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AN EXPLICIT MERTENS' TYPE INEQUALITY FOR ARITHMETIC PROGRESSIONS

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ABSTRACT. We give an explicit Mertens type formula for primes in arithmetic progressions using mean values of Dirichlet L-functions at $s = 1$.

Key words and phrases: Mertens' formula, Arithmetic progressions, Mean values of Dirichlet $L-$functions.

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1. INTRODUCTION AND MAIN RESULT

The very useful Mertens' formula states that

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} \left\{1 + O\left(\frac{1}{\log x}\right)\right\}$$

for any real number $x \geq 2$, where $\gamma \approx 0.577215664\ldots$ denotes the Euler constant. Some explicit inequalities have been given in [4] where it is showed for example that

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right)^{-1} < e^{\gamma} \delta\left(x\right) \log x, \tag{1.1}$$

where

$$\delta\left(x\right) := 1 + \frac{1}{\left(\log x\right)^2}. \tag{1.2}$$

Let $1 \le l \le k$ be positive integers satisfying $(k, l) = 1$. The aim of this paper is to provide an explicit upper bound for the product

$$\prod_{\substack{p \le x \\ p \equiv l \,(\mathrm{mod}\, k)}} \left(1 - \frac{1}{p}\right)^{-1}. \tag{1.3}$$

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In [2, 5], the authors gave asymptotic formulas for (1.3) in the form

$$\prod_{\substack{p\le x\\ p\equiv l\ (\mathrm{mod}\ k)}}\left(1-\frac{1}{p}\right)\sim c\left(k,l\right)\left(\log x\right)^{-1/\varphi(k)},$$

where φ is the Euler totient function and $c\left(k,l\right)$ is a constant depending on l and k. Nevertheless, because of the non-effectivity of the Siegel-Walfisz theorem, one cannot compute the implied constant in the error term. Moreover, the constant $c\left(k,l\right)$ is given only for some particular cases in [2], whereas K.S. Williams established a quite complicated expression of $c\left(k,l\right)$ involving a product of Dirichlet $L-$functions $L\left(s;\chi\right)$ and a function $K\left(s;\chi\right)$ at $s=1$, where $K\left(s;\chi\right)$ is the generating Dirichlet series of the completely multiplicative function k_χ defined by

$$k_\chi\left(p\right):=p\left\{1-\left(1-\frac{\chi\left(p\right)}{p}\right)\left(1-\frac{1}{p}\right)^{-\chi(p)}\right\}$$

for any prime number p and any Dirichlet character χ modulo k. The author then gave explicit expressions of $c\left(k,l\right)$ in the case $k=24$.

It could be useful to have an explicit upper bound for (1.3) valid for a large range of k and x. Indeed, we shall see in a forthcoming paper that such a bound could be used to estimate class numbers of certain cyclic number fields. We prove the following result:

Theorem 1.1. *Let $1\le l\le k$ be positive integers satisfying $(k,l)=1$ and $k\ge 37$, and x be a positive real number such that $x>k$. We have:*

$$\prod_{\substack{p\le x\\ p\equiv l\ (\mathrm{mod}\ k)}}\left(1-\frac{1}{p}\right)^{-1}<e^{2(\gamma-B)}\sqrt{\zeta\left(2\right)\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)}\cdot\left(\frac{e^\gamma\varphi\left(k\right)}{k}\log x\right)^{\frac{1}{\varphi(k)}}\cdot\Phi\left(x,k\right),$$

where

$$\Phi\left(x,k\right):=\exp\left\{\frac{2}{\log x}\left(\frac{2\sqrt{k}\log k}{\varphi\left(k\right)}\sum_{\chi\ne\chi_0}\left|\frac{L'}{L}\left(1;\chi\right)\right|+2\sqrt{k}\log k+E-\gamma\right)\right\},$$

$B\approx 0.2614972128476 43\ldots$ *and* $E\approx 1.332582275733221\ldots$

The restriction $k\ge 37$ is given here just to use a simpler expression of the Polyá-Vinogradov inequality, but one can prove a similar result with $k\ge 9$ only, the constants in $\Phi\left(x,k\right)$ being slightly larger.

