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Pseudo-Injective Modules and Principally Pseudo-Injective Modules

DU Xian-neng, ZHAO Chun-e

(School of Mathematics and Computational Sciences, Anhui University, Hefei 230039, China)
(E-mail: xndu@ahu.edu.cn)

Abstract: Considered in this paper are pseudo-injective modules and principally pseudo-injective modules, which are generalizations of quasi-injective modules and PQ-injective modules. Pseudo-injective modules are dual to pseudo-projective modules. We study their properties and endomorphism rings, and obtain some properties of the Jacobson radical of such rings.

Key words: pseudo-injective; principally pseudo-injective; idempotent principal self-generator; endomorphism ring; Jacobson radical.
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1. Pseudo-injective modules

A module M_R is called pseudo-injective if for every R-monomorphism $\beta: 0 \to A \to M$ and $\alpha: 0 \to A \to M$ there exists a $\gamma \in \mathrm{End}(M_R)$, such that $\beta = \gamma\alpha$. Our first task is to describe several characterizations of these modules.

Proposition 1.1 *Let M_R be a module, then the following statements are equivalent:*

(1) M_R is a (principally) pseudo-injective module;

(2) For every R-monomorphism $\beta: 0 \to A \to M$ and $\alpha: 0 \to A \to N$ (A is principal) where N embeds in M, there exists $\gamma \in \mathrm{Hom}_R(N, M)$ such that $\beta = \gamma\alpha$;

(3) For every R-monomorphism $\beta: 0 \to A \to M$ and $\alpha: 0 \to A \to N$ (A is principal) where N is a submodule of M, there exists $\gamma \in \mathrm{Hom}_R(N, M)$ such that $\beta = \gamma\alpha$;

(4) Every R-monomorphism $\beta: 0 \to N \to M$ (N is principal) where N is a submodule of M, can be extended to an endomorphism of M.

Proof (1)$\Rightarrow$ (2). Let $\beta: 0 \to A \to M$ and $\alpha: 0 \to A \to N$ where N embeds in M be R-homomorphisms. Then there exists an R-homomorphisms $\gamma_1: 0 \to N \to M$. It is not difficult to check that $\gamma_1\alpha: 0 \to A \to M$ is monic. Then there exists $\gamma_2 \in \mathrm{End}(M_R)$ such that $\beta = \gamma_2\gamma_1\alpha$ by (1). Let $\gamma_2\gamma_1 = \gamma: N \to M$. Then $\beta = \gamma\alpha$.

(2)$\Rightarrow$ (3)$\Rightarrow$ (4). Clearly.

(4)$\Rightarrow$ (1). Let $\alpha: 0 \to A \to M$ and $\beta: 0 \to A \to M$ be R-monomorphisms. Then $\alpha: A \to \mathrm{Im}\alpha$ is an isomorphism, so there exists $\alpha^{-1}: \mathrm{Im}\alpha \to A$ such that $\alpha^{-1}\alpha = 1_A$. Then

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$\beta\alpha^{-1}: 0 \to \mathrm{Im}\alpha \to M$ is monic. Hence there exists $\gamma \in \mathrm{End}(M_R)$ such that $\gamma|_{\mathrm{Im}\alpha} = \beta\alpha^{-1}$, for every $a \in A$, $\gamma\alpha(a) = \beta\alpha^{-1}\alpha(a) = \beta(a)$, that is $\gamma\alpha = \beta$.

Corollary 1.2 *Let M_R be a pseudo-injective module. Then*

(1) Every R-monomorphism $\alpha \in \mathrm{End}(M_R)$ splits.

(2) For every R-monomorphism $\beta: 0 \to A \to M$ and $\alpha: 0 \to A \to A$, there exists $\gamma \in \mathrm{Hom}_R(A, M)$ such that $\beta = \gamma\alpha$.

(3) Every R-monomorphism $\alpha \in \mathrm{Hom}_R(M, N)$ where N embeds in M splits.

Proof (1) For R-monomorphism $\alpha \in \mathrm{End}(M_R)$ and $1_M \in \mathrm{End}(M_R)$, there exists $\beta \in \mathrm{End}(M_R)$ such that $1_M = \beta\alpha$. So α splits.

(2) Let $\beta: 0 \to A \to M$ and $\alpha: 0 \to A \to A$ be R-monomorphisms. Then A embeds in M. So there exists a $\gamma \in \mathrm{Hom}_R(A, M)$ such that $\beta = \gamma\alpha$ by Proposition 1.1 (2).

