A Framework for Mesoscale Eddy Parameterization Based on Density-Weighted Averaging at Fixed Height

RICHARD J. GREATBATCH

Department of Oceanography, Dalhousie University, Halifax, Nova Scotia, Canada

27 October 2000 and 26 March 2001

ABSTRACT

A framework for mesoscale eddy parameterization based on density-weighted averaging at fixed height is developed. The method uses the fully non-Boussinesq equations of motion and is connected to the equations carried by Boussinesq ocean models only after the averaged equations have been developed. The framework applies to the continuity, tracer, and momentum equations within a single formalism. Two methods for applying parameterizations in ocean models are obtained. The first, based on the tracer equation, corresponds to the approach commonly taken when including eddy effects in ocean models. The second puts the forcing for the eddy-induced transport into the averaged momentum equation where it appears as the divergence of a generalized Eliassen–Palm flux.

It is then shown how to solve for the tracer transport velocity. The solutions form a family closely related to the temporal residual mean (TRM) velocity of McDougall and McIntosh, valid to $O(\alpha^3)$, where α is perturbation amplitude. The analysis is extended to obtain a family of exact solutions for the eddy-induced mass transport, valid at any order in perturbation amplitude. It is also shown how to obtain a generalization of the TRM to take account of diffusion and time dependence in the instantaneous equations. The solution suggests that the tracer transport velocity could be different for different tracers, depending primarily on the structure of the mean field. This conclusion also applies in the case of isopycnal averaging; it is not a result that is peculiar to averaging at fixed height.

Finally, it is shown how the non-Boussinesq analysis presented in the paper can be modified to analyze output from eddy-resolving, Boussinesq ocean models.

1. Introduction

As parameterizations are developed for the transport of tracers by mesoscale eddies, different ways of averaging the equations of motion are being considered. For example, the widely used Gent and McWilliams (1990, hereafter GM90) parameterization is often interpreted in terms of averaging on an isopycnal surface (Gent et al. 1995), and other approaches to eddy parameterization have used thickness-weighted averaging on an isopycnal surface (e.g., de Szoeke and Bennett 1993; Dukowicz and Smith 1997; Greatbatch 1998; Dukowicz and Greatbatch 1999; Smith 1999). While averaging on an isopycnal surface has been justified on the grounds that we believe mesoscale eddies mix along isopycnal surfaces, measurements obtained from mooring arrays are generally available at fixed height, rather than at fixed density, and averaging is commonly done at fixed height. In addition, many ocean models, including the widely used Modular Ocean Model (MOM) code (Pacanowski and Griffies 1999), use the height z as their vertical coordinate. There is also the difficulty in the ocean of defining exactly what is meant by isopycnal averaging. This is because the ocean is compressible (be it only weakly), and although progress has been made by thinking in terms of ''neutral density'' (McDougall 1987), problems remain because ''neutral density'' cannot be defined globally, and an approximate form must be used (e.g., Jackett and McDougall 1997; Eden and Willebrand 1999). There is, therefore, considerable motivation to develop an approach to mesoscale eddy parameterization based on averaging at fixed height rather than averaging at fixed density. Very little work has been done, however, to investigate averaging at fixed height. McDougall and McIntosh (1996, hereafter MM) introduced the ''temporal residual mean'' (TRM) velocity using averaging at fixed height, but later modified their approach to mimic isopycnal averaging (McDougall and McIntosh 2001). More recently, McDougall et al. (2001, manuscript submitted to *J. Phys.*

Corresponding author address: Dr. Richard J. Greatbatch, Dept. of Oceanography, Dalhousie University, Halifax, NS B3H 4J1, Canada.
E-mail: richard.greatbatch@dal.ca

Oceanogr.; hereafter MGL) have suggested a method of interpreting the variables in Boussinesq ocean models based on averaging at fixed height, but questions remain as to how to connect their work to approaches for parameterizing mesoscale eddies. This is especially problematic since all the most widely known approaches to eddy parameterization in the ocean are based on the Boussinesq equations of motion.

Here, we extend the analysis of MGL by showing, in section 2, how an approach to mesoscale eddy parameterization fits within their framework. The analysis is based on the non-Boussinesq equations of motion and uses density-weighted averaging at fixed height. Only after the basic framework has been set up within the non-Boussinesq system do we then show how the Boussinesq approximation can be applied to the averaged equations of motion for use in Boussinesq ocean models. The analysis also has the advantage that the continuity, tracer, and momentum equations are treated together within a single framework. In section 3, we show how to solve for the tracer transport velocity. The solutions form a family that are all related to the TRM velocity introduced by MM. The original TRM solution is valid only to $O(\alpha^3)$ in perturbation amplitude. In section 4, the analysis is extended to find an exact solution, valid at any order in perturbation amplitude. The results suggest that the tracer transport velocity may be different for different tracers, depending mostly on the structure of the mean field. In section 5, it is shown how the analysis in sections 2, 3, and 4, all of which is based on the non-Boussinesq equations of motion, can be applied to the analysis of eddy-resolving, Boussinesq ocean models. Finally, section 6 provides a summary and conclusions.