2. NOTATION

p denotes always a prime, $1\le l\le k$ are positive integers satisfying $(k,l)=1$ and $k\ge 37$, $x>k$ is a real number,

$$\gamma:=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)\approx 0.5772156649015328\ldots$$

is the Euler constant and

$$\gamma_1:=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\log k}{k}-\frac{\left(\log n\right)^2}{2}\right)\approx -0.07281584548367\ldots$$

is the first Stieltjes constant. Similarly,

$$E:=\lim_{n\to\infty}\left(\log n-\sum_{p\le n}\frac{\log p}{p}\right)\approx 1.332582275733221\ldots$$

and

$$B := \lim_{n\to\infty}\left(\sum_{p\le n}\frac{1}{p} - \log\log n\right) \approx 0.261497212847643\ldots$$

χ denotes a Dirichlet character modulo k and χ_0 is the principal character modulo k. For any Dirichlet character χ modulo k and any $s \in \mathbb{C}$ such that $\operatorname{Re} s > 1$, $L(s;\chi) := \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$ is the Dirichlet $L-$ function associated to χ. $\sum_{\chi\neq\chi_0}$ means that the sum is taken over all non-principal characters modulo k. Λ is the Von Mangoldt function and $f * g$ denotes the usual Dirichlet convolution product.

3. Sums with Primes

From [4] we get the following estimates:

Lemma 3.1.

$$\sum_p \log p \sum_{\alpha=2}^{\infty}\frac{1}{p^\alpha} = E - \gamma \quad \textit{and} \quad \sum_p \sum_{\alpha=2}^{\infty}\frac{1}{\alpha p^\alpha} = \gamma - B.$$

4. The Polyá-Vinogradov Inequality and Character Sums with Primes

Lemma 4.1. *Let χ be any non-principal Dirichlet character modulo $k \ge 37$.*

(i) *For any real number $x \ge 1$,*

$$\left|\sum_{n\le x}\chi(n)\right| < \frac{9}{10}\sqrt{k}\log k.$$

(ii) *Let $F \in C^1\left([1;+\infty[\,,\,[0;+\infty[\right)$ such that $F(t) \underset{t\to\infty}{\searrow} 0$. For any real number $x \ge 1$,*

$$\left|\sum_{n>x}\chi(n)F(n)\right| \le \frac{9}{5}F(x)\sqrt{k}\log k.$$

(iii) *For any real number $x > k$,*

$$\left|\sum_{p>x}\frac{\chi(p)}{p}\right| < \frac{2}{\log x}\left\{2\sqrt{k}\log k\left(\left|\frac{L'}{L}(1;\chi)\right| + 1\right) + E - \gamma\right\}.$$

Proof.

(i) The result follows from Qiu's improvement of the Polyá-Vinogradov inequality (see [3, p. 392]).

(ii) Abel summation and (i).

(iii) Let $\chi \neq \chi_0$ be a Dirichlet character modulo $k \ge 37$ and $x > k$ be any real number.

(a) Since $\chi(\mu * \mathbf{1}) = \varepsilon$ where $\varepsilon(n) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{otherwise}\end{cases}$ and $\mathbf{1}(n) = 1$, we get:

$$\sum_{d\le x}\frac{\mu(d)\chi(d)}{d}\sum_{m\le x/d}\frac{\chi(m)}{m} = 1$$

and hence, since $\chi \neq \chi_0$,

$$\sum_{d\le x}\frac{\mu(d)\chi(d)}{d} = \frac{1}{L(1;\chi)}\left(\sum_{d\le x}\frac{\mu(d)\chi(d)}{d}\sum_{m>x/d}\frac{\chi(m)}{m} + 1\right)$$

and thus, using (ii),

$$\left|\sum_{d\le x}\frac{\mu(d)\chi(d)}{d}\right| \le \frac{\frac{9}{5}\sqrt{k}\log k+1}{|L(1;\chi)|} < \frac{2\sqrt{k}\log k}{|L(1;\chi)|}. \tag{4.1}$$

