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Periodic Solutions of Linear Neutral Functional Differential Equations with Infinite Delay

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Abstract: In this paper, we chose space C_g as phase space. It had been proved that for linear neutral functional differential equations of D-operator type with infinite delay, there was a periodic solution if and only if there was a bounded solution. Our results were different from the ones given in Acta Mathematica Sinica, 4(2000)695-702.

Key words: linear neutral functional differential equation; infinite delay; bounded solution; periodic solution; necessary and sufficient condition.
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1. Introduction

The existence of periodic solution is an important topic in the theoretical research of functional differential equations. The work for the existence of periodic solution is still challengeable in theory meaning and application field. It had been paid high attention during the past two decades, and many experts and scholars had made extensive and creative contribution, they obtained some good and interesting results. In recent years, periodic retarded functional differential equations with finite or infinite delay, and periodic neutral functional differential equations with finite or infinite delay are discussed extensively, respectively. Among these works, we mention [1], where Fan and Wang made a brief and complete discussion for weakening the conditions of functional differential equations and widening the scopes of type of functional differential equations. They systematically summarized some important results made by other scholars in recent half century. Details can be founded in [1] and the references therein.

Since the theory for phase space was built by Hale and Kato in 1978[6], for functional differential equations with infinite delay, some new progress had been made on the existence and uniqueness, periodic solutions, global stability and persistence etc. So far, the effective phase space form are C_h space[5] and C_g space[2], The bounded continuous functions space $BC((-\infty,0];R^n)$ can also be found in recent literature. For functional differential equations with infinite delay, different problems use different phase spaces. Therefore, choosing a suitable phase space is a critical step for solving the problems. It will do for neutral type functional

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differential equations with infinite delay. Consequently, we choose C_g space as our phase space to investigate the problem given in [1], for D-operator type linear neutral functional differential equations with infinite delay, we have shown that there is a periodic solution if and only if there is a bounded solution. Moreover, our results are different from the ones given by [1]. They are non-inclusive results. We will begin our discussion in the next section.

2. The existence of periodic solutions

We will consider neutral functional differential equations with infinite delay

$$\frac{\mathrm{d}Dx_t}{\mathrm{d}t} = f(t, x_t), \tag{2.1}$$

where $x_t(s) = x(t+s), s \in (-\infty, 0], f : R \times C_g \to R^n$, C_g is phase space of (2.1). Let $C := C(R^-, R^n)$ represent all continuous functions mapping from R^- to R^n and $g : R^- \to [1, +\infty)$ be a continuous and nonincreasing function such that $g(0) = 1, g(-\infty) = +\infty$. Define $C_g = \{\varphi \in C : \frac{\varphi}{g}$ uniformly continuous on R^-, and $\sup_{s\leq 0} \frac{|\varphi(s)|}{g(s)} < \infty\}$. For any $\varphi \in C_g$, the norm is defined by $|\varphi|_g = \sup_{s\leq 0} \frac{|\varphi(s)|}{g(s)}$, where $R^- = (-\infty, 0]$. [2] had shown that $|\cdot|_g$ is norm and $(C_g, |\cdot|_g)$ is a Banach space.

For any $\varphi \in C_g$, we denote $\|\varphi\| = \sup_{s\leq 0} |\varphi(u)|$. For any given $\alpha > 0, \varphi \in C_g, t_0 \in R$, A denotes the set of all functions which satisfies $x : (-\infty, t_0 + \alpha] \to R^n, x_{t_0} = \varphi, x$ is continuous function on $[t_0, t_0 + \alpha]$.

We claim that $(C_g, |\cdot|_g)$ satisfies the assumptions $(B_1) - (B_4)$ of space $B^{[3]}$:

(1) For any $\varphi \in C_g$, then $|\varphi(0)| \leq |\varphi|_g$. By definition, $|\varphi|_g \geq |\varphi(0)|/g(0) = |\varphi(0)|$.

