}{dq}.$$

These represent coupled and highly nonlinear differential equations for Ψ and h, to be solved in the domain inside the $h = 0$ curve (which is part of the solution), and such that c is an eigenvalue (hidden in the definitions of both b' and q) to be determined by the requirement of steady and well-behaved fields. This is hardly a problem to be solved ''on the back of an envelope.''

Solutions are thus found making an expansion in R^{-1} and working with (2.3), which uses (x', y') as *planar* coordinates. To lowest order it is

$$\nabla_0' \cdot (h_0 u_0') = 0,$$

$$\mathbf{u}_0' \cdot \nabla_0'\mathbf{u}_0' + f_0\hat{\mathbf{z}} \times \mathbf{u}_0' + g\nabla_0' h_0 = \mathbf{0}, \quad (3.1)$$

and to first order in R^{-1}, it is

$$\mathcal{D}\begin{pmatrix} h_1 \\ \mathbf{u}_1' \end{pmatrix} := \begin{pmatrix} \nabla_0' \cdot (h_0\mathbf{u}_1' + h_1\mathbf{u}_0') \\ \mathbf{u}_0' \cdot \nabla_0'\mathbf{u}_1' + \mathbf{u}_1' \cdot \nabla_0'\mathbf{u}_0' + f_0\hat{\mathbf{z}} \times \mathbf{u}_0' + g\nabla_0' h_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \mathbf{F} \end{pmatrix}, \quad (3.2)$$

where in

$$\mathbf{F} = -cf_0\hat{\mathbf{y}}' - \beta y'\hat{\mathbf{z}} \times \mathbf{u}_0'$$

are grouped the leading term of $-\nabla\Phi'$ and $-(f' - f_0)\hat{\mathbf{z}} \times \mathbf{u}'$. These are the equations to solve in order to find the lowest order correction to the vortex structure. The a priori formula for the drift velocity (2.4a) follows from the condition $\langle \mathbf{F} \rangle = \mathbf{0}$. Clearly, (3.1) is no more than the equations for a steady solution on the f plane: A reasonable choice for $(h_0, \mathbf{u}_0')$ is any stable equilibrium and appendix A shows how to use its (f plane) normal modes in order to calculate $(h_1, \mathbf{u}_1')$.

The simplest form of the lowest order solution is probably a circular vortex in (anticyclonic) solid-body rotation, $\omega = -\nu f_0 = \text{const}$, which implies

$$u_0' = \nu f_0 y', \qquad v_0' = -\nu f_0 x',$$

$$gh_0 = \frac{1}{2}\nu(1 - \nu)f_0^2(a^2 - r^2), \quad (3.3)$$

where obviously $0 < \nu < 1$ and $0 \le r \le a$. The a priori formula (2.4a) gives

$$c = -\frac{1}{6}\nu\beta a^2. \quad (3.4)$$

In appendix B it is shown that the $O(R^{-1})$ solution can be written in compact form as

$$\left.\begin{aligned} gh &= \frac{1}{12}f_0[6\nu(1-\nu)f_0 + (1+3\nu)\beta y'](a^2 - r^2) \quad &(3.5a)\\ \psi &= \frac{gf_0h^2}{2(1-\nu)[f_0 + \beta y'/(3-3\nu)]^2} \quad &(3.5b) \end{aligned}\right\}$$

$+ O(R^{-2})$. Notice that, if h and $\mathbf{u}'$ $(=h^{-1}\hat{\mathbf{z}} \times \nabla'\psi)$ are written as polynomials in the coordinates (x', y'), neglecting $o(R^{-1})$ terms, these polynomials are one order larger than that of the exact f-plane solutions discussed in the introduction. Two nondimensional parameters characterize the vortex (3.5), ν and

$$\varepsilon = \frac{\beta a(1 + 3\nu)}{18f_0\nu(1 - \nu)},$$

in addition to the environmental parameters f_0, β, and τ_0; as in the analysis of the disk presented in Part I, τ_0 only enters in the transformation back to spherical coordinates $(x', y') \mapsto (x, y)$.

To second order, the total vertical vorticity is given by

$$f + \xi = (1 - 2\nu)\left(f_0 + \frac{\beta y'}{3 - 3\nu}\right) + O(R^{-2}). \quad (3.6)$$

The gradient of relatively vorticity $-\beta(2 - \nu)/(3 - 3\nu)$ is opposite to the planetary vorticity gradient β, and is very important: its smallest value, corresponding to $\nu \to 0$, equals $-(2/3)\beta$; for solutions with anticyclonic absolute vorticity ($\nu > 1/2$) changes in the relative vorticity are larger, in magnitude, than those of the ambient vorticity.