2. The governing equations

We begin by writing down the instantaneous equations governing conservation of mass, a conservative scalar, C, and momentum. Following Batchelor (1967) and Gill (1982), these are

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}$$

$$(\rho C)_t + \nabla \cdot (\rho \mathbf{u} C) = \nabla \cdot (\rho k_C \nabla C), \tag{2}$$

$$\begin{aligned}(\rho \mathbf{u})_t &+ \nabla \cdot (\rho \mathbf{u}\mathbf{u}) + \mathbf{f} \times (\rho \mathbf{u}) \\ &= -\nabla p - \mathbf{k} g \rho + \nabla \cdot (\mu \nabla \mathbf{u}) + \frac{1}{3}\nabla(\mu \nabla \cdot \mathbf{u}).\end{aligned} \tag{3}$$

The terminology here is standard, with μ being the viscosity and k_C being the diffusivity of tracer, C. Here $\mathbf{f} = f\hat{\mathbf{k}}$ where $\hat{\mathbf{k}}$ is a unit vector in the upward vertical direction, and f is the Coriolis parameter (for simplicity, we neglect the horizontal component of the earth's rotation vector). It should be noted that C is defined as the mass of tracer contained in unit mass of fluid (Gill 1982). Throughout the following, C is usually either salinity or potential temperature. In the case of potential temperature, the diffusion term strictly requires modification from the form given in (2) [see Gill 1982, Eq. (4.4.7)], a difference that is not important for the analysis here and will be ignored. It should also be noted that McDougall and Jackett (unpublished manuscript) have argued that a more accurate form of the heat equation is obtained using potential enthalpy, rather than potential temperature, as the prognostic variable. Apart from these issues regarding potential temperature, (1)–(3) are the fully non-Boussinesq equations of motion. None of the analysis that follows depends on making the Boussinesq approximation.

We now consider what happens when the instantaneous equations are time averaged. Rather than use the normal Reynolds averaging, we use density-weighted, or Favre, averaging at fixed height (after Favre 1965a,b). We therefore define

$$\overline{\mathbf{u}}^{\rho} = \overline{\rho \mathbf{u}}/\overline{\rho}, \quad \overline{C}^{\rho} = \overline{\rho C}/\overline{\rho}, \quad \mathbf{u}'_{\rho} = \mathbf{u} - \overline{\mathbf{u}}^{\rho}, \quad \text{and}$$
$$C'_{\rho} = C - \overline{C}^{\rho}, \tag{4}$$

where $\overline{\mathbf{u}}^{\rho}$, $\overline{C}^{\rho}$ are the density-weighted averages of velocity $\mathbf{u}$ and tracer concentration, C, respectively, and $\overline{\rho \mathbf{u}'_{\rho}} = 0$ and $\overline{\rho C'_{\rho}} = 0$. Averaging the instantaneous conservation equations, (1)–(3), then leads to

$$\overline{\rho}_t + \nabla \cdot (\overline{\rho}\, \overline{\mathbf{u}}^{\rho}) = 0, \tag{5}$$

$$(\overline{\rho}\, \overline{C}^{\rho})_t + \nabla \cdot (\overline{\rho}\, \overline{\mathbf{u}}^{\rho} \overline{C}^{\rho}) = -\nabla \cdot (\overline{\rho \mathbf{u}'_{\rho} C'_{\rho}}), \tag{6}$$

$$\begin{aligned}(\overline{\rho}\, \overline{\mathbf{u}}^{\rho})_t &+ \nabla \cdot (\overline{\rho}\, \overline{\mathbf{u}}^{\rho} \overline{\mathbf{u}}^{\rho}) + \mathbf{f} \times (\overline{\rho}\, \overline{\mathbf{u}}^{\rho}) \\ &= -\nabla \overline{p} - \mathbf{k} g \overline{\rho} - \nabla \cdot (\overline{\rho \mathbf{u}'_{\rho} \mathbf{u}'_{\rho}}).\end{aligned} \tag{7}$$

For simplicity, the molecular diffusion and viscosity terms have been neglected in comparison with the turbulent correlation terms. Perhaps the major advantage of using density-weighted averaging is that no turbulent correlation terms appear in the averaged mass conservation equation (5), as appear with conventional Reynolds averaging. Indeed, the form of (5)–(7) closely follows that of the instantaneous equations, (1)–(3).