(2) For any $t_0 \in R, \alpha \geq 0, x \in A, t \in [t_0, t_0+\alpha]$, it follows that $x_t \in C_g$, and x_t is continuous on $[t_0, t_0 + \alpha]$ about t.

First to check that $\frac{x_t(s)}{g(s)}$ is a uniformly continuous function of $s \in R^-$ for fixed $t \geq t_0$.

$$x(t+s) = x(t - t_0 + t_0 + s) = \begin{cases} \varphi(t - t_0 + s), & s \leq -(t - t_0), \\ x(t+s), & s > -(t - t_0). \end{cases}$$

For fixed $t = t_0$, $\frac{x_{t_0}(s)}{g(s)} = \frac{\varphi(s)}{g(s)}$. By the fact that $\frac{\varphi}{g}$ is a uniformly continuous function on R^-. Furthermore, $\frac{x_t(s)}{g(s)}$ is a uniformly continuous function on R^- for fixed $t = t_0$.

For fixed $t > t_0$, set $u = t - t_0 > 0$.

$$x(t+s) = \begin{cases} \varphi(u+s), & s \leq -u, \\ x(t+s), & s > -u. \end{cases}$$

It is discussed from the following cases:

(i) As $s_1, s_2 \in [-(u+1), 0]$; since $x(t+s)$ is continuous on $[-(u+1), 0]$, then yields that $\frac{x(t+s)}{g(s)}$ is continuous on $[-(u+1), 0]$, so it is uniformly continuous. Thus, for any $\varepsilon > 0$, there exists a $\delta_1 > 0$, such that for any $s_1, s_2 \in [-(u+1), 0] : |s_1 - s_2| < \delta_1$, it deduces that

$$\left| \frac{x(t+s_1)}{g(s_1)} - \frac{x(t+s_2)}{g(s_2)} \right| < \varepsilon;$$

(ii) As $s_1, s_2 \in (-\infty, -u]$; it implies that

$$\begin{aligned}
&\left|\frac{x(t+s_1)}{g(s_1)} - \frac{x(t+s_2)}{g(s_2)}\right| \\
&\quad = \left|\frac{x_{t_0}(t+s_1-t_0)}{g(s_1)} - \frac{x_{t_0}(t+s_2-t_0)}{g(s_2)}\right| = \left|\frac{\varphi(u+s_1)}{g(s_1)} - \frac{\varphi(u+s_2)}{g(s_2)}\right| \\
&\quad \le \left|\frac{\varphi(u+s_1)}{g(u+s_1)} \cdot \frac{g(u+s_1)}{g(s_1)} - \frac{\varphi(u+s_2)}{g(u+s_2)} \cdot \frac{g(u+s_2)}{g(s_2)}\right| \\
&\quad \le \left|\frac{\varphi(u+s_1)}{g(u+s_1)} - \frac{\varphi(u+s_2)}{g(u+s_2)}\right| \cdot \frac{g(u+s_1)}{g(s_1)} + \left|\frac{g(u+s_1)}{g(s_1)} - \frac{g(u+s_2)}{g(s_2)}\right| \cdot \left|\frac{\varphi(u+s_2)}{g(u+s_2)}\right|.
\end{aligned}$$

Noting the fact that $\frac{\varphi(u+s)}{g(u+s)}$ is uniformly continuous on $(-\infty, -u]$, then for any $\varepsilon > 0$, there exists a $\delta_2 > 0$ such that for any $s_1, s_2 \in (-\infty, -u] : |s_1 - s_2| < \delta_2$, one yields that

$$\left|\frac{\varphi(u+s_1)}{g(u+s_1)} - \frac{\varphi(u+s_2)}{g(u+s_2)}\right| < \varepsilon.$$