The height field can be written as $h = (1 + 3\varepsilon y'/a)h_0(r)$; solutions will be restricted to $|\varepsilon| < 1/3$ (although formally it is $\varepsilon \ll 1$), so that the boundary is the circle $h_0 = 0$. This condition can be seen as limiting the allowed radii a as a function of ν, namely, $\beta a/|f_0| < 6\nu(1 - \nu)/(1 + 3\nu)$; the right hand side reaches a maximum of 2/3 at $\nu = 1/3$. The total vorticity can be written as

$$f + \xi = \left(1 + \frac{6\varepsilon\nu}{1 + 3\nu}\frac{y'}{a}\right)(f_0 + \xi_0),$$

and thus it has the same sign all over the domain of the vortex, because

$$\left|\frac{6\varepsilon\nu}{1 + 3\nu}\right| < \frac{1}{2}.$$

Notably, to the order resolved the relationship between Bernoulli head, transport function, and potential vorticity is the same as that of the $O(R^0)$ solution (valid on the f plane), namely

$$\left.\begin{aligned} B(q) &= A/q + b_0 \\ \Psi(q) &= \frac{1}{2}A/q^2 \end{aligned}\right\}, \qquad (3.7)$$

where $A = gf_0(1 - 2\nu)^2/(1 - \nu)$ and $b_0 = \nu^2 f_0^2 a^2/2$. Notice that $d\Psi/dq = -A/q^3$ is negative (positive) if $\nu < 1/2$ ($\nu > 1/2$), that is, if the total vorticity is cyclonic (anticyclonic). These vortices are circular but not axisymmetric. The applicable formal sufficient stability conditions (derived from the conservation of pseudoenergy) take the form $d\Psi/dq > 0$ and $\mathbf{u}'^2 < gh$ (Ripa 1991), and are violated somewhere for all $\varepsilon \neq 0$. Consequently, it is not possible to say anything a priori on the stability of these solutions, unlike the f-plane lens, which is proved stable using conservation of pseudoenergy and vertical angular pseudomomentum.

The vortex boundary is a circle with radius a. Some points of interest are

	x'	y'
Deepest ($\nabla h = 0$):	0	εa
No motion ($\nabla \psi = 0$):	0	$\varepsilon a/(1 + 3\nu)$
Center of mass:	0	0
Center:	0	$-\varepsilon a/2$

$+ O(\varepsilon^2 a)$. To lowest order in ε, the depth contours h = const and particle orbits ψ = const belong to different families of nonconcentric circles; see Fig. 1. Since isobaths and orbits are different circles, a water column changes its height along its trajectory, in order to compensate for the changes of total vorticity (3.6), so that the potential vorticity $q = (f + \xi)/h$ remains constant.

Consider a general water column whose orbit has a radius r_0 ($\leq a$); it can be shown that the center is at

$$y_0 = \frac{2a^2 - 3r_0^2(1 + \nu)}{2(1 + 3\nu)a}\varepsilon.$$

Evaluating $\sqrt{u'^2 + v'^2}/r_0$ along $x'^2 + (y' - y_0)^2 = r_0^2$ it follows that the angular velocity is anticyclonic and varies linearly with y', as it does ξ in (3.6). More precisely, the Lagrangian trajectory and relative vorticity of a generic fluid element are given by

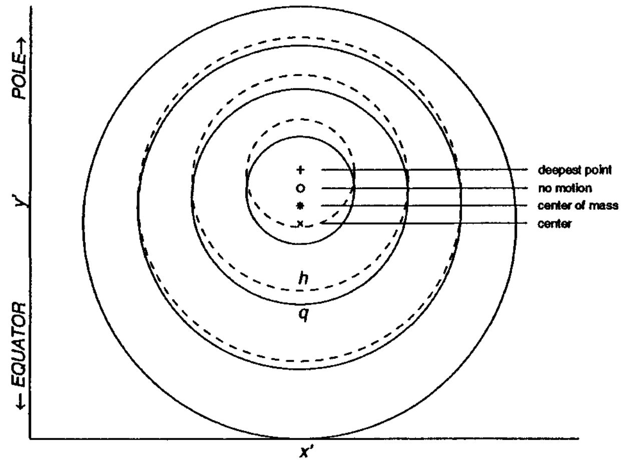

FIG. 1. Structure of a circular and almost solid-body rotating vortex, Eq. (3.5), in a stereographic projection following the secular drift. The effects of the earth's curvature are exaggerated. The orbits (solid lines) are contours of potential vorticity, transport function, or Bernoulli head; the three fields are functionally related in Eq. (3.7). Dashed contours are isobaths. Notice that in each orbit the depth increases with the distance to the equator; the absolute vorticity changes likewise, in order to conserve potential vorticity.