The tracer transport velocity is the effective velocity by which time-averaged tracer fields are advected (Plumb and Mahlman 1987; Gent et al. 1995). We shall begin by assuming the tracer transport velocity is the same for all conservative tracers, and then show, in sections 3 and 4, how this assumption may require modification. By analogy with Gent et al. (1995), we expect the tracer transport velocity $\mathbf{u}^{\#}$ to satisfy

$$\overline{\rho}_t + \nabla \cdot (\overline{\rho}\, \mathbf{u}^{\#}) = 0, \tag{8}$$

thereby ensuring conservation of mass by the transport velocity, $\mathbf{u}^{\#}$. Comparing with (5), it follows immediately that

$$\nabla \cdot [\overline{\rho}(\mathbf{u}^{\#} - \overline{\mathbf{u}}^{\rho})] = 0, \tag{9}$$

and hence that

$$\overline{\rho}\, \mathbf{u}^{\#} = \overline{\rho}\, \overline{\mathbf{u}}^{\rho} + \nabla \times \mathbf{B}, \tag{10}$$

where $\mathbf{B}$ is a three-dimensional vector field. [In this paper, the terminology ''tracer transport velocity'' is used to refer to the total effective velocity by which mean tracer fields are advected, i.e., $\mathbf{u}^{\#}$, and the ''eddy-induced transport velocity'' to $(\mathbf{u}^{\#} - \overline{\mathbf{u}}^{\rho})$. Similarly the ''eddy-induced mass transport'' refers to $\overline{\rho}(\mathbf{u}^{\#} - \overline{\mathbf{u}}^{\rho})$.] We note that, without loss of generality, we can always write $\mathbf{B}$ in the form

$$\mathbf{B} = (A_2, -A_1, 0). \tag{11}$$

This is because we are interested only in the curl of $\mathbf{B}$, so the gradient of a scalar field can always be added to $\mathbf{B}$ to obtain the form in (11). The two-dimensional vector $\mathbf{A} = (A_1, A_2)$ is then the vector streamfunction for the mass flux associated with $\nabla \times \mathbf{B}$ in (10). The form of $\mathbf{B}$ given by (11) will prove useful in the later analysis.

Let us begin by considering the tracer equation (6). Using (10), this can be written

$$(\overline{\rho}\overline{C}^{\rho})_t + \nabla \cdot (\overline{\rho}\mathbf{u}^{\#}\overline{C}^{\rho}) = -\nabla \cdot [\overline{\rho \mathbf{u}'_\rho C'_\rho} - \overline{C}^{\rho}\nabla \times \mathbf{B}]. \tag{12}$$

Since $\mathbf{u}^{\#}$ is the tracer transport velocity, it is usual to assume that the right-hand side of (12) can be parameterized in terms of a symmetric diffusion tensor K so that

$$(\overline{\rho}\overline{C}^{\rho})_t + \nabla \cdot (\overline{\rho}\mathbf{u}^{\#}\overline{C}^{\rho}) = \nabla \cdot (\overline{\rho}\mathsf{K}\nabla\overline{C}^{\rho}). \tag{13}$$

This is equivalent to adopting a Fickian diffusion parameterization for the flux of tracer $\overline{\rho \mathbf{u}'_\rho C'_\rho}$ in terms of an antisymmetric tensor associated with $\mathbf{u}^{\#}$ and a symmetric tensor K. Written in terms of $\overline{\mathbf{u}}^{\rho}$, (13) becomes

$$(\overline{\rho}\overline{C}^{\rho})_t + \nabla \cdot (\overline{\rho}\overline{\mathbf{u}}^{\rho}\overline{C}^{\rho}) = \nabla \cdot [\overline{\rho}\mathsf{K}\nabla\overline{C}^{\rho} - \overline{C}^{\rho}\nabla \times \mathbf{B}]. \tag{14}$$