Since $\frac{g(s+u)}{g(s)}$ is a uniformly continuous function of s for fixed $u > 0$, that is for any $\varepsilon > 0$, there exists a $\bar{\delta} > 0$, such that for any $s_1, s_2 \in (-\infty, 0] : |s_1 - s_2| < \bar{\delta}$, it follows that

$$\left|\frac{g(u+s_1)}{g(s_1)} - \frac{g(u+s_2)}{g(s_2)}\right| < \varepsilon.$$

Moreover, $\frac{g(s+u)}{g(s)}$ is a bounded function of s for fixed $u > 0$. Let its boundary be M. So

$$\left|\frac{x(t+s_1)}{g(s_1)} - \frac{x(t+s_2)}{g(s_2)}\right| < \varepsilon M + \varepsilon |\varphi|_g = (|\varphi|_g + M)\,\varepsilon.$$

Taking $\delta = \min\{\delta_1, \delta_2, 1\}$. For any given $t \ge t_0$, for any $\varepsilon > 0$, there exists a $\delta > 0$, such that for any $s_1, s_2 \in R^-$, $|s_1 - s_2| < \delta$, it follows that

$$\left|\frac{x_t(s_1)}{g(s_1)} - \frac{x_t(s_2)}{g(s_2)}\right| = \left|\frac{x(t+s_1)}{g(s_1)} - \frac{x(t+s_2)}{g(s_2)}\right| < \varepsilon.$$

This shows that for given $t \ge t_0$, $\frac{x_t(s)}{g(s)}$ is a uniformly continuous function on R^-.

Next for $t \in [t_0, t_0 + \alpha]$, and $g(s)$ is monotonously nonincreasing function, then

$$\begin{aligned}
|x_t|_g &\le \sup_{s \le t_0 - t} \frac{|x_t(s)|}{g(s)} + \sup_{t_0 - t \le s \le 0} \frac{|x_t(s)|}{g(s)} \\
&\le \sup_{s+t-t_0 \le 0} \frac{|x_{t_0}(s+t-t_0)|}{g(s+t-t_0)} + \sup_{t_0 \le s+t \le t} |x(s+t)| = |\varphi|_g + \sup_{t_0 \le s+t \le t} |x(s+t)|.
\end{aligned}$$

Since x is continuous on $[t_0, t_0+\alpha]$, then it is bounded. We denote $|x(s+t)| \le b < \infty, s+t \in [t_0, t_0+\alpha]$. It follows that $|x_t|_g \le |\varphi|_g + b < \infty$, it proves that $x_t \in C_g, t \in [t_0, t_0+\alpha]$.

Finally, we will prove that x_t is continuous on $[t_0, t_0+\alpha]$. Let $t_1, t_2 \in [t_0, t_0+\alpha]$, without loss of generality, let $t_0 \le t_1 < t_2$.

$$\begin{aligned}
|x_{t_1}-x_{t_2}|_g &= \sup_{s\le 0}\frac{|x_{t_1}(s)-x_{t_2}(s)|}{g(s)} = \sup_{s\le 0}\frac{|x(s+t_1)-x(s+t_2)|}{g(s)}\\
&\le \sup_{s\le t_0-t_2}\frac{|x(s+t_1)-x(s+t_2)|}{g(s)} + \sup_{t_0-t_2<s\le 0}\frac{|x(s+t_1)-x(s+t_2)|}{g(s)}\\
&= \sup_{s-t_0+t_2\le 0}\frac{|x_{t_0}(s-t_0+t_2-t_2+t_1)-x_{t_0}(s-t_0+t_2)|}{g(s)} + \sup_{t_0<s+t_2\le t_2}\frac{|x(s+t_1)-x(s+t_2)|}{g(s)}\\
&\le \sup_{r\le 0}\frac{|\varphi(r+t_1-t_2)-\varphi(r)|}{g(r+t_0-t_2)} + \sup_{t_0<s+t_2<t_0+\alpha}|x(s+t_1)-x(s+t_2)|\\
&= \sup_{r\le 0}\left|\frac{\varphi(r+t_1-t_2)}{g(r+t_1-t_2)}\cdot\frac{g(r+t_1-t_2)}{g(r+t_0-t_2)} - \frac{\varphi(r)}{g(r)}\cdot\frac{g(r)}{g(r+t_0-t_2)}\right| +\\
&\quad \sup_{t_0<s+t_2<t_0+\alpha}|x(s+t_1)-x(s+t_2)|,
\end{aligned}$$