$$\text{Water column} = \begin{cases} (x', y') \sim (0, y_0) + r_0(\cos\kappa, \sin\kappa) \\ \dot{\kappa} \sim -\nu f_0 - \dfrac{1}{6}\beta y' \\ \xi' \sim -2\nu f_0 - \dfrac{1}{3}\dfrac{2 - \nu}{1 - \nu}\beta y'. \end{cases}$$

This is qualitatively similar to the results obtained in Part I for a disk, namely the coordinates (X', Y') of its center of mass and the internal rotation ω round this point are given by

$$\text{Disk} = \begin{cases} (X', Y') \sim \rho(\cos\kappa, \sin\kappa) \\ \dot{\kappa} \sim -f_0 - \dfrac{1}{2}\beta Y' \\ \omega \sim \overline{\omega} - \dfrac{1}{2}\beta Y', \end{cases}$$

where ρ is the radius inertial oscillation and $\overline{\omega}$ is the temporal mean of ω.

In the case of the symmetric disk, changes of the intrinsic rotation are related to meridional motions through the law of vertical angular momentum conservation $f + \omega$ = const. On the other hand, changes in the relative vorticity of a water column are more complicated because they are produced by both the meridional displacements and the divergence field

$$\nabla' \cdot \mathbf{u}' \sim \frac{1}{3 - 3\nu}\beta x'.$$

The acceleration and driving forces for water particles in any orbit are presented in Table 1, where $\hat{\mathbf{r}}_0$ is the

TABLE 1. Uniformly translating vortex, as seen in a stereographic projection following the secular drift. (The entries are the factors that multiply the vectors defined on top of each column.) For a particle in an orbit of radius r_0 ($\leq a$), the radial and meridional components of the acceleration are produced by the ageostrophic imbalance (sum of the Coriolis and pressure forces) and the "geoforce" (sum of the equatorward centrifugal force and the poleward gravitational one); see Fig. 2.

	$O(R^0)$	$O(R^{-1})$	
	$r_0 f_0^2 \hat{\mathbf{r}}_0$	$r_0 f_0 \beta \delta y' \hat{\mathbf{r}}_0$	$f_0 \beta \hat{\mathbf{y}}'$
Acceleration	$-\nu^2$	$-\frac{1}{2}\nu$	$\frac{1}{6}\nu r_0^2$
Coriolis	$-\nu$	$-\frac{1}{6} - \nu$	0
Pressure	$\nu - \nu^2$	$\frac{1}{6} + \frac{1}{2}\nu$	$-\frac{1}{6}\nu(a^2 - r_0^2)$
Geoforce	0	0	$\frac{1}{6}\nu a^2$

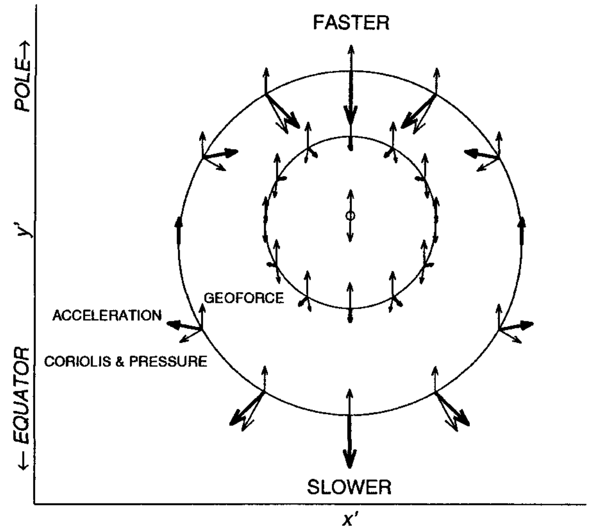

FIG. 2. Excess acceleration and forces, from the f-plane balance, for the boundary, one internal orbit, and the point of no motion [see $O(R^{-1})$ contributions in Table 1]. This diagram is similar to that corresponding to the inertial oscillation of a particle or disk, except that the Coriolis force is here replaced by the ageostrophic imbalance, that is, the sum of the Coriolis and pressure forces. The (larger) smaller the orbit the more meridional (central) the pressure forces are; in the limit of the point of no motion, there is an exact balance between the "geoforce" and the pressure force.

radial unit vector with respect to the center of that particular orbit, and exemplified in Fig. 2.