(b) Since $\log = \Lambda * \mathbf{1}$, we get:

$$\begin{aligned}\sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n} &= \sum_{d\le x}\frac{\mu(d)\chi(d)}{d}\sum_{m\le x/d}\frac{\chi(m)\log m}{m}\\ &= \left(\sum_{d\le x/e}+\sum_{x/e<d\le x}\right)\sum_{m\le x/d}\frac{\chi(m)\log m}{m}\\ &= \sum_{d\le x/e}\frac{\mu(d)\chi(d)}{d}\sum_{m\le x/d}\frac{\chi(m)\log m}{m}+\frac{\chi(2)\log 2}{2}\sum_{x/e<d\le x}\frac{\mu(d)\chi(d)}{d}\\ &= -L'(1;\chi)\sum_{d\le x/e}\frac{\mu(d)\chi(d)}{d}-\sum_{d\le x/e}\frac{\mu(d)\chi(d)}{d}\sum_{m>x/d}\frac{\chi(m)\log m}{m}\\ &\quad+\frac{\chi(2)\log 2}{2}\sum_{x/e<d\le x}\frac{\mu(d)\chi(d)}{d}\end{aligned}$$

and, by using (ii), (4.1) and the trivial bound for the third sum, we get:

$$\begin{aligned}\left|\sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n}\right| &< \sqrt{k}\log k\left\{2\left|\frac{L'}{L}(1;\chi)\right|+\frac{9}{5x}\sum_{d\le x/e}\log\left(\frac{x}{d}\right)+\frac{\log 2}{2}\left(1+\frac{e}{x}\right)\right\}\\ &\le \sqrt{k}\log k\left\{2\left|\frac{L'}{L}(1;\chi)\right|+\frac{18}{5e}+\frac{\log 2}{2}\left(1+\frac{e}{37}\right)\right\}\\ &< 2\sqrt{k}\log k\left\{\left|\frac{L'}{L}(1;\chi)\right|+1\right\}\end{aligned} \tag{4.2}$$

since $x > q \ge 37$.

(c) By Abel summation, we get:

$$\left|\sum_{p>x}\frac{\chi(p)}{p}\right| \le \frac{2}{\log x}\max_{t\ge x}\left|\sum_{p\le t}\frac{\chi(p)\log p}{p}\right|.$$

Moreover,

$$\sum_{p\le t}\frac{\chi(p)\log p}{p} = \sum_{n\le t}\frac{\chi(n)\Lambda(n)}{n}-\sum_{\substack{p\\ p^\alpha\le t}}\sum_{\alpha=2}^{\infty}\frac{\chi(p^\alpha)\log p}{p^\alpha}$$

and then:

$$\begin{aligned}\left|\sum_{p\le t}\frac{\chi(p)\log p}{p}\right| &\le \left|\sum_{n\le t}\frac{\chi(n)\Lambda(n)}{n}\right| + \sum_p \log p\sum_{\alpha=2}^{\infty}\frac{1}{p^\alpha}\\ &= \left|\sum_{n\le t}\frac{\chi(n)\Lambda(n)}{n}\right| + E-\gamma\\ &< 2\sqrt{k}\log k\left\{\left|\frac{L'}{L}(1;\chi)\right|+1\right\}+E-\gamma\end{aligned}$$

by (4.2). This concludes the proof of Lemma 4.1.

□

5. MEAN VALUE ESTIMATES OF DIRICHLET $L-$FUNCTIONS

Lemma 5.1.