(3) Let $\alpha \in \mathrm{Hom}_R(M, N)$ be a R-monomorphism. Then for $\alpha: 0 \to M \to N$ and $1_M: 0 \to M \to M$, there exists $\beta \in \mathrm{Hom}_R(N, M)$ such that $1_M = \beta\alpha$ by Proposition 1.1 (2).

Proposition 1.3 *Let $(U_a)_{a\in I}$ be an indexed set of right R-modules. If $\oplus_I U_a$ is (principally) pseudo-injective, then for every R-monomorphism (K is principal) $\beta: 0 \to K \to U_a$ and $\alpha: 0 \to K \to U_b$ where $a \in I$, $b \in I$, there exists $\gamma \in \mathrm{Hom}_R(U_b, U_a)$ such that $\beta = \gamma\alpha$.*

Proof Let $\beta: 0 \to K \to U_a$ and $\alpha: 0 \to K \to U_b$ be R-monomorphisms. For $i_a\beta: 0 \to K \to \oplus_I U_a$ and $\alpha: 0 \to K \to U_b$, there exists $\gamma\bar{\gamma} \in \mathrm{Hom}_R(U_b, \oplus_I U_a)$ such that $i_a\beta = \bar{\gamma}\alpha$ by Proposition 1.1 (3). Let $\gamma = \pi_a\bar{\gamma}: U_a \to U_a$. Then $\gamma\alpha = \pi_a\bar{\gamma}\alpha = \pi_a i_a\beta = \beta$.

Corollary 1.4 *Every direct summand of a (principally) pseudo-injective module is also (principally) pseudo-injective.*

Proof Let $U_a = U_b$ in Proposition 1.3. It is clear.

Proposition 1.5 *M_R is a pseudo-injective module with $S = \mathrm{End}(M_R)$. Let $\alpha \in S$. Then*

$$\alpha \in J(S) \Leftrightarrow \ker\alpha \subseteq^{ess} M.$$

Proof It is similar with the case of injective modules and the details of the proof are ommitted.

Remark The Jacobson radicals of the endomorphism rings of the injective modules, quasi-injective modules and the pseudo-injective modules have the same property. The reason is that such modules have the same characterization: Every R-monomorphism $\alpha \in \mathrm{End}(M_R)$ splits. So, if M_R is a module such that for every $\beta: K \to M \to 0$ and $\alpha: 0 \to K \to M$, there exists $\gamma \in \mathrm{End}(M_R)$ such that $\beta = \gamma\alpha$, then its Jacobson radical has the same property.

2. Principally pseudo-injective modules

An R-module M is called principally pseudo-injective if each R-monomorphism from a principal submodule of M to M can be extended to an endomorphism of M. If M_R is a module, we write $l_M(r) = \{m \in M \,|\, mr = 0\}$ for all $r \in R$, $r_R(m) = \{r \in R \,|\, mr = 0\}$ for all $m \in M$, $A_m = \{n \in M \,|\, r_R(n) = r_R(m)\}$, $S_{(\alpha,m)} = \{\beta \in S \,|\, \ker\beta \cap mR = \ker\alpha \cap mR\}$ for all $m \in M$ and $B_m = \{\alpha \in S \,|\, \ker\alpha \bigcap mR = 0\}$ for all $m \in M$.

Proposition 2.1 *For a given module M_R with $S = \mathrm{End}(M_R)$, the following conditions are equivalent for an element $m \in M$:*

(1) M_R is principally pseudo-injective;

(2) $A_m = B_m m$;

(3) If $A_m = A_n$, then $B_m m = B_n n$;

(4) For every R-monomorphism $\alpha : 0 \to mR \to M$ and $\beta : 0 \to mR \to M$, there exists $\gamma \in \mathrm{End}(M_R)$ such that $\alpha = \gamma\beta$.

Proof (1)$\Rightarrow$ (2). If $n \in A_m$, then $A_m = A_n$, hence $\alpha : mR \to M$ is well defined by $\alpha(mr) = nr$ and α is an R-monomorphism. So let $s \in S$ extend α by (1). Then $s(m) = \alpha(m) = n = sm$ where $s \in B_m$. (Indeed, if $mr \in \{\ker s \cap mR\}$, then $s(mr) = \alpha(mr) = 0$, so $mr = 0$.) Conversely, if $sm \in B_m m$, then $s \in B_m$, that is $\{\ker s \cap mR\} = 0$. It is clear that $r_R(sm) \supseteq r_R(m)$. If $r \in r_R(m)$, then $smr = 0$, so $mr \in \{\ker s \cap mR\} = 0$, and $r \in r_R(m)$. Now we have $r_R(sm) = r_R(m)$. Then $sm \in A_m$.