In the case of the disk, the acceleration,

$$-\left(f_0 + \frac{3}{2}\beta Y'\right)\rho f_0 \hat{\mathbf{r}} + \frac{1}{2}\beta \rho^2 f_0 \hat{\mathbf{y}}',$$

is produced by the Coriolis forces due to the orbital motion $-[f_0 + (3/2)\beta Y']\rho f_0 \hat{\mathbf{r}}$ and the internal motion $\frac{1}{2}\beta I \overline{\omega} \hat{\mathbf{y}}'$, as well as the geoforce

$$-f_0 c \hat{\mathbf{y}}' = \frac{1}{2}\beta(\rho^2 f_0 - I\overline{\omega})\hat{\mathbf{y}}'.$$

The acceleration and driving forces for the water columns near the boundary, $r_0 \sim a$, are similar to those of the disk's center of mass (see Table 2 in Part I), with the ageostrophic imbalance instead of the Coriolis force and without the internal Coriolis force. For water columns near the point of no motion, $r_0 \ll a$, on the other hand, there is a balance between the pressure and geoforce, sea level slopes down toward the equator, maintaining the uniform translation c.

In order to compute a Lagrangian time average of the balances of Table 1, recall that $\hat{\mathbf{r}}_0$ rotates at a nonuniform rate. In particular, it can be shown that

$$\overline{\hat{\mathbf{r}}_0} = -\frac{1}{12}\frac{\beta r_0}{\nu f_0}\hat{\mathbf{y}}' \quad \text{and} \quad \overline{\delta y' \hat{\mathbf{r}}_0} = \frac{1}{2} r_0 \hat{\mathbf{y}}';$$

these equations imply that the time averaged acceleration vanishes, as it should.

The classical β plane

This approximation is formally equivalent to making $\gamma = 1$ (and thus $\tau_0 = 0$) in spherical coordinates. Its prediction for $c \sim -\beta f_0^{-1}\langle y u_0 \rangle$ (Nof 1981; Killworth 1983) coincides with ((2.4a)), but the value of $\langle u \rangle$ predicted by this approximation, $\langle u \rangle = c$ instead of (2.4b), is wrong by a factor of $\cos^2\theta_0$. Allowing for oscillations of the center of mass, as done in Part III, the prediction for the time-averaged center of mass zonal velocity $\overline{U}$ is also incorrect (see also Part I). With respect to the vorticity, the classical β-plane approximation predicts

$$\partial_x v - \partial_y u \sim -2\nu f_0 - \frac{1}{3}\frac{2 - \nu}{1 - \nu}\beta y,$$

whereas the correct value is obtained using $\xi \sim (1 + \tau_0 y)(\partial_x v - \partial_y u) + \tau_0 u$ in (3.6), which gives an extra term, $\nu f_0 \tau_0 y$, of the same order as βy.

In the classical β-plane model, the transformation to the frame moving with the vortex is $x'' = x - ct$ and $y'' = y$. The equations of motion also take the form (2.2), except that the transformed variables are $\mathbf{u}'' = \mathbf{u} - c\hat{\mathbf{x}}$, $f'' = f$, $\xi'' = \xi$, and $\Phi'' = c f_0 y + \frac{1}{2}c\beta y^2$ (Nof 1981; Killworth 1983). These differ from the exact ones at $O(R^{-2})$ [for instance, the effective potential should be $\Phi'' = c f_0 y''(1 - \frac{1}{2}\tau_0 y'') + \frac{1}{2}c\beta y''^2 + O(R^{-3})$]. Consequently, the lowest order nontrivial solutions of (3.1) and (3.2) are formally the same, except that they are posed in different coordinate systems, which differ in $O(R^{-1})$. A similar situation is encountered in the simpler problems of the particle and the disk. In the three cases, the classical β-plane approximation gives the correct value of c because it uses the same set of equations even though in the incorrect frame, (x'', y'') instead of (x', y'). Consequently, the vortex structure calculated by Killworth (1983) can be rendered valid reinterpreting

$(u'', v'', h'')(x'', y'') \mapsto (u', v', h')(x', y')$ (see last paragraph in appendix B). Benilov (1996) calculated the structure of $(u, v, h)(x'', y'')$ for a nonisolated vortex ($h \to h_\infty$ as $r \to \infty$). If the decay is fast enough so that the geometric terms proportional to $\tilde{\gamma} r$ can be neglected, then this solution could also be rendered valid reinterpreting $(u - c, v, h)(x'', y'') \mapsto (u', v', h')(x', y')$.