that is

$$\begin{aligned}
|x_{t_1}-x_{t_2}|_g \le & \sup_{r\le 0}\left\{\left|\frac{\varphi(r+t_1-t_2)}{g(r+t_1-t_2)} - \frac{\varphi(r)}{g(r)}\right|\cdot\frac{g(r+t_1-t_2)}{g(r+t_0-t_2)} +\right.\\
&\left.\left|\frac{\varphi(r)}{g(r)}\right|\cdot\left|\frac{g(r+t_1-t_2)}{g(r+t_0-t_2)} - \frac{g(r)}{g(r+t_0-t_2)}\right|\right\} +\\
&\sup_{t_0<s+t_2<t_0+\alpha}|x(s+t_1)-x(s+t_2)| := I_1 + I_2.
\end{aligned}$$

Noting the fact that $\frac{\varphi}{g}$ is uniformly continuous on R^- and x is uniformly continuous on $[t_0, t_0+\alpha]$, then for any given $\varepsilon > 0$, there exists a $\delta_3 > 0$, such that as $|t_1 - t_2| < \delta_3$, it follows that

$$\left|\frac{\varphi(r+t_1-t_2)}{g(r+t_1-t_2)} - \frac{\varphi(r)}{g(r)}\right| < \varepsilon, \quad |x(s+t_1)-x(s+t_2)| < \varepsilon.$$

Set $v = r + t_0 - t_2 < 0$. By the fact that $\frac{g(s+u)}{g(s)}$ is a continuous function of u for fixed $s \le 0$. For the above $\varepsilon > 0$, there exists a $\delta_4 > 0$, such that as $|t_1 - t_2| < \delta_4$, we get

$$\left|\frac{g(r+t_1-t_2)}{g(r+t_0-t_2)} - \frac{g(r)}{g(r+t_0-t_2)}\right| = \left|\frac{g(v-t_0+t_1)}{g(v)} - \frac{g(v-t_0+t_2)}{g(v)}\right| < \varepsilon.$$

Therefore, for any given $\varepsilon > 0$, there exists a $\delta = \min\{\delta_3, \delta_4\}$, such that as $|t_1 - t_2| < \delta$, one obtains that $I_1 \le \varepsilon \cdot M + |\varphi|_g \cdot \varepsilon = (M + |\varphi|_g)\,\varepsilon$, that is $|x_{t_1} - x_{t_2}|_g \le (M + |\varphi|_g)\,\varepsilon + \varepsilon = (M + |\varphi|_g + 1)\,\varepsilon$. It means that x_t is continuous on $[t_0, t_0+\alpha]$ about t.

(3) For any $t_0 \in R, \alpha \ge 0, x \in A$, then $|x_{t_0+\alpha}|_g \le |x_{t_0}|_g + \sup_{s\in[t_0,t_0+\alpha]}|x(s)|$.

In fact, since $g(s)$ is a monotonously nonincreasing function, then

$$\begin{aligned}
|x_{t_0+\alpha}|_g &\le \sup_{s\le -\alpha}|x_{t_0+\alpha}(s)|/g(s) + \sup_{-\alpha\le s\le 0}|x_{t_0+\alpha}(s)|/g(s)\\
&\le \sup_{s+\alpha\le 0}|x_{t_0}(s+\alpha)|/g(s+\alpha) + \sup_{0\le s+\alpha\le\alpha}|x(t_0+s+\alpha)|\\
&= |x_{t_0}|_g + \sup_{u\in[t_0,t_0+\alpha]}|x(u)|.
\end{aligned}$$

Since $(C_g, |\cdot|_g)$ satisfies the conditions (1)-(3), it easily proves that $(C_g, |\cdot|_g)$ is the special case of phase space defined by [4].

