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A Two-Sided Inequality of Gamma Function

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Abstract: This note shows that the inequality

$$(\frac{x}{e})^x\sqrt{2\pi x}(1+\frac{1}{12x}) < \Gamma(x+1) < (\frac{x}{e})^x\sqrt{2\pi x}(1+\frac{1}{12x-0.5})$$

holds for all $x \geq 1$.

Key words: two-sided inequality; Gamma function; Stirling formula.
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1. Introduction

In [1], the authors proved the following sharp inequality for all the positive integers n:

$$(\frac{n}{e})^n\sqrt{2\pi n}(1+\frac{1}{12n}) < n! < (\frac{n}{e})^n\sqrt{2\pi n}(1+\frac{1}{12n-0.5}).$$

Considering the relation

$$\Gamma(n+1) = n!$$

a nature problem is whether the inequality

$$(\frac{x}{e})^x\sqrt{2\pi x}(1+\frac{1}{12x}) < \Gamma(x+1) < (\frac{x}{e})^x\sqrt{2\pi x}(1+\frac{1}{12x-0.5})$$

holds for all $x \geq 1$. The answer is yes.

In this note, we will prove the following theorem.

Theorem *For all real numbers $x \geq 1$, the following inequality holds:*

$$(\frac{x}{e})^x\sqrt{2\pi x}(1+\frac{1}{12x}) < \Gamma(x+1) < (\frac{x}{e})^x\sqrt{2\pi x}(1+\frac{1}{12x-0.5}), \tag{1}$$

where Γ denotes the Gamma function.

2. Proof of the Theorem

In order to prove the theorem, we need the following two Lemmas.

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Lemma 1 *The following inequality*

$$(x+\frac{1}{2})\log(1+\frac{1}{x})-1>\frac{1}{12x(x+1)}-\frac{1}{72}\frac{1}{x^2(x+1)^2}+\frac{4}{45}\frac{1}{(2x+1)^4}$$

holds for $x>0$.

Proof We first prove that

$$(x+\frac{1}{2})\log(1+\frac{1}{x})-1=\sum_{k=1}^{\infty}\frac{1}{2k+1}(\frac{1}{2x+1})^{2k} \tag{2}$$

for all $x>0$. In fact,

$$\begin{aligned}(x+\frac{1}{2})\log(1+\frac{1}{x})-1&=\frac{2x+1}{2}\log\frac{1+\frac{1}{2x+1}}{1-\frac{1}{2x-1}}-1\\&=\frac{2x+1}{2}2\sum_{k=0}^{\infty}\frac{1}{2k+1}(\frac{1}{2x+1})^{2k+1}-1=\sum_{k=0}^{\infty}\frac{1}{2k+1}(\frac{1}{2x+1})^{2k}-1\\&=\sum_{k=1}^{\infty}\frac{1}{2k+1}(\frac{1}{2x+1})^{2k}.\end{aligned}$$

Equality (2) implies

$$\begin{aligned}(x+\frac{1}{2})\log(1+\frac{1}{x})-1&=\frac{1}{3}\frac{1}{(2x+1)^2}+\frac{1}{5}(\frac{1}{(2x+1)^2})^2+\frac{1}{7}(\frac{1}{(2x+1)^2})^3+\cdots\\&>\frac{1}{3}\frac{1}{(2x+1)^2}+(\frac{1}{3(2x+1)^2})^2+(\frac{1}{3(2x+1)^2})^3+\cdots+\frac{1}{5}\frac{1}{(2x+1)^4}-\frac{1}{9}\frac{1}{(2x+1)^4}\\&=\sum_{k=1}^{\infty}(\frac{1}{3(2x+1)^2})^k+\frac{1}{5}\frac{1}{(2x+1)^4}-\frac{1}{9}\frac{1}{(2x+1)^4}\\&=\frac{1}{12x^2+12x+2}+\frac{4}{45}\frac{1}{(2x+1)^4}\\&=\frac{1}{12x^2+12x}\cdot\frac{1}{1+\frac{1}{6x^2+6x}}+\frac{4}{45}\frac{1}{(2x+1)^4}\\&>\frac{1}{12x^2+12x}(1-\frac{1}{6x^2+6x})+\frac{4}{45}\frac{1}{(2x+1)^4}\\&=\frac{1}{12x(x+1)}-\frac{1}{72}\frac{1}{x^2(x+1)^2}+\frac{4}{45}\frac{1}{(2x+1)^4}.\end{aligned}$$

Lemma 1 is proved.

Lemma 2[2] *For any* $x>0$, *there exists* $\theta(x)\in(0,1)$, *such that*

$$\Gamma(x+1)=\sqrt{2\pi x}(\frac{x}{e})^x\exp(\frac{\theta(x)}{12x}),$$

where

$$\theta(x)=12x\int_0^{+\infty}\frac{[t]-t+\frac{1}{2}}{t+x}\mathrm{d}t.$$

Proof of Theorem By Lemma 2, Inequality (1) is equivalent to the following inequality

$$\log(1+\frac{1}{12x})<\int_0^{+\infty}\frac{[t]-t+1/2}{t+x}\mathrm{d}t<\log(1+\frac{1}{12x-0.5}). \tag{3}$$

We first prove the inequality on the right hand side of (3). It is known[2] that for any $x>0$,

$$0<\int_0^{+\infty}\frac{[t]-t+1/2}{t+x}\mathrm{d}t<\frac{1}{12x}.$$

Therefore, for $x\geq 1$, we have

$$\begin{aligned}&\log(1+\frac{1}{12x-0.5})-\int_0^{+\infty}\frac{[t]-t+1/2}{t+x}\mathrm{d}t>\log(1+\frac{1}{12x-0.5})-\frac{1}{12x}\\&>\sum_{j=1}^{4}\frac{(-1)^{j-1}}{j}(\frac{1}{12x-0.5})^j-\frac{1}{12x}=\frac{12x^2-2x-(0.5)^4}{12x\cdot(12x-0.5)^4}>0.\end{aligned}$$