4. Conclusions

An isolated vortex in a rotating planet experiences a secular westward drift, along a latitude circle, and consequently the natural coordinates to describe the problem are spherical ones. The curvature of the planet has two effects of similar importance: the change of Coriolis parameter with latitude (the ''β effect'') and a geometric one (the convergence of the meridians towards the poles). For a small vortex both effects are $O(a/R)$, where a and R are the radius of the vortex and that of the planet; the classical β-plane approximation represents only the first effect, therefore making errors of the same order of magnitude as the difference between the f plane and exact solutions. The β effect is best described in stereographic coordinates that move with the secular drift of the vortex, for which nonplanar geometric corrections are then $O(R^{-2})$, but their drift speed c must be determined a posteriori.

A very simple purely translating solution of the shallow-water equations is found in these coordinates, which has the form of the well-known solid body rotating in the f plane, with $O(a/R)$ corrections. The vortex is circular but not axisymmetric. The isobaths are nonconcentric circles with their centers slightly shifted toward the nearest pole with depth. The orbits belong to a different set of nonconcentric circles. A water column makes an anticyclonic rotation, decreasing its speed and the magnitude of its vorticity (which could be either cyclonic or anticyclonic), while shrinking its height, when approaching the equator. The changes of relative vorticity are opposite to those of the ambient vorticity and large enough to keep a constant potential vorticity. Two forces produce the acceleration of a water column: the ageostrophic imbalance (sum of the pressure and Coriolis forces) in the radial direction of the orbit and the meridional ''geoforce,'' which is the imbalance between the poleward gravitational force (due to the deviation of the geoid from a perfect sphere and to inhomogeneities in the mass distribution within the planet) and the centrifugal force due to the planet's rotation.

The model used here is far too idealized to be compared with observations. Real vortices in the ocean or planetary atmospheres are not isolated; they usually ride on a external field, whose shear may be important, and exchange properties with the environment. Nevertheless, the results of this paper may be used as a scaling guide of what to expect from observations. Consider, for instance, $\theta_0 = \pi/4$, and thus $f_0 = 1.0 \times 10^{-4}$ s^{-1}, $\beta = 1.6 \times 10^{-11}$ m^{-1} s^{-1}, and $\tau_0 = 1.6 \times 10^{-7}$ m^{-1}. Choosing as zeroth-order solution a uniform potential vorticity lens, $\nu = 0.5$, with a radius $a = 100$ km, Eq. (3.4) gives $c = -1.3$ cm s^{-1}. The $O(R^{-1})$ solution studied in section 3 is a very good approximation of the whole solution. For instance, a physically meaningful $O(R^{-2})$ parameter is the relative change of the frame angular velocity when using coordinates following the secular drift; in this case $\delta\Omega/\Omega = -3 \times 10^{-9}$, suggesting that corrections beyond $O(R^{-1})$ are not necessary. Nondimensional numbers measuring the importance of the earth's curvature on the geometry (convergence of the meridians towards the pole) and the dynamics (drift speed over maximum particle velocity) are $\varepsilon_g := \tau_0 a = 1.6 \times 10^{-2}$ and $\varepsilon_d := c/(-\nu a f_0) = \frac{1}{6}\beta a/f_0 = 2.6 \times 10^{-3}$, respectively. Since $\varepsilon_g > \varepsilon_d$, it is clear that geometric effects in spherical-like coordinates cannot be ignored, that is, the classical β-plane approximation is quantitatively incorrect. Consequently, any attempt to describe the physics beyond the f-plane scenario is best done in the stereographic coordinate frame, following the secular drift. Notice that this conclusion is independent of the size of the vortex, since $\varepsilon_g, \varepsilon_d \propto a$ (their ratio is, in general, $\varepsilon_g/\varepsilon_d = 6 \tan^2\theta_0$), and thus is expected to hold for solutions large enough for their difference with the f-plane structure to be more significant. [A similar conclusion is reached using ε, defined in the text, instead of ε_d; indeed $\varepsilon_g/\varepsilon = 18 \tan^2\theta_0 \nu(1 - \nu)/(1 + 3\nu)$.]