(i) *For any positive integers* j,k,

$$\sum_{\substack{n=1\\(n,k)=1}}^{jk}\frac{1}{n}=\frac{\varphi(k)}{k}\left\{\log(jk)+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right\}+\frac{c_0(j,k)\,2^{\omega(k)}}{jk}$$

where $\omega(k):=\sum_{p|k}1$ *and* $|c_0(j,k)|\le 1$.

(ii) *For any positive integer* $k\ge 9$,

$$\left(\frac{k}{\varphi(k)}\right)^2\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)+2\gamma_1+\gamma+\frac{\pi^2}{3}-\left(\log k+\sum_{p|k}\frac{\log p}{p-1}\right)^2\le 0.$$

(iii) *For any positive integer* $k\ge 9$,

$$\prod_{\chi\ne\chi_0}|L(1;\chi)|^{1/\varphi(k)}\le\sqrt{\zeta(2)}\prod_{p|k}\left(1-\frac{1}{p^2}\right)^{\frac{1}{2}}.$$

Proof.

(i)

$$\sum_{\substack{n=1\\(n,k)=1}}^{jk}\frac{1}{n}=\sum_{d|k}\frac{\mu(d)}{d}\sum_{n\le jk/d}\frac{1}{n}=\sum_{d|k}\frac{\mu(d)}{d}\left(\log\left(\frac{jk}{d}\right)+\gamma+\frac{\varepsilon(d)\,d}{jk}\right)$$

where $|\varepsilon(d)|\le 1$ and hence:

$$\sum_{\substack{n=1\\(n,k)=1}}^{jk}\frac{1}{n}=\{\log(jk)+\gamma\}\sum_{d|k}\frac{\mu(d)}{d}-\sum_{d|k}\frac{\mu(d)\log d}{d}+\frac{1}{jk}\sum_{d|k}\varepsilon(d)\,\mu(d)$$

and we conclude by noting that

$$\sum_{d|k}\frac{\mu(d)}{d}=\frac{\varphi(k)}{k},$$

$$\sum_{d|k}\frac{\mu(d)\log d}{d}=-\frac{\varphi(k)}{k}\sum_{p|k}\frac{\log p}{p-1}$$

and

$$\left|\sum_{d|k}\varepsilon\left(d\right)\mu\left(d\right)\right|\leq\sum_{d|k}\mu^{2}\left(d\right)=2^{\omega(k)}.$$

(ii) Define

$$A\left(k\right):=\left(\frac{k}{\varphi\left(k\right)}\right)^{2}\zeta\left(2\right)\prod_{p|k}\left(1-\frac{1}{p^{2}}\right)+2\gamma_{1}+\gamma+\frac{\pi^{2}}{3}-\left(\log k+\sum_{p|k}\frac{\log p}{p-1}\right)^{2}.$$

Using [1] we check the inequality for $9\leq k\leq 513$ and then suppose $k\geq 514$. Since

$$\frac{k}{\varphi\left(k\right)}=\prod_{p|k}\frac{p}{p-1}\leq\prod_{p|k}p^{\frac{1}{p-1}}$$

we have taking logarithms

$$\sum_{p|k}\frac{\log p}{p-1}\geq\log\left(\frac{k}{\varphi\left(k\right)}\right)\geq\log\left(\frac{k}{k-1}\right)$$

and from the inequality ([4])

$$\frac{k}{\varphi\left(k\right)}<e^{\gamma}\log\log k+\frac{2.50637}{\log\log k}$$

valid for any integer $k\geq 3$, we obtain

$$A\left(k\right)\leq\zeta\left(2\right)\left(e^{\gamma}\log\log k+\frac{2.50637}{\log\log k}\right)^{2}+2\gamma_{1}+\gamma+\frac{\pi^{2}}{3}-\left(\log\left(\frac{k^{2}}{k-1}\right)\right)^{2}<0$$

if $k\geq 514$.