We now follow MGL and introduce a new velocity variable,

$$\tilde{\mathbf{u}} = \frac{\overline{\rho \mathbf{u}}}{\rho_o} = \frac{\overline{\rho}}{\rho_o}\overline{\mathbf{u}}^{\rho}, \tag{15}$$

where ρ_o is a representative density for seawater. (Note that $\tilde{\mathbf{u}}$ here corresponds to $\tilde{\overline{\mathbf{u}}}$ in MGL.) Writing (5), (14), and (7) in terms of $\tilde{\mathbf{u}}$ yields

$$(\overline{\rho}/\rho_o)_t + \nabla \cdot \tilde{\mathbf{u}} = 0, \tag{16}$$

$$\left(\frac{\overline{\rho}}{\rho_o}\overline{C}^{\rho}\right)_t + \nabla \cdot (\tilde{\mathbf{u}}\overline{C}^{\rho}) = \frac{1}{\rho_o}\nabla \cdot [\overline{\rho}\mathsf{K}\nabla\overline{C}^{\rho} - \overline{C}^{\rho}\nabla \times \mathbf{B}], \tag{17}$$

$$\tilde{\mathbf{u}}_t + \nabla \cdot \left(\frac{\rho_o}{\overline{\rho}}\tilde{\mathbf{u}}\tilde{\mathbf{u}}\right) + \mathbf{f} \times \tilde{\mathbf{u}} = -\frac{1}{\rho_o}\nabla\overline{p} - \mathbf{k}g\frac{\overline{\rho}}{\rho_o} - \frac{1}{\rho_o}\nabla \cdot (\overline{\rho \mathbf{u}'_\rho \mathbf{u}'_\rho}). \tag{18}$$

Following MGL, we note that making the Boussinesq approximation is equivalent to replacing $\overline{\rho}$ everywhere by ρ_o, except in the buoyancy forcing term in the vertical momentum equation, in which case (16)–(18) reduce to

$$\nabla \cdot \tilde{\mathbf{u}} = 0, \tag{19}$$

$$\overline{C}^{\rho}_t + \nabla \cdot (\tilde{\mathbf{u}}\overline{C}^{\rho}) = \nabla \cdot \left[\mathsf{K}\nabla\overline{C}^{\rho} - \overline{C}^{\rho}\frac{\nabla \times \mathbf{B}}{\rho_o}\right], \tag{20}$$

$$\tilde{\mathbf{u}}_t + \nabla \cdot (\tilde{\mathbf{u}}\tilde{\mathbf{u}}) + \mathbf{f} \times \tilde{\mathbf{u}} = -\frac{1}{\rho_o}\nabla\overline{p} - \mathbf{k}g\frac{\overline{\rho}}{\rho_o} - \frac{1}{\rho_o}\nabla \cdot (\overline{\rho \mathbf{u}'\mathbf{u}'}). \tag{21}$$

The hydrostatic version of (19)–(21) are the equations commonly integrated by Boussinesq ocean models. It is usual to parameterize $-\rho_o^{-1}\nabla \cdot (\overline{\rho \mathbf{u}'_\rho \mathbf{u}'_\rho})$ on the right-hand side of (21) as a Fickian diffusion of momentum, and to use the GM90 parameterization to represent $\rho_o^{-1}\nabla \times \mathbf{B}$. In fact, GM90 can be obtained by putting $\mathbf{A} = (A_1, A_2) = \kappa\rho_o\hat{\mathbf{k}} \times \nabla_H\overline{\rho}/\overline{\rho}_z$ in (11), where κ is the thickness diffusivity and ∇_H is the horizontal gradient operator. In practice, GM90 is implemented either by taking the $\nabla \times \mathbf{B}$ term to the left-hand side of (20), in which case it appears as an additional advective velocity (e.g., Danabasoglu and McWilliams 1995), or by incorporating it with the diffusion tensor K (Griffies 1998). In either case, K in (20) combines diapycnal mixing with the isopycnal mixing tensor introduced by Redi (1982).

An alternative approach is to keep the tracer equation in the form (13) and write the momentum and continuity equations in terms of $\mathbf{u}^{\#}$ to give

$$\overline{\rho}_t + \nabla \cdot (\overline{\rho}\mathbf{u}^{\#}) = 0, \tag{22}$$

$$(\overline{\rho}\overline{C}^{\rho})_t + \nabla \cdot (\overline{\rho}\mathbf{u}^{\#}\overline{C}^{\rho}) = \nabla \cdot (\overline{\rho}\mathsf{K}\nabla\overline{C}^{\rho}), \tag{23}$$

$$\begin{aligned}(\overline{\rho}\,\overline{\mathbf{u}}^{\rho})_t + \nabla \cdot (\overline{\rho}\mathbf{u}^{\#}\overline{\mathbf{u}}^{\rho}) + \mathbf{f} \times (\overline{\rho}\mathbf{u}^{\#}) &= -\nabla\overline{p} - \mathbf{k}g\overline{\rho} + \mathbf{f} \times (\nabla \times \mathbf{B}) \\ &\quad - \nabla \cdot [\overline{\rho \mathbf{u}'_\rho \mathbf{u}'_\rho} - \overline{\mathbf{u}}^{\rho}\nabla \times \mathbf{B}].\end{aligned} \tag{24}$$