(2)$\Rightarrow$ (3). Let $A_m = A_n$. Then $A_m = B_m m, A_n = B_n n$. So $B_m m = B_n n$.

(3)$\Rightarrow$ (4). Let $\alpha : 0 \to mR \to M$ and $\beta : 0 \to mR \to M$ be R-monomorphisms. Then $r_R(\beta m) = r_R(\alpha m)$. So $A_{\alpha m} = A_{\beta m}$, $B_{\alpha m}\alpha m = B_{\beta m}\beta m$ by (3). Because $\{\ker 1_M \cap \alpha mR\} = 0$, $1_M \in B_{\alpha m}$. Then $\alpha m \in B_{\beta m}\beta m$. There exists $\gamma \in B_{\beta m}$ such that $\alpha = \gamma\beta$.

(4)$\Rightarrow$ (1). Let $\beta = i_{mR}$. It is clear.

Proposition 2.2 *Let M_R be a principally pseudo-injective module with $S = \mathrm{End}(M_R)$. Then*

$$S_{(\alpha,m)} = B_{\alpha m}\alpha + l_S(m).$$

Proof If $\beta \in S_{(\alpha,m)}$, then $\ker\beta \cap mR = \ker\alpha \cap mR$. We claim $r_R(\alpha m) = r_R(\beta m)$. (Indeed, if $\alpha(m)r = 0$, then $mr \in \ker\alpha \cap mR = \ker\beta \cap mR$, so $\beta(m)r = 0$. If $\beta(m)r_1 = 0$, then $mr_1 \in \ker\beta \cap mR = \ker\beta \cap mR$, so $\alpha(m)r_1 = 0$.) Hence $\beta m \in B_{\alpha m}\alpha m$ by Proposition 2.1. Say $\beta m = b\alpha m, b \in B_{\alpha m}$. This means that $\beta - b\alpha \in l_S(m)$. Conversely, let $b\alpha + s \in B_{\alpha m}\alpha + l_S(m)$ with $b \in B_{\alpha m}, s \in l_S(m)$. If $mr \in \ker(b\alpha + s) \cap mR$, then $(b\alpha + s)(mr) = b\alpha mr + smr = bb\alpha mr = 0$. Hence $\alpha mr \in \ker b \cap \alpha mR = 0$. So $mr \in \ker\alpha \cap mR$. If $mr_1 \in \ker\alpha \cap mR$, then $\alpha mr_1 = 0$, so $(b\alpha + s)(mr_1) = b\alpha mr_1 + smr_1 = b\alpha mr_1 = 0$. This means $b\alpha + s \in S_{(\alpha,m)}$. Thus, $S_{(\alpha,m)} = B_{\alpha m}\alpha + l_S(m)$.

Proposition 2.3 *Let M_R be a principally pseudo-injective module with $S = \mathrm{End}(M_R)$ and $\alpha \in S$, $m \in M$. Then*

$$\alpha \in B_m \Leftrightarrow B_m = B_{\alpha m}\alpha + l_S(m).$$

Proof ($\Rightarrow$). If $\alpha \in B_m$, then $S_{(\alpha,m)} = B_m$. So $B_m = B_{\alpha m}\alpha + l_S(m)$ by Proposition 2.2.

($\Leftarrow$). Suppose $\alpha \in S - B_m$ such that $B_m = B_{\alpha m}\alpha + l_S(m)$, then there exists $0 \neq mr \in mR$ such that $\alpha(mr) = 0$. Because $1_M \in B_m$, then $1 = b\alpha + s$ with $b \in B_{\alpha m}$, $s \in l_S(m)$, so $mr = b\alpha mr + smr = b\alpha mr = 0$, contradicting $\alpha \in S - B_m$.

Proposition 2.4 *Let M_R be a principally pseudo-injective module with $S = \mathrm{End}(M_R)$. Then we have the following conclusions.*

(1) If K is a simple submodule of M_R, then $\mathrm{soc}_K(M_R) = SK$.

(2) If kR is a simple R-module, $k \in M$, then Sk is simple S-module.

(3) $\mathrm{soc}(M_R) = \mathrm{soc}(_SM)$.

Proof (1) Let $\sigma : K \to K_1$ be an R-isomorphism where $K_1 \subseteq M$. If $K = kR$ then $r_R(k) = r_R(\sigma k)$. So $B_k k = B_{\sigma k}\sigma k$ by Proposition 2.1 (3). Thus $\sigma k \in B_k k \subseteq Sk \subseteq SK$. So if $\hat{\sigma}$ is an extension of σ to S, and we have $K_1 = \sigma kR = \hat{\sigma}kR \subseteq SK$. This shows $\mathrm{soc}_K(M) \subseteq SK$. The other inclusion always holds.