Ball (1963) showed that in the context of the f-plane approximation (constant Coriolis parameter and flat geometry), the internal and center of mass motions are decoupled (the latter being a pure inertial oscillation). Including planet curvature effects, both motions are coupled and more complicated. The solutions for the vortex in pure translation (done here) and the general solution for the disk (presented in Part I) are a first step toward the understanding of this problem. Part III will be devoted to the more difficult task of addressing Ball's problem in the sphere, that is, including the inertial oscillations of the center of mass and their interaction with the internal motion.

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APPENDIX A

Derivation of the $O(R^{-1})$ Uniformly Translating Vortex Fields

As explained in the main text, the solution $(h_0, \mathbf{u}'_0)$ of (3.1) is any stable equilibrium of the f-plane equations. Its perturbation normal modes $(\hat{h}_a, \hat{\mathbf{u}}_a)$ are calculated, in the f plane, as the eigensolutions of

$$(\mathcal{D} - i\hat{\omega}_a)\begin{pmatrix}\hat{h}_a\\ \hat{\mathbf{u}}_a\end{pmatrix} = \begin{pmatrix}0\\ \mathbf{0}\end{pmatrix}, \qquad \text{(A.1)}$$

where the operator $\mathcal{D}$ is defined in (3.2). These eigenmodes satisfy an orthogonality condition of the form

$$(\hat{h}_a^*, \hat{\mathbf{u}}_a^*) \cdot (\hat{h}_b, \hat{\mathbf{u}}_b) = 0 \text{ if } a \neq b, \qquad \text{(A.2)}$$

where the centered dot denotes a linear operator. A very simple example will be given, and used, shortly: $(\delta h, \delta\mathbf{u}) \cdot (\delta h, \delta\mathbf{u})$ is the *pseudoenergy* of the perturbation $(\delta h, \delta\mathbf{u})$ superimposed to the basic flow $(h_0, \mathbf{u}_0')$. Since these modes span a complete basis, the solution of (3.2) is easily found to be of the form

$$\begin{pmatrix}h_1\\ \mathbf{u}_1'\end{pmatrix} = \sum_a \frac{i}{\hat{\omega}_a}\frac{(\hat{h}_a^*, \hat{\mathbf{u}}_a^*) \cdot (0, \mathbf{F})}{(\hat{h}_a^*, \hat{\mathbf{u}}_a^*) \cdot (\hat{h}_a, \hat{\mathbf{u}}_a)}\begin{pmatrix}\hat{h}_a\\ \hat{\mathbf{u}}_a\end{pmatrix}. \qquad \text{(A.3)}$$

There are two important points to make about this expansion.

- The solution of (3.2) is defined modulo modes for which $(\hat{h}_a^*, \hat{\mathbf{u}}_a^*) \cdot (0, \mathbf{F})$ vanishes trivially.
- For all modes such that $\hat{\omega}_a = 0$, it must be required $(\hat{h}_a^*, \hat{\mathbf{u}}_a^*) \cdot (0, \mathbf{F}) = 0$; this is the condition that gives the eigenvalue c. These modes are also part of the null space of the solution to (3.2). Since the origin of f-plane solutions can be changed at will, two of these $\hat{\omega}_a = 0$ modes are $(\hat{h}_a, \hat{\mathbf{u}}_a) \propto \partial_{x'}(h_0, \mathbf{u}_0)$ and $(\hat{h}_a, \hat{\mathbf{u}}_a) \propto \partial_{y'}(h_0, \mathbf{u}_0)$.

In the particular case of the basic flow (3.3), the eigenmodes of (A.1) are derived in (Ripa 1992):[3] the pressure field $g\hat{h}_a$ and the polar components of the velocity field $\hat{\mathbf{u}}_a$ take the form of a polynomial in r times $e^{im\phi}$, where m is an integer, and the dispersion relations are given by the roots of

$m \neq 0$, $n > 0$:

$$\frac{\omega_*^2 - f_*^2}{\nu(1 - \nu)f_0^2} + \frac{mf_*}{\omega_*} = 2n(n + |m| + 1) + |m|,$$

$m = 0$, $n > 0$:

$$\begin{cases}\omega_* = 0\\ \omega_* = \pm f_0\sqrt{1 + 2\nu(1 - \nu)(n - 1)(n + 2)},\end{cases}$$