Definition 2.1 *We called the operator D of (2.1) uniformly g-stable, if there exist constants $k_1 > 0, k_2 > 0$ such that the solution $x_t(t_0, \varphi)$ of general difference equation $Dx_t = e(t), t \geq t_0, x_{t_0} = \varphi$ satisfies*

$$|x(t_0, \varphi)(t)| \leq k_1 \sup_{\theta \in [t_0, t]} |e(\theta)| + k_2 |x_{t_0}|_g, \varphi \in C_g, x_t \in C_g, e \in C([t_0, +\infty), R^n).$$

We assume that $D : C_g \to R^n$ is linear, continuous, g-uniformly stable; $f : R \times C_g \to R^n$ is continuous and it is linear about φ, and $f(t + \omega, \varphi) = f(t, \varphi)$ in (2.1). We denote the solution of (2.1) through $(t_0, \varphi) \in R \times C_g$ by $x_t(t_0, \varphi)$. The basic theory of neutral functional differential equation with infinite delay was builded in [4]. Under the above assumptions, by [4], one gets that for any $(t_0, \varphi) \in R \times C_g$, (2.1) exists a unique solution $x_t(t_0, \varphi)$, and it satisfies continuous dependence of the initial condition. In addition, if the solution is bounded, then it can be extended to infinity.

Lemma 2.1[5] *For any constants $a, c > 0, L \geq 0$, the set*

$$S := \{\varphi \in C_g : |\varphi|_g \leq c, \|\varphi\| \leq a, |\varphi(\theta_1) - \varphi(\theta_2)| \leq L|\theta_1 - \theta_2|, \theta_1, \theta_2 \in R^-\}$$

is convex and $|\cdot|_g$ compact.

Lemma 2.2 *Let D be linear, continuous, g- uniformly stable; for $e \in C([\alpha, +\infty), R^n)(\alpha > -\infty)$, there exists a constant $E > 0$ such that for any $t_1, t_2 \in [\alpha, +\infty)$ it follows that $|e(t_1) - e(t_2)| \leq E|t_1 - t_2|$; If $x_t(t_0, \varphi)$ is the solution of $Dx_t = e(t), t \geq 0, x_{t_0} = \varphi, \varphi \in C_g$, then there exists a constant $N(E) > 0$ such that for any $t_1, t_2 \in [t_0, +\infty)(t_0 \geq \alpha)$ then*

$$|x(t_0, \varphi)(t_1) - x(t_0, \varphi)(t_2)| \leq N(E)|t_1 - t_2|, \quad N(E) = k_1(1 + k_2)E.$$

Proof For any $t \in [t_0, +\infty), \Delta > 0$, by the linearity of D, we have

$$D(x_{t+\Delta}(t_0, \varphi) - x_t(t_0, \varphi)) = Dx_{t+\Delta}(t_0, \varphi) - Dx_t(t_0, \varphi) = e(t + \Delta) - e(t).$$

By g-uniform stability of D yields that

$$\begin{aligned} |x(t_0, \varphi)(t + \Delta) - x(t_0, \varphi)(t)| &\leq k_1 \sup_{\theta \in [t_0, t]} |e(\theta + \Delta) - e(\theta)| + k_2 |x_{t_0+\Delta} - x_{t_0}|_g \\ &\leq k_1 E\Delta + k_2 |x_{t_0+\Delta} - x_{t_0}|_g. \end{aligned} \tag{2.2}$$