(iii) First,

$$\sum_{\chi\neq\chi_{0}}\left|L\left(1;\chi\right)\right|^{2}=\lim_{N\to\infty}S\left(N\right)$$

where

$$S\left(N\right):=\sum_{m,n=1}^{Nk}\frac{\chi\left(n\right)\overline{\chi}\left(m\right)}{nm}-\left(\sum_{\substack{n=1\\(n,k)=1}}^{Nk}\frac{1}{n}\right)^{2}.$$

Following a standard argument, we have using (i):

$$\begin{aligned}S\left(N\right)&=\varphi\left(k\right)\sum_{\substack{m\neq n=1\\m\equiv n\ (\mathrm{mod}\ k)\\(n,k)=(m,k)=1}}^{Nk}\frac{1}{mn}-\left(\sum_{\substack{n=1\\(n,k)=1}}^{Nk}\frac{1}{n}\right)^{2}\\&=\varphi\left(k\right)\sum_{\substack{n=1\\(n,k)=1}}^{Nk}\frac{1}{n^{2}}+\varphi\left(k\right)\sum_{\substack{m\neq n=1\\m\equiv n\ (\mathrm{mod}\ k)\\(n,k)=(m,k)=1}}^{Nk}\frac{1}{mn}-\left(\sum_{\substack{n=1\\(n,k)=1}}^{Nk}\frac{1}{n}\right)^{2}\end{aligned}$$

$$\leq \varphi(k)\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)+2\varphi(k)\sum_{j=1}^{N}\sum_{\substack{n=1\\(n,k)=1}}^{(N-j)k}\frac{1}{n(n+jk)}-\left(\sum_{\substack{n=1\\(n,k)=1}}^{Nk}\frac{1}{n}\right)^2$$

$$=\varphi(k)\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)+\frac{2\varphi(k)}{k}\sum_{j=1}^{N}\frac{1}{j}\left(\sum_{\substack{n=1\\(n,k)=1}}^{(N-j)k}\frac{1}{n}-\sum_{\substack{n=1+jk\\(n,k)=1}}^{Nk}\frac{1}{n}\right)-\left(\sum_{\substack{n=1\\(n,k)=1}}^{Nk}\frac{1}{n}\right)^2$$

$$\leq \varphi(k)\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)+\frac{2\varphi(k)}{k}\sum_{j=1}^{N}\frac{1}{j}\sum_{\substack{n=1\\(n,k)=1}}^{jk}\frac{1}{n}-\left(\sum_{\substack{n=1\\(n,k)=1}}^{Nk}\frac{1}{n}\right)^2$$

$$=\varphi(k)\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)$$
$$+\frac{2\varphi(k)}{k}\sum_{j=1}^{N}\frac{1}{j}\left(\frac{\varphi(k)}{k}\left\{\log(jk)+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right\}+\frac{c_0(j,k)\,2^{\omega(k)}}{jk}\right)$$
$$-\left(\frac{\varphi(k)}{k}\left\{\log(Nk)+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right\}+\frac{c_0(N,k)\,2^{\omega(k)}}{Nk}\right)^2.$$

We now neglect the dependance of c_0 in k. Since

$$\sum_{m=1}^{M}\frac{1}{m}=\log M+\gamma+\frac{c_1(M)}{M}$$

and

$$\sum_{m=1}^{M}\frac{\log m}{m}=\frac{(\log M)^2}{2}+\gamma_1+\frac{c_2(M)\log M}{M},$$

where $0<c_1(M)\leq\frac{1}{2}$ and $|c_2(M)|\leq 1$, we get:

$$S(N)\leq\varphi(k)\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)+\left(\frac{\varphi(k)}{k}\right)^2\left\{(\log N)^2+2\gamma_1+\frac{2c_2(N)\log N}{N}\right.$$
$$+2\left(\log k+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right)\left(\log N+\gamma+\frac{c_1(N)}{N}\right)$$
$$\left.-\left(\log(Nk)+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right)^2\right\}$$
$$+\frac{2^{\omega(k)+1}\varphi(k)}{k^2}\left\{\sum_{j=1}^{N}\frac{c_0(j)}{j^2}-\frac{c_0(N)}{N}\left(\log(Nk)+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right)\right\}$$
$$-\frac{2^{2\omega(k)}c_0^2(N)}{N^2k^2}$$