In writing the momentum equation, we have retained $\overline{\mathbf{u}}^{\rho}$ as the primary velocity variable (apart from in the Coriolis term), but have used $\mathbf{u}^{\#}$ as the advective velocity, in keeping with the continuity equation (22). This is because, as noted by Greatbatch et al. (2001) the kinetic energy is naturally defined in terms of $\overline{\mathbf{u}}^{\rho}$. However, if desired, it is simple matter to use (10) to write the momentum equation entirely in terms of the single velocity variable $\mathbf{u}^{\#}$. The appearance of two different velocity variables in (24) is, nevertheless, commonly the case when writing the averaged momentum equation in a form to take account of the eddy forcing. It is a feature, for example, of the transformed Eulerian mean under zonal averaging (Andrews et al. 1987), Eq. (2.8a) in Tung (1986), and Eqs. (54) and (55) in Greatbatch (1998).

We next note that writing **B** in terms of the vector streamfunction **A**, as in (11), we can express the right-hand side of (24) as the divergence of a generalized Eliassen–Palm flux, analogous to that in Gent and McWilliams (1996), that is,

$$(\overline{\rho}\,\overline{\mathbf{u}}^{\rho})_t + \nabla\cdot(\overline{\rho}\mathbf{u}^{\#}\overline{\mathbf{u}}^{\rho}) + \mathbf{f}\times(\overline{\rho}\mathbf{u}^{\#}) = -\nabla\overline{p} - \mathbf{k}g\overline{\rho} + \nabla\cdot\mathbf{E}, \tag{25}$$

where

$$\mathbf{E} = -(\overline{\rho\mathbf{u}'_{\rho}\mathbf{u}'_{\rho}} - \overline{\mathbf{u}}^{\rho}\nabla\times\mathbf{B}) + \hat{\mathbf{k}}\mathbf{f}\times\mathbf{A}. \tag{26}$$

In (25), the term arising from **A** takes the form $\partial(\mathbf{f}\times\mathbf{A})/\partial z$, indicating that $\mathbf{f}\times\mathbf{A}$ has the form of a horizontally acting stress [in fact, this term corresponds to what Greatbatch (1998) called the ''eddy stress'']. In the case of the GM90 parameterization, $\partial(\mathbf{f}\times\mathbf{A})/\partial z$ is the term appearing on the right-hand side of Eq. (23) in Gent et al. (1995) and corresponds to the divergence of a vertical flux of geostrophic momentum.

Written in terms of $\mathbf{u}^{\#}$, the continuity, tracer, and momentum equations are

$$\overline{\rho}_t + \nabla\cdot(\overline{\rho}\mathbf{u}^{\#}) = 0, \tag{27}$$

$$(\overline{\rho}\overline{C}^{\rho})_t + \nabla\cdot(\overline{\rho}\mathbf{u}^{\#}\overline{C}^{\rho}) = \nabla\cdot(\overline{\rho}\mathsf{K}\nabla\overline{C}^{\rho}), \tag{28}$$

$$(\overline{\rho}\overline{\mathbf{u}}^{\rho})_t + \nabla\cdot(\overline{\rho}\mathbf{u}^{\#}\overline{\mathbf{u}}^{\rho}) + \mathbf{f}\times(\overline{\rho}\mathbf{u}^{\#}) = -\nabla\overline{p} - \mathbf{k}g\overline{\rho} + \nabla\cdot\mathbf{E}. \tag{29}$$

To see what form these equations take when the Boussinesq approximation is applied, we again follow MGL, except that this time we define the new velocity variable in terms of $\mathbf{u}^{\#}$ rather than $\overline{\mathbf{u}}^{\rho}$. We therefore define

$$\tilde{\mathbf{u}}^{\#} = \frac{\overline{\rho}}{\rho_o}\mathbf{u}^{\#}. \tag{30}$$

In terms of $\tilde{\mathbf{u}}^{\#}$, (27)–(29) become

$$(\overline{\rho}/\rho_o)_t + \nabla\cdot\tilde{\mathbf{u}}^{\#} = 0, \tag{31}$$

$$\left(\frac{\overline{\rho}}{\rho_o}\overline{C}^{\rho}\right)_t + \nabla\cdot(\tilde{\mathbf{u}}^{\#}\overline{C}^{\rho}) = \frac{1}{\rho_o}\nabla\cdot(\overline{\rho}\mathsf{K}\nabla\overline{C}^{\rho}), \tag{32}$$

$$\left(\frac{\overline{\rho}}{\rho_o}\overline{\mathbf{u}}^{\rho}\right)_t + \nabla\cdot(\tilde{\mathbf{u}}^{\#}\overline{\mathbf{u}}^{\rho}) + \mathbf{f}\times\tilde{\mathbf{u}}^{\#} = -\frac{1}{\rho_o}\nabla\overline{p} - \mathbf{k}g\frac{\overline{\rho}}{\rho_o} + \frac{1}{\rho_o}\nabla\cdot\mathbf{E}. \tag{33}$$