Furthermore, for any $s \in [t_0, t]$, by g- uniform stability of D thus implies that

$$|x(t_0, \varphi)(s) - x(t_0, \varphi)(t_0)| \leq k_1 \sup_{\tau \in [t_0, s]} |e(\tau) - e(t_0)| + k_2 |x_{t_0} - x_{t_0}|_g \leq k_1 E|s - t_0|.$$

By (3), $|x_s(t_0, \varphi) - x_{t_0}(t_0, \varphi)|_g \leq \sup_{\tau \in [t_0, s]} |x(t_0, \varphi)(\tau) - x(t_0, \varphi)(t_0)| + |x_{t_0}(t_0, \varphi) - x_{t_0}(t_0, \varphi)|_g \leq k_1 E|s - t_0|$, then

$$|x_{t_0+\Delta} - x_{t_0}|_g \leq k_1 E\Delta. \tag{2.3}$$

From (2.2) and (2.3), $|x(t_0,\varphi)(t+\Delta)-x(t_0,\varphi)(t)|\le k_1E\Delta+k_1k_2E\Delta:=N(E)\Delta$. So, for any $t_1,t_2\in[t_0,+\infty)$, there exists a constant $N(E)>0$ such that

$$|x(t_0,\varphi)(t_1)-x(t_0,\varphi)(t_2)|\le N(E)|t_1-t_2|,$$

where $N(E)=k_1(1+k_2)E$. The proof is completed.

Theorem 2.1 *Equation* (2.1) *has an ω periodic solution if and only if there is a bounded solution which defined on R* (*its norm is super norm*).

Proof Since periodic solution itself is a bounded solution, we just need to prove that the existence of bounded implies the existence of periodic solution. For simplicity, without loss of generality, we assume that $t_0=0$.

Let x_t be the bounded solution of (2.1), which is defined on R and its boundedness is B, that is $\|x_t\|\le B$, then $|x_t|_g=\sup_{s\le 0}|x_t(s)|/g(s)\le B$. By the linearity and periodicity of f, there exists a constant $E\ge 0$ such that for any $\varphi\in C_g$ and $|\varphi|_g\le B$, it follows that $|f(t,\varphi)|\le E$. The set

$$\begin{aligned}\Omega:=\{\varphi\in C_g:|\varphi|_g\le B,\|\varphi\|\le B,&\qquad\text{(a)}\\ |\varphi(s_1)-\varphi(s_2)|\le N(E)|s_1-s_2|,s_1,s_2\in R^-,&\qquad\text{(b)}\\ \|x_t(0,\varphi)\|\le B,t\ge 0\},&\qquad\text{(c)}\end{aligned}$$

where $x_t(0,\varphi)$ is the unique solution of (2.1) through $(0,\varphi)$, $N(E)=k_1(1+k_2)E$.

First to prove Ω is not empty. We consider the bounded solution x_t, and define $\varphi_0(s):=x_0(s)=x(s),s\in R^-$, by the definition we get $x_t=x_t(0,\varphi_0)$, so φ_0 satisfies the conditions (a) and (c) in Ω. Next to prove φ_0 satisfies (b). For any given $s_1,s_2\in R^-$, we can choose $\eta>0$ such that $s_1,s_2>-\eta$. Let $\varphi_{-\eta}:=x_{-\eta}(s)=x(s-\eta),s\in R^-$, then $x_t=x_t(0,\varphi_0)=x_t(-\eta,\varphi_{-\eta})$. The bounded solution x_t of (2.1) satisfies

$$Dx_t=Dx_t(-\eta,\varphi_{-\eta})=D\varphi_{-\eta}+\int_{-\eta}^t f(s,x_s(-\eta,\varphi_{-\eta}))\mathrm{d}s:=e(t),$$

then from the boundedness of x_t and the linearity of f, for any $t_1,t_2\in[-\eta,+\infty)$, we have

$$|e(t_1)-e(t_2)|=\Big|\int_{t_1}^{t_2}f(s,x_s(-\eta,\varphi_{-\eta}))\mathrm{d}s\Big|\le\Big|\int_{t_1}^{t_2}|f(s,x_s(-\eta,\varphi_{-\eta}))|\mathrm{d}s\Big|\le E|t_1-t_2|.$$