(2) Let $0 \neq \alpha k \in Sk$. Then $\alpha : kR \to \alpha(kR)$ is an isomorphism by hypothesis. So let $\delta : \alpha(kR) \to kR$ be the inverse. If $\hat{\delta} \in S$ extends δ, then $\hat{\delta}(\alpha k) = \delta(\alpha k) = k$, and so $k \in S\alpha k$. Therefore, $Sk \subseteq S\alpha k$. Then $Sk = S\alpha k$. So Sk is simple S-module.

(3) This follows from (2).

Proposition 2.5 *Let M_R be a principally pseudo-injective module with $S = \mathrm{End}(M_R)$ and let $m_1, m_2, \dots, m_n$ denote the elements of M. If $\oplus_i Sm_i$ is direct then any R-monomorphism $\alpha : 0 \to m_1R + m_2R + \cdots + m_nR \to M$ has an extension in S.*

Proof Let α_i and β denote the restriction of α to m_iR and $(m_1 + m_2 + \cdots + m_n)R$, respectively, and let $\hat{\alpha_i}$ and $\hat{\beta}$ extend α_i and β to M. Then

$$\begin{aligned}\hat{\beta}(m_1 + m_2 + \cdots + m_n) &= \hat{\beta}(m_1) + \hat{\beta}(m_2) + \cdots + \hat{\beta}(m_n)\\ &= \alpha(m_1) + \alpha(m_2) + \cdots + \alpha(m_n)\\ &= \hat{\alpha}(m_1) + \hat{\alpha}(m_2) + \cdots + \hat{\alpha}(m_n).\end{aligned}$$

Because $\oplus_i Sm_i$ is direct, we obtain $\hat{\beta}(m_i) = \hat{\alpha}(m_i) = \alpha(m_i)$.

Proposition 2.6 *If M_R is a principally pseudo-injective module with $S = \mathrm{End}(M_R)$, then $W(S) = \{w \in S \mid 1 - \beta w$ is monomorphism for all $\beta \in S\}$.*

Proof Assume that $1 - \beta w$ is monomorphism for all $\beta \in S$ and let $\ker(w) \bigcap mR = 0$, $m \in M$. Then $r_R(wm) \subseteq r_R(m)$. And $r_R(m) \subseteq r_R(wm)$. So $A_{wm} = A_m$, and $B_{wm}wm = B_m m$. So $m \in B_{wm}wm$ by Proposition 2.1. This means that $m \in \ker(1 - \beta m)$ for some $\beta \in B_{wm}$. So $m = 0$. This proves that $w \in W(S)$. Conversely, if $w \in W(s)$, then $\ker w \cap \ker(1 - \beta w) = 0$ for all $\beta \in S$ implies that $1 - \beta w$ is monomorphism.

Proposition 2.7 *Let M_R be a principally pseudo-injective module with $S = \mathrm{End}(M_R)$. Then*

$$J(S) \subseteq W(S) \subseteq Z(S_S).$$

Proof $J(S) \subseteq W(S)$ will be showed in Proposition 3.2. Because $W(S)$ is an ideal of S, suppose $\alpha \in W(S) - Z(S_S)$. Then kerα is not essential in M_R, so let ker$\alpha \cap mR = 0$ where $0 \neq m \in M$. Hence $\alpha : mR \to M$ is monomorphism and $1_{mR} : mR \to M$ is also monomorphism. So, by Proposition 2.1, there exists $\beta : M \to M$ such that $\beta\alpha = 1_{mR}$. Thus $(1 - \beta\alpha)(m) = 0$, contradicting Proposition 3.2.

A module M_R is said to satisfy the C_2-condition if every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M.

Proposition 2.8 *Let M_R be a principally pseudo-injective module with $S = \mathrm{End}(M_R)$. Then we have the following conclusions*

(1) If N and K are isomorphic principal submodule of M and K is a direct summand of M, then N is also a direct summand of M.

(2) Every principal principally pseudo-injective module has the C_2-condition.

Proof Clearly, (1) implies (2). Let $\sigma : N \to K$ be an isomorphism and let $\pi : M \to K$ be a projection. If $\hat{\sigma} : M \to M$ is an extension of σ, define $\alpha = \sigma^{-1}\pi\hat{\sigma} : M \to N$. If $n \in N$ write $\sigma n = k \in K$, so $\alpha n = \sigma^{-1}[\pi(\hat{\sigma}n)] = \sigma^{-1}[\pi(\sigma n)] = \sigma^{-1}[\pi(k)] = \sigma^{-1}(k) = \sigma^{-1}(\sigma n) = n$. Hence the inclusion map $N \hookrightarrow M$ splits, which proves (1).