$m \neq 0$, $n = 0$:

$$\omega_* = -\frac{1}{2}f_* \operatorname{sgn}(m) \pm \frac{1}{2}f_0\sqrt{1 + 4\nu(1 - \nu)(|m| - 1)},$$

where n is another integer, $\omega_* = \hat{\omega}_a + m\nu f_0$, and $f_* = f_0(1 - 2\nu)$. The pseudoenergy integral for this basic state is simply

$$(\hat{h}_a^*, \hat{\mathbf{u}}_a^*) \cdot (\hat{h}_b, \hat{\mathbf{u}}_b) = \iint_{h_0>0} (h_0\hat{\mathbf{u}}_a^* \cdot \hat{\mathbf{u}}_b + g\hat{h}_a^*\hat{h}_b)\, dS.$$

Given the inner product and the form of (A.3), it is clear that:

- All $\hat{\omega}_a = 0$ modes with $m = 0$ are orthogonal to the forcing $\langle\hat{\mathbf{u}}_a^* \cdot \mathbf{F}\rangle = 0$; these axisymmetric steady modes span the trivial part of the null solution of (3.2), which can be absorbed in a change in the angular velocity $\omega(r)$. These modes will not be included, keeping a solid body rotating as the structure of the lowest order solution: $\omega = -\nu f_0$.
- Orthogonality of the forcing $(\iint_{h_0>0} \hat{\mathbf{u}}_a^* \cdot \mathbf{F} = 0)$ in (3.2) to the nonsymmetric $\hat{\omega}_a = 0$ modes $(\hat{h}_a, \hat{\mathbf{u}}_a) \propto \partial_{x'}(h_0, \mathbf{u}_0)$ and $(\hat{h}_a, \hat{\mathbf{u}}_a) \propto \partial_{y'}(h_0, \mathbf{u}_0)$ give the eigenvalue (3.4) and $\langle x'y'\rangle = 0$.
- The $O(R^{-1})$ solution is then obtained making the expansion (A.3) in the normal modes, or by the method explained in appendix B, which gives (B.4).

The expansion in normal modes is useful for two things. First, imposing orthogonality of the forcing with mode $\partial_{x'}(h_0, \mathbf{u}_0)$ gives the drift velocity (3.4). Second, the solution is defined modulo the addition of this mode that taken as a wave sustained by the lens in the f plane, is related to the freedom of changing the origin of coordinates. Addition of mode $\partial_{x'}(h_0, \mathbf{u}_0)$ in the present context is not a trivial result, though, because the set (3.1)–(3.2) is *not* invariant under a change of the origin of coordinates (the f-plane dynamics was used only to derive the lowest order solution and the normal modes basis).

APPENDIX B

Structure of an Almost Solid-Body Rotating Vortex

The solution of (3.2) in the particular case of the zeroth-order flow in the form of the "lens" (3.3) could be obtained by the general method of making the expansion (A.3) derived in appendix A. However, a more direct method is presented next for this particular case. Equation (3.2) is of the type

$$\mathcal{D}\begin{pmatrix}\tilde{h}\\ \tilde{\mathbf{u}}\end{pmatrix} = \begin{pmatrix}0\\ \mathbf{F}\end{pmatrix}, \qquad \text{(B.1)}$$

where the forcing $\mathbf{F}$ and the response $(\tilde{h}, \tilde{\mathbf{u}})$ are assumed to be proportional to $e^{im\phi}$. A peculiarity of the lens basic state is that these perturbation equations are like the linearized shallow-water equations in paraboloid topography, with a Doppler shifted frequency and modified Coriolis parameter, namely

[3] There are two typos on p. 404 of (Ripa 1992): a right parenthesis is missing in the equation for $\hat{P}$, it should finish "$\ldots g_r H_\nu)\hat{P} = 0$," and a + sign is missing in the line after dispersion relation, it should read "where $\hat{\gamma}^2 := 2n(n + |m| + 1) + |m|$ (see . . ."