Making the Boussinesq approximation then reduces these equations to

$$\nabla\cdot\tilde{\mathbf{u}}^{\#} = 0, \tag{34}$$

$$\overline{C}^{\rho}_t + \nabla\cdot(\tilde{\mathbf{u}}^{\#}\overline{C}^{\rho}) = \nabla\cdot(\mathsf{K}\nabla\overline{C}^{\rho}), \tag{35}$$

$$\overline{\mathbf{u}}^{\rho}_t + \nabla\cdot(\tilde{\mathbf{u}}^{\#}\overline{\mathbf{u}}^{\rho}) + \mathbf{f}\times\tilde{\mathbf{u}}^{\#} = -\frac{1}{\rho_o}\nabla\overline{p} - \mathbf{k}g\frac{\overline{\rho}}{\rho_o} + \frac{1}{\rho_o}\nabla\cdot\mathbf{E}. \tag{36}$$

To implement (34)–(36) [or, more generally, (31)–(33)] in a numerical model, we need to parameterize $\nabla\cdot\mathbf{E}$, **B** (or equivalently, the vector streamfunction, **A**) and the diffusivity tensor K. Knowing **B**, $\overline{\mathbf{u}}^{\rho}$ and $\tilde{\mathbf{u}}^{\#}$ are related by (10) and (30). Likewise, to implement (19)–(21) [or, more generally, (16)–(18)] in a numerical model we must parameterize **B**, K, and $-\rho_o^{-1}\nabla\cdot(\overline{\rho\mathbf{u}'_{\rho}\mathbf{u}'_{\rho}})$ on the right-hand side of (21) and (18). As noted earlier, the common approach is to work with the form (19)–(21), that is, to parameterize the eddy-induced transport through the tracer equation. Equations (34)–(36), on the other hand, have the advantage of revealing how eddies influence the mean flow, analogous to the approach advocated by Wardle and Marshall (2000).

3. The connection between the vector streamfunction A and the TRM velocity of McDougall and McIntosh (1996)

In this section, we show how the vector streamfunction **A** for the eddy-induced mass flux, and hence the vector **B** in (10), is related to the TRM velocity introduced by MM. We work with a conservative tracer C, which, as before, could be potential temperature or salinity. We note that MM choose C to be neutral density, and that our analysis differs from MM in that we do not make the Boussinesq approximation. Rather, we work with the fully non-Boussinesq governing equations.

To begin, we assume there are no sources and sinks, and that we are in a statistically steady state. The effect of adding diffusion to the right-hand side of (2) will be discussed later. The instantaneous equation governing C is then

$$(\rho C)_t + \nabla\cdot(\rho\mathbf{u}C) = 0. \tag{37}$$

Density-weighted averaging leads, as before, to (6), which in statistically steady state reduces to

$$\nabla\cdot(\overline{\rho}\,\overline{\mathbf{u}}^{\rho}\overline{C}^{\rho}) = -\nabla\cdot(\overline{\rho\mathbf{u}'_{\rho}C'_{\rho}}). \tag{38}$$

We also have the eddy variance equation, the statistically steady version of which is

$$\nabla\cdot(\overline{\rho}\,\overline{\mathbf{u}}^{\rho}\overline{\phi}) = -(\overline{\rho\mathbf{u}'_{\rho}C'_{\rho}})\cdot\nabla\overline{C}^{\rho} + O(\alpha^3), \tag{39}$$

where $\phi = ½\,C'^2_{\rho}$. The triple correlation term is shown as $O(\alpha^3)$, where α measures perturbation amplitude. (An exact solution, including the triple correlation, is derived in section 4.)