$$= \varphi(k)\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)+\left(\frac{\varphi(k)}{k}\right)^2\left\{2\gamma_1+\gamma-\left(\log k+\sum_{p|k}\frac{\log p}{p-1}\right)^2\right.$$
$$\left.+\frac{2c_1(N)}{N}\left(\log k+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right)+\frac{2c_2(N)\log N}{N}\right\}$$
$$+\frac{2^{\omega(k)+1}\varphi(k)}{k^2}\left\{\sum_{j=1}^{N}\frac{c_0(j)}{j^2}-\frac{c_0(N)}{N}\left(\log(Nk)+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right)\right\}$$
$$-\frac{2^{2\omega(k)}c_0^2(N)}{N^2k^2}$$

and then

$$\lim_{N\to\infty}S(N)\leq\varphi(k)\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)$$
$$+\left(\frac{\varphi(k)}{k}\right)^2\left\{2\gamma_1+\gamma-\left(\log k+\sum_{p|k}\frac{\log p}{p-1}\right)^2\right\}+\frac{2^{\omega(k)}\varphi(k)\pi^2}{3k^2}$$

and the inequality $2^{\omega(k)}\leq\varphi(k)$ (valid for any integer $k\geq 3$ and $\neq 6$) implies

$$\lim_{N\to\infty}S(N)\leq\varphi(k)\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)$$
$$+\left(\frac{\varphi(k)}{k}\right)^2\left\{2\gamma_1+\gamma+\frac{\pi^2}{3}-\left(\log k+\sum_{p|k}\frac{\log p}{p-1}\right)^2\right\}$$
$$=(\varphi(k)-1)\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)+\left(\frac{\varphi(k)}{k}\right)^2\left\{\left(\frac{k}{\varphi(k)}\right)^2\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)\right.$$
$$\left.+2\gamma_1+\gamma+\frac{\pi^2}{3}-\left(\log k+\sum_{p|k}\frac{\log p}{p-1}\right)^2\right\}$$
$$\leq(\varphi(k)-1)\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)$$

if $k\geq 9$ by (ii). Hence

$$\frac{1}{\varphi(k)-1}\sum_{\chi\neq\chi_0}|L(1;\chi)|^2\leq\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right).$$

Now the IAG inequality implies:

$$\begin{aligned}\prod_{\chi\neq\chi_0}\left|L(1;\chi)\right|^{\frac{1}{\varphi(k)}} &= \exp\left\{\frac{1}{2\varphi(k)}\sum_{\chi\neq\chi_0}\log\left|L(1;\chi)\right|^2\right\}\\ &\leq \exp\left\{\frac{\varphi(k)-1}{2\varphi(k)}\log\left(\frac{1}{\varphi(k)-1}\sum_{\chi\neq\chi_0}\left|L(1;\chi)\right|^2\right)\right\}\\ &\leq \left(\zeta(2)\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)\right)^{\frac{\varphi(k)-1}{2\varphi(k)}}\\ &\leq \left(\zeta(2)\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)\right)^{\frac{1}{2}}.\end{aligned}$$

□

6. PROOF OF THE THEOREM

Lemma 6.1. *If χ_0 is the principal character modulo k and if $x>k$, then:*

$$\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-\chi_0(p)} < \frac{e^{\gamma}\varphi(k)\,\delta(x)}{k}\cdot\log x,$$

where δ is the function defined in (1.2).