3. Principally pseudo-injective modules

We say that M_R is a principal self-generator if every element $m \in M$ has the form $m = \alpha(m_1)$ for some $\alpha : M_R \to mR$.

Lemma 3.1 *M_R is a principal self-generator, then every principal submodule is in the form of mR where $r_R(m) \supseteq r_R(m_0)$, $M = m_0R$.*

Proof Let nR be a principal submodule of M. Then there exists $\alpha : M \to nR$ such that $n = \alpha(m_1)$. It is clear that Im$\alpha = nR$. Note $\alpha(m_0) = m$. Then Im$\alpha = mR = nR$ and $m \in l_M r_R(m_0)$.

Proposition 3.2 *Let M_R be a principal module which is a principal self-generator and let $S = \mathrm{End}(M_R)$. The following conditions are equivalent:*

(1) M_R is principally pseudo-injective;

(2) $S_{(\alpha,m)} = B_{\alpha m} + l_S(m)$ for all $\alpha \in S$ and all $m \in M$;

(3) If $A_{\alpha m} = A_{\beta m}$ then $\beta \in B_{\alpha m}\alpha + l_S(m)$.

Proof (1)$\Rightarrow$ (2). This follows from Proposition 2.2.

(2)$\Rightarrow$ (3). Let $A_{\alpha m} = A_{\beta m}$, then $r_R(\alpha m) = r_R(\beta m)$ and $S_{(\alpha,m)} = S_{(\beta,m)}$, so $B_{\alpha m}\alpha + l_S(m) = B_{\beta m} + l_S(m)$. Let $1_M \in B_{\beta m}, 0 \in l_S(m)$, then $\beta \in B_{\alpha m}\alpha + l_S(m)$.

(3)$\Rightarrow$ (1). Let $\gamma : 0 \to mR \to M$ be an R-monomorphism. Because M is principal, there exists $m_0 \in M$ such that $M = m_0R$. And M is also idempotent principal self-generator, so there exists $\alpha : M \to mR$ with $\alpha(m_0) = m$ by Lemma 3.1. Similarly, we can find $\beta : M \to$

$\gamma(n)R$ such that $\gamma(n)=\beta(m_0)$. Because γ is a monomorphism, $r_R(\gamma(m))=r_R(m)$, that is $r_R(\beta m_0)=r_R(\alpha m_0)$. This means $\ker\alpha=\ker\beta$. So $\ker\alpha\cap mR=\ker\beta\cap mR$, $S_{(\alpha,m)}=S_{(\beta,m)}$ and $A_{\alpha m}=A_{\beta m}$. So $\beta\in B_{\alpha m}\alpha+l_S(m)$ by condition (3), then $\beta=\theta\alpha+s$ where $\theta\in B_{\alpha m}$, $s\in l_S(m)$. Then $s\alpha=0$ because $0=s(m)=s(\alpha(m_0))=s\alpha(m_0)$. So $\beta\alpha=\theta\alpha^2$ and α is an epimorphism, then $\beta=\theta\alpha$, $\theta(m)=\theta\alpha(m_0)=\beta(m_0)=\gamma(m)$.

Proposition 3.3 *Let m_R be a principally pseudo-injective module with $S=\mathrm{End}(M_R)$. If M is nonsingular, then $J(S)=0$.*

Proof Since $J(S)\subseteq W(S)$ by Proposition 2.6, we show that $W(S)=0$. If $w\in W(S)$, then $\ker(w)\subseteq^{ess} M_R$. But $\ker(w)$ is closed in M_R because M_R is nonsingular, so $\ker(w)=M_R$ and $w=0$.

Proposition 3.4 *If M_R is a principal, principally pseudo-injective module, then $J(S)=W(S)$ where $S=\mathrm{End}(M_R)$.*

Proof $J(S)=W(S)$ is shown in Proposition 2.5. Because M_R is principally pseudo-injective, M_R is pseudo-injective. So every R-monomorphism $\alpha\in S$ splits that is a left inverse. So $W(s)\subseteq J(S)$ by Proposition 2.1 (2) in [4].

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伪内射模与主伪内射模

杜先能， 赵春娥
(安徽大学数学与计算科学学院，安徽 合肥 230039)

摘要: 本文研究了伪内射模与主伪内射模，它们分别是拟内射模与 PQ- 内射模的推广. 伪内射模是对偶于伪投射模的. 我们讨论了伪内射模与主伪内射模的性质及其自同态环, 并得到了自同态环的 Jacobson 根的若干性质.

关键词: 伪内射模；主伪内射模；主自生成子；自同态环； Jacobon 根.