In order to prove the left hand side of (3), we note[2]

$$\int_0^{+\infty}\frac{[t]-t+1/2}{t+x}\mathrm{d}t=\sum_{k=0}^{\infty}\{(k+\frac{1}{2}+x)\log(1+\frac{1}{k+x})-1\},x>0.$$

Replacing x by $k+x$ in Lemma 1, we obtain

$$\begin{aligned}&\int_0^{+\infty}\frac{[t]-t+1/2}{t+x}\mathrm{d}t-\log(1+\frac{1}{12x})\\
&>-\log(1+\frac{1}{12x})+\frac{1}{12}\sum_{k=0}^{\infty}\frac{1}{(k+x)(k+x+1)}-\frac{1}{72}\sum_{k=0}^{\infty}\frac{1}{(k+x)^2(k+x+1)^2}+\\
&\quad\frac{4}{45}\sum_{k=0}^{\infty}\frac{1}{(2(k+x)+1)^4}\\
&>-\frac{1}{12x}+\frac{1}{2}(\frac{1}{12x})^2-\frac{1}{3}(\frac{1}{12x})^3+\frac{1}{12x}-\frac{1}{72}\sum_{k=0}^{\infty}\frac{1}{(k+x)^2(k+x+1)^2}+\frac{4}{45}\frac{1}{(2x+1)^4}\\
&=\frac{1}{288x^2}-\frac{1}{5184x^3}-\frac{1}{72}(\frac{1}{x^2(x+1)^2}+\frac{1}{(x+1)^2(x+2)^2}+\sum_{k=2}^{\infty}\frac{1}{(k+x)^2(k+x+1)^2})+\\
&\quad\frac{4}{45}\frac{1}{(2x+1)^4}\\
&>\frac{1}{288x^2}-\frac{1}{5184x^2}-\frac{1}{72}(\frac{1}{x^2(x+1)^2}+\frac{1}{(x+1)^2(x+2)^2}+\int_1^{+\infty}\frac{\mathrm{d}y}{(y+x)^2(y+x+1)^2})+\\
&\quad\frac{4}{45}\frac{1}{(2x+1)^4}\\
&=\frac{17}{5184x^2}+\frac{4}{45}\frac{1}{(2x+1)^4}-\frac{1}{72}(\frac{1}{x^2(x+1)^2}+\frac{1}{(x+1)^2(x+2)^2}+\frac{1}{x+1}+\frac{1}{x+2}-\\
&\quad 2\log(1+\frac{1}{x+1}))\\
&>\frac{17}{5184x^2}+\frac{4}{45}\frac{1}{(2x+1)^4}-\frac{1}{72}(\frac{1}{x^2(x+1)^2}+\frac{1}{(x+1)^2(x+2)^2}+\frac{1}{x+1}+\frac{1}{x+2}-\end{aligned}$$

$$2(\frac{1}{x+1}-\frac{1}{2}\frac{1}{(x+1)^2}+\frac{1}{3}\frac{1}{(x+1)^3}-\frac{1}{4}\frac{1}{(x+1)^4}))$$

$$=(\frac{17}{5184x^2}+\frac{4}{45(1+2x)^4})+\frac{1}{72}(\frac{1}{1+x}-\frac{1}{(1+x)^2}+\frac{2}{3(1+x)^3}-\frac{1}{2(1+x)^4})-$$

$$\frac{1}{72}(\frac{1}{(2+x)}+\frac{1}{x^2(1+x)^2}+\frac{1}{(1+x)^2(2+x)^2})$$

$$=\frac{85+680x+4344x^2+2720x^3+1360x^4}{25920x^2(1+2x)^4}+\frac{1}{72}(\frac{1+10x+12x^2+6x^3}{6(1+x)^4}-$$

$$\frac{4+4x+4x^2+5x^3+4x^4+x^5}{x^2(1+x)^2(2+x)^2})$$

$$=\frac{85+680x+4344x^2+2720x^3+1360x^4}{25920x^2(1+2x)^4}+\frac{-24-72x-92x^2-58x^3-19x^4-2x^5}{432x^2(1+x)^4(2+x)^2}$$

$$=\frac{-1100-11420x-40179x^2-68020x^3-52666x^4+9696x^5+65909x^6}{25920x^2(1+x)^4(2+x)^2(1+2x)^4}+$$

$$\frac{70952x^7+39384x^8+11680x^9+1360x^{10}}{25920x^2(1+x)^4(2+x)^2(1+2x)^4}.$$

Rewrite the numerator of (3) as follows to see that for $x \geq 1$,

$$x^3(70952x^4-68020)+x^4(65909x^2-52666)+x(11680x^8-11420)+$$
$$(1360x^{10}-1100)+x^2(9696x^3+39384x^6-40179)>0.$$

This completes the proof of the Theorem. □

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有关 Gamma 函数的一个双不等式

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摘要: 本文将文献 [1] 中的双边不等式从自然数推广至实数，证明了下面不等式成立：

$$(\frac{x}{e})^x\sqrt{2\pi x}(1+\frac{1}{12x})<\Gamma(x+1)<(\frac{x}{e})^x\sqrt{2\pi x}(1+\frac{1}{12x-0.5}),$$

其中 $x \geq 1$.

关键词: 双边不等式； Γ 函数； Stirling 公式.