$$-i\omega_*\tilde{h} + \nabla \cdot (h_0\tilde{\mathbf{u}}) = 0,$$

$$-i\omega_*\tilde{\mathbf{u}} + f_*\hat{\mathbf{z}} \times \tilde{\mathbf{u}} + g\nu\tilde{\mathbf{h}} = \mathbf{F},$$

where $gh_0 = \frac{1}{2}\nu(1 - \nu) f_0^2(a^2 - r^2)$, $\omega_* = m\nu f_0$ and $f_* = f_0(1 - 2\nu)$. The second equation gives

$$(\omega_*^2 - f_*^2)\tilde{\mathbf{u}} = (i\omega_* + f_*\hat{\mathbf{z}}\times)(\mathbf{F} - g\nabla\tilde{h}); \quad \text{(B.2)}$$

substituting in the first one it is obtained

$$\nabla \cdot \left[\frac{1}{2}(a^2 - r^2)\nabla\tilde{h}\right] + \mu\tilde{h} = F, \quad \text{(B.3)}$$

where

$$\mu = \frac{\omega_*^2 - f_*^2}{\nu(1 - \nu) f_0^2} + \frac{mf_*}{\omega_*}$$

$$gF = \nabla \cdot \left[\frac{1}{2}(a^2 - r^2)\left(1 - i\frac{f_*}{\omega_*}\hat{\mathbf{z}} \times\right) \mathbf{F}\right].$$

The normal modes, solution of the $F = 0$ equation, correspond to the eigenvalues $\mu = 2n(n + |m| + 1) + |m|$.

For $m = 1$ it is $\mu = 1$ and $F = \frac{1}{6}i\beta f(1 + 3\nu)(2a^2 - 3r^2)r$. The solution of (B.2)/(B.3) is

$$gh_1 = \frac{1}{12} f_0\beta(1 + 3\nu)(a^2 - r^2 + \kappa a^2)y', \quad \text{(B.4a)}$$

$$u_1' = \frac{1}{6}\beta y'^2 - \frac{1}{12}\frac{\beta}{1 - \nu} \times [(a^2 - r^2)(1 - \nu) + \kappa a^2(1 + 3\nu)], \quad \text{(B.4b)}$$

$$v_1' = -\frac{1}{6}\beta x'y', \quad \text{(B.4c)}$$

in Cartesian coordinates, where the terms proportional to the arbitrary parameter κ represent the freedom mentioned above, namely adding to $(\tilde{h}, \tilde{\mathbf{u}})$ a term proportional to $\partial_{y'}(h_0, \mathbf{u}_0)$, which is an homogeneous solution of (B.1). [The polar coordinates of $\mathbf{u}'_1$; that is, the real and imaginary parts of $(u_1' + iv_1')e^{-i\phi}$ have only terms proportional to $\cos\phi$ and $\sin\phi$, as it should. The solution in spherical coordinates, fixed to the Earth, is more complicated, namely, $u = c + \nu f_0[y - \frac{1}{2}\tau_0(x - ct)^2] + u_1'$ for the zonal component and $v = -\nu f_0(x - ct) + v_1'$ for the meridional one, where $(x', y') \sim (x, y)$ in the expression of u_1' and v_1', since they are $O(R^{-1})$ terms.]

The second-order solution is then $h = h_0 + h_1 + O(R^{-2})$ (and similarly for the velocity fields u' and v'), and is obviously valid in the domain determined by $h \geq 0$. Notice that if $\kappa = 0$ the boundary of the vortex is the circle $r = a$. A simpler representation of the solution is obtained as follows. First, the absolute vorticity is given by (3.6), independent of κ. Second, an appropriate transport function is given by

$$12(1 - \nu) f_0\psi = gh_0^2 - gh_0 f_0\beta[(1 + \nu)(r^2 - a^2) - \kappa a^2(1 + 3\nu)]y';$$

the rotated gradient of this function gives $(h_0 + h_1)\mathbf{u}_0' + h_0\mathbf{u}_1'$ not $(h_0 + h_1)(\mathbf{u}_0' + \mathbf{u}_1')$ but, since the difference between both vector fields is $O(R^{-2})$, ψ can be redefined to be equal to the expression in (3.5b) so that $\hat{\mathbf{z}} \times \nabla\psi \equiv (h_0 + h_1)(\mathbf{u}_0' + \mathbf{u}_1')$. Finally, for $\xi = O(1)$ and $\varepsilon \ll 1$, the parameter κ gives only a trivial displacement $(0, (3/2)\kappa\varepsilon a)$ of the whole solution; for example, $(3/2)\kappa\varepsilon a$ can be subtracted from y' in the formula (3.6) for the total vorticity, within the same order of accuracy. The second-order solution is then given by equations (3.5), where it has been chosen $\kappa = 0$, for simplicity. Killworth (1983) obtained the equivalent of solution (B.4) for $\kappa = -1$ and $\nu = (0, ½, 1)$, but in the incorrect frame used by the classical β-plane approximation: (x'', y'') instead of (x', y'). [There are discrepancies between the solutions published by Killworth (1983) and those of this paper, though.]

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