Proof. Since $x>k$,

$$\prod_{\substack{p\leq x\\ p\mid k}}\left(1-\frac{1}{p}\right)=\prod_{p\mid k}\left(1-\frac{1}{p}\right)=\frac{\varphi(k)}{k}$$

and then

$$\begin{aligned}\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-\chi_0(p)} &= \prod_{\substack{p\leq x\\ p\nmid k}}\left(1-\frac{1}{p}\right)^{-1}\\ &= \prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-1}\prod_{\substack{p\leq x\\ p\mid k}}\left(1-\frac{1}{p}\right)\\ &= \frac{\varphi(k)}{k}\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-1}\end{aligned}$$

and we use (1.1). □

Proof of the theorem. Let $1\leq l\leq k$ be positive integers satisfying $(k,l)=1$ and $k\geq 37$, and x be a positive real number such that $x>k$. We have:

$$\prod_{\substack{p\leq x\\ p\equiv l\,(\mathrm{mod}\,k)}}\left(1-\frac{1}{p}\right)^{-\varphi(k)}=\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-\chi_0(p)}\cdot\prod_{\chi\neq\chi_0}\left(\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-\chi(p)}\right)^{\overline{\chi}(l)}:=\Pi_1\times\Pi_2$$

with $\Pi_1 < \dfrac{e^{\gamma}\varphi(k)\,\delta(x)}{k} \cdot \log x$ by Lemma 6.1. Moreover,

$$\begin{aligned}
\Pi_2 &= \exp\left\{\sum_{\chi\neq\chi_0} \overline{\chi}(l)\left(-\sum_{p\le x}\chi(p)\log\left(1-\frac{1}{p}\right)\right)\right\} \\
&= \exp\left(\sum_{\chi\neq\chi_0} \overline{\chi}(l)\sum_{p\le x}\sum_{\alpha=1}^{\infty}\frac{\chi(p)}{\alpha p^{\alpha}}\right) \\
&= \exp\left\{\sum_{\chi\neq\chi_0} \overline{\chi}(l)\left(\sum_{p\le x}\frac{\chi(p)}{p}+\sum_{p\le x}\sum_{\alpha=2}^{\infty}\frac{\chi(p)}{\alpha p^{\alpha}}\right)\right\}
\end{aligned}$$

and if $\chi \neq \chi_0$, we have

$$L(1;\chi) = \prod_p \left(1-\frac{\chi(p)}{p}\right)^{-1} = \exp\left(\sum_{p\le x}\frac{\chi(p)}{p}+\sum_{p>x}\frac{\chi(p)}{p}+\sum_p\sum_{\alpha=2}^{\infty}\frac{\chi(p^{\alpha})}{\alpha p^{\alpha}}\right)$$

and thus

$$\Pi_2 = \prod_{\chi\neq\chi_0} L(1;\chi)^{\overline{\chi}(l)} \cdot \exp\left\{\sum_{\chi\neq\chi_0}\overline{\chi}(l)\left(-\sum_{p>x}\frac{\chi(p)}{p}+\sum_{p\le x}\sum_{\alpha=2}^{\infty}\frac{\chi(p)}{\alpha p^{\alpha}}-\sum_p\sum_{\alpha=2}^{\infty}\frac{\chi(p^{\alpha})}{\alpha p^{\alpha}}\right)\right\}$$

and hence

$$\begin{aligned}
|\Pi_2| &\le \prod_{\chi\neq\chi_0}|L(1;\chi)| \cdot \exp\left\{\sum_{\chi\neq\chi_0}\left|\sum_{p>x}\frac{\chi(p)}{p}\right| + 2(\varphi(k)-1)\sum_p\sum_{\alpha=2}^{\infty}\frac{1}{\alpha p^{\alpha}}\right\} \\
&= e^{2(\varphi(k)-1)(\gamma-B)}\prod_{\chi\neq\chi_0}|L(1;\chi)| \cdot \exp\left\{\sum_{\chi\neq\chi_0}\left(\left|\sum_{p>x}\frac{\chi(p)}{p}\right|\right)\right\}
\end{aligned}$$

and we use Lemma 4.1 (iii) and Lemma 5.1 (iii). We conclude the proof by noting that, if $x > 37$, $\frac{\delta(x)}{e^{2(\gamma-B)}} < 1$. □

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