We shall begin by seeking the vector **B** such that the right-hand side of (12) is zero. Then, later, when we add diffusion to (37), we shall show that a solution can be found for which K in (13) is nonzero. It follows that at this stage, the problem is reduced to finding **B** such that

$$\overline{\rho\mathbf{u}'_{\rho}C'_{\rho}} - \overline{C}^{\rho}\nabla\times\mathbf{B} = \nabla\times\mathbf{D}, \tag{40}$$

where **D** is a "gauge" that enters because all we require is that the divergence on the left-hand side of (12) be zero. We next note that (40) can be rewritten as

$$\overline{\rho \mathbf{u}'_\rho C'_\rho} = \nabla \times \mathbf{\Theta} + \mathbf{B} \times \nabla \overline{C}^\rho, \qquad (41)$$

where

$$\mathbf{\Theta} = (\overline{C}^\rho \mathbf{B} + \overline{\mathbf{D}}). \qquad (42)$$

Using (11) to write **B** in terms of **A**, and noting that, without loss of generality, we can write

$$\mathbf{\Theta} = (\theta_2, -\theta_1, 0) \qquad (43)$$

we obtain

$$\overline{\rho \mathbf{u}'_\rho C'_\rho} = \left(\frac{\partial \mathbf{T}}{\partial z} - \frac{\partial \overline{C}^\rho}{\partial z}\mathbf{A}, -\nabla_H \cdot \mathbf{T} + \mathbf{A} \cdot \nabla_H \overline{C}^\rho\right), \qquad (44)$$

where

$$\mathbf{T} = (\theta_1, \theta_2). \qquad (45)$$

Equating the horizontal components in (44), we obtain

$$\mathbf{A} = \frac{1}{\overline{C}^\rho_z}\frac{\partial \mathbf{T}}{\partial z} - \frac{1}{\overline{C}^\rho_z}\overline{\rho \mathbf{v}'_\rho C'_\rho}, \qquad (46)$$

where $\mathbf{v}$ is the horizontal component of **u**.

Equation (46) gives an expression for **A**. To complete the solution, we must now find a solution for **T**, or, equivalently, $\mathbf{\Theta}$. To do so, we first take the scalar product of $\nabla \overline{C}^\rho$ with (41). The term on the far right-hand side of (41) then drops out, and we are left with

$$(\overline{\rho \mathbf{u}'_\rho C'_\rho}) \cdot \nabla \overline{C}^\rho = \nabla \cdot (\mathbf{\Theta} \times \nabla \overline{C}^\rho). \qquad (47)$$

We now substitute for the left-hand side of (47) from the eddy variance equation, (39), and use (43) and (45) to obtain (dropping the triple correlation)

$$\nabla \cdot \left\{(\overline{\rho}\, \overline{\mathbf{u}}^\rho \overline{\phi}) + \left(-\mathbf{T}\frac{\partial \overline{C}^\rho}{\partial z}, \mathbf{T} \cdot \nabla_H \overline{C}^\rho\right)\right\} = 0. \qquad (48)$$

The general solution is

$$(\overline{\rho}\, \overline{\mathbf{u}}^\rho \overline{\phi}) + \left(-\mathbf{T}\frac{\partial \overline{C}^\rho}{\partial z}, \mathbf{T} \cdot \nabla_H \overline{C}^\rho\right) = \left(\frac{\partial \mathbf{F}}{\partial z}, -\nabla_H \cdot \mathbf{F}\right), \qquad (49)$$

where **F** is a two-dimensional vector.

Let us seek a solution by putting $\mathbf{F} = 0$. Then

$$(\overline{\rho}\, \overline{\mathbf{u}}^\rho \overline{\phi}) + \left(-\mathbf{T}\frac{\partial \overline{C}^\rho}{\partial z}, \mathbf{T} \cdot \nabla_H \overline{C}^\rho\right) = 0. \qquad (50)$$

Equating the horizontal components, we obtain

$$\mathbf{T} = \frac{1}{\overline{C}^\rho_z}(\overline{\rho}\, \overline{\mathbf{v}}^\rho \overline{\phi}). \qquad (51)$$

Using (38), it is easy to show that with this choice for **T**, the vertical component of (50) is also satisfied to $O(\alpha^4)$, where, as before, α measures perturbation amplitude. Since we have already neglected terms of $O(\alpha^3)$, it follows that (51) gives a consistent solution for **T** to this order in α. Substituting back in (46), we obtain

$$\mathbf{A} = \frac{1}{\overline{C}^\rho_z}\frac{\partial}{\partial z}\left\{\frac{1}{\overline{C}^\rho_z}(\overline{\rho}\, \overline{\mathbf{v}}^\rho \overline{\phi})\right\} - \frac{1}{\overline{C}^\rho_z}\overline{\rho \mathbf{v}'_\rho C'_\rho}. \qquad (52)$$