Thus by Lemma 2.2, we get

$$|x(-\eta,\varphi_\eta)(t_1)-x(-\eta,\varphi_{-\eta})(t_2)|\le N(E)|t_1-t_2|,t_1,t_2\in[-\eta,+\infty).$$

So

$$\begin{aligned}|\varphi_0(s_1)-\varphi_0(s_2)|&=|x_0(-\eta,\varphi_{-\eta})(s_1)-x_0(-\eta,\varphi_{-\eta})(s_2)|\\&=|x(-\eta,\varphi_{-\eta})(s_1)-x(-\eta,\varphi_{-\eta})(s_2)|\le N(E)|s_1-s_2|.\end{aligned}$$

Since s_1, s_2 are arbitrary, φ_0 satisfies the condition (b) in Ω, then $\varphi_0 \in \Omega$, Ω is not empty.

Next to prove Ω is convex and compact. In fact, it easily verifies that Ω is closed set, by Lemma 2.1, Ω is compact. For any $\varphi_1, \varphi_2 \in \Omega, \alpha \in [0,1]$, we get

$$\|\alpha\varphi_1 + (1-\alpha)\varphi_2\| \le \alpha\|\varphi_1\| + (1-\alpha)\|\varphi_2\|, \quad |\alpha\varphi_1 + (1-\alpha)\varphi_2|_g \le \alpha|\varphi_1|_g + (1-\alpha)|\varphi_2|_g \le B,$$

$$|\alpha\varphi_1(s_1) + (1-\alpha)\varphi_2(s_1) - (\alpha\varphi_1(s_2) + (1-\alpha)\varphi_2(s_2))| \le N(E)|s_1 - s_2|, \quad s_1, s_2 \in R^-.$$

From $\varphi_1, \varphi_2 \in \Omega$, then $x_t(0,\varphi_1)$ and $x_t(0,\varphi_2)$ are the solutions of (2.1). For any $\alpha \in [0,1]$, by the linearity of f and D, $\alpha x_t(0,\varphi_1) + (1-\alpha)x_t(0,\varphi_2)$ is still a solution of (2.1), by the uniqueness of solution, $x_t(0, \alpha\varphi_1 + (1-\alpha)\varphi_2) = \alpha x_t(0,\varphi_1) + (1-\alpha)x_t(0,\varphi_2)$, therefore

$$\|x_t(0, \alpha\varphi_1 + (1-\alpha)\varphi_2)\| = \|\alpha x_t(0,\varphi_1) + (1-\alpha)x_t(0,\varphi_2)\| \le B.$$

So we prove Ω is convex and compact.

Now define $P : \Omega \to \Omega$ as follows $P\varphi := x_\omega(0,\varphi)$, that is

$$P\varphi(s) := x_\omega(0,\varphi)(s) = x(0,\varphi)(s+\omega), s \in R^-.$$

From the continuous dependence of solution on initial condition, obviously, P is continuous. In fact, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that as $|\varphi - \bar{\varphi}|_g < \delta$, it follows that $|x(t;0,\varphi) - x(t;0,\bar{\varphi})| < \varepsilon$, taking $t = \omega$, that is $|x_\omega(0,\varphi) - x_\omega(0,\bar{\varphi})| < \varepsilon$.