Apart from the factors of ρ that appear because we have used density-weighted averaging, (52) is identical to the formula given by MM for the vector streamfunction $\mathbf{\Psi}$ associated with the temporal residual mean velocity [cf. (52) with Eq. (11) in MM). It follows that we have found a strong connection between the TRM introduced by MM, and the eddy-induced mass transport associated with the eddies in (10) and (14). It should also be noted that repeating the analysis leading to (52) with $\rho = 1$ throughout, and using Reynolds averaging instead of density-weighted averaging, gives an alternative derivation for $\mathbf{\Psi}$ to that given by MM. Indeed, in the Boussinesq system usually integrated by models (see section 5), (52) corresponds to an eddy-induced transport velocity with vector streamfunction, $\mathbf{A}^\#$, given by

$$\mathbf{A}^\# = \frac{1}{\overline{C}_z}\frac{\partial}{\partial z}\left\{\frac{1}{\overline{C}_z}(\overline{\mathbf{v}}\overline{\phi})\right\} - \frac{1}{\overline{C}_z}\overline{\mathbf{v}'C'}. \qquad (53)$$

Equation (52) is a special solution that was obtained by putting $\mathbf{F} = 0$ in (49). For $\mathbf{F} \neq 0$, we can use the horizontal component of (49) to obtain an expression for **T**, as before, that is,

$$\mathbf{T} = \frac{1}{\overline{C}^\rho_z}\left(\overline{\rho}\, \overline{\mathbf{v}}^\rho \overline{\phi} - \frac{\partial \mathbf{F}}{\partial z}\right), \qquad (54)$$

and use the vertical component of (49) to place a consistency condition on **F**. Using (38), the consistency condition, to $O(\alpha^4)$, is

$$-\mathbf{F}_z \cdot \nabla_H \overline{C}^\rho + \nabla_H \cdot \mathbf{F}\, \overline{C}^\rho_z = 0. \qquad (55)$$

In other words, **F** is the vector streamfunction for a flux of eddy variance that must lie in surfaces of $\overline{C}^\rho$. Any **F** satisfying this condition can be used to obtain a possible solution for **T**, and hence **A**. The lack of uniqueness implied by this result has already been noted by MM who argue that the solution given by (52) is, nevertheless, the physically relevant one.

Let us now seek a solution when diffusion is added to the right-hand side of (37), so that we now work with (2). In a statistically steady state, the eddy variance equation (39) becomes

$$\begin{aligned}\nabla \cdot (\overline{\rho}\, \overline{\mathbf{u}}^\rho \overline{\phi}) = &-(\overline{\rho \mathbf{u}'_\rho C'_\rho}) \cdot \nabla \overline{C}^\rho \\ &+ \kappa_C \overline{\rho}\{\overline{\nabla(C'_\rho \nabla C'_\rho)}^\rho - \overline{\nabla C'_\rho \cdot \nabla C'_\rho}^\rho\} \\ &+ O(\alpha^3). \end{aligned} \qquad (56)$$

The term $\overline{\nabla(C'_\rho \nabla C'_\rho)}^\rho$ is small in its effect compared to $\overline{\nabla C'_\rho \cdot \nabla C'_\rho}$ (McDougall and Garrett 1992) and will be dropped. We now seek a solution such that the right-hand side of (12) can be written as a diffusion, as in (13). Then (40) is replaced by

$$\overline{\rho \mathbf{u}'_\rho C'_\rho} - \overline{C}^\rho \boldsymbol{\nabla} \times \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{D} - \overline{\rho}\mathsf{K}\boldsymbol{\nabla}\overline{C}^\rho, \qquad (57)$$

where K is a symmetric, positive definite diffusion tensor [in models, K combines diapycnal mixing with the isopycnal diffusion tensor of Redi (1982)]. As in the previous analysis, we write (57) as

$$\overline{\rho \mathbf{u}'_\rho C'_\rho} = \boldsymbol{\nabla} \times \boldsymbol{\Theta} + \mathbf{B} \times \boldsymbol{\nabla}\overline{C}^\rho - \overline{\rho}\mathsf{K}\boldsymbol{\nabla}\overline{C}^\rho. \qquad (58)$$

Taking the scalar product with $\boldsymbol{\nabla}\overline{C}^\rho$ and using the eddy variance equation (56) (dropping the $\overline{\boldsymbol{\nabla}(C'_\rho \boldsymbol{\nabla} C'_\rho)}^\rho$ term), we obtain

$$\kappa_C \overline{\rho}[\overline{\boldsymbol{\nabla} C'_\rho \cdot \boldsymbol{\nabla} C'_\rho}^\rho] - \overline{\rho}\boldsymbol{\nabla}\overline{C}^\rho \cdot \mathsf{K}\boldsymbol{\nabla}\overline{C}^\rho = -\boldsymbol{\nabla} \cdot \left\{(\overline{\rho}\,\overline{\mathbf{u}}^\rho \overline{\phi