Next to show P maps Ω to Ω. Since φ satisfies the condition (c) in Ω, we easily know that $P\varphi$ satisfies the condition (a) in Ω. For $t \ge 0$, by the periodicity of f and the uniqueness of solution,

$$\|x_t(0, P\varphi)\| = \|x_t(0, x_\omega(0,\varphi))\| = \|x_{t+\omega}(\omega, x_\omega(0,\varphi))\| = \|x_{t+\omega}(0,\varphi)\| \le B.$$

We say $P\varphi$ satisfies the condition (c) in Ω. And we notice that the initial value problem of (2.1) with $x_0 = \varphi$ equivalent to $Dx_t = D\varphi + \int_0^t f(s, x_s(0,\varphi))\mathrm{d}s := e(t), t \ge 0$. For any $t_1, t_2 \in [0, +\infty)$, by $\varphi \in \Omega$ and $x_t(0,\varphi)$ is bounded, one obtains that

$$|e(t_1) - e(t_2)| = \Big|\int_{t_1}^{t_2} f(s, x_s(0,\varphi))\Big| \le \Big|\int_{t_1}^{t_2} |f(s, x_s(0,\varphi))|\mathrm{d}s\Big| \le E|t_1 - t_2|.$$

From Lemma 2.2, we have

$$|x(0,\varphi)(t_1) - x(0,\varphi)(t_2)| \le N(E)|t_1 - t_2|, t_1, t_2 \in [0, +\infty). \tag{2.4}$$

By $\varphi \in \Omega$, we get

$$|x(0,\varphi)(t_1) - x(0,\varphi)(t_2)| = |\varphi(t_1) - \varphi(t_2)| \le N(E)|t_1 - t_2|, t_1, t_2 \in R^-. \tag{2.5}$$

So far, we can claim that:

$$|P\varphi(s_1) - P\varphi(s_2)| \le N(E)|s_1 - s_2|, s_1, s_2 \in R^-.$$

If $s_1, s_2 \in [-\omega, 0]$ (or $s_1, s_2 \in (-\infty, -\omega]$), by (2.4) (or (2.5)), then

$$|P\varphi(s_1)-P\varphi(s_2)| \le |x_\omega(0,\varphi)(s_1)-x_\omega(0,\varphi)(s_2)| = |x(0,\varphi)(s_1+\omega)-x(0,\varphi)(s_2+\omega)| \le N(E)|s_1-s_2|;$$

If $s_1(-\infty, -\omega], s_2 \in [-\omega, 0]$, obviously, $s_1 \le -\omega \le s_2$, by (2.4) and (2.5), we have

$$\begin{aligned}|P\varphi(s_1) - P\varphi(s_2)| &= |x_\omega(0,\varphi)(s_1) - x_\omega(0,\varphi)(s_2)| = |x(0,\varphi)(s_1+\omega) - x(0,\varphi)(s_2+\omega)| \\ &\le |x(0,\varphi)(s_1+\omega) - x(0,\varphi)(0)| + |x(0,\varphi)(0) - x(0,\varphi)(s_2+\omega)| \\ &\le N(E)|s_1+\omega| + N(E)|s_2+\omega| = -N(E)(s_1+\omega) + N(E)(s_2+\omega) = N(E)|s_1-s_2|.\end{aligned}$$

Thus, $P\varphi$ satisfies the condition (b) in Ω. So, P maps Ω to Ω. By Schauder fixed theorem, P has a unique fixed point in Ω, i.e. there exists a $\psi \in \Omega$ such that $P\psi = \psi$, that is to say $x_\omega(0,\psi) = x_0(0,\psi)$. From the periodicity of f and the uniqueness of the solution,

$$x_{t+\omega}(0,\psi) = x_t(0,\psi), t \ge 0.$$

Therefore, $x_t(0,\psi)$ is an ω periodic solution of (2.1). The proof is completed.

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无限时滞线性中立型泛函微分方程周期解

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摘要: 本文以 C_g 空间为相空间，证明了具无限时滞 D 算子型线性中立型泛函微分方程存在周期解当且仅当存在有界解，得到了与以往结论互不包含的结果.

关键词: 线性中立型泛函微分方程; 无限时滞; 有界解; 周期解; 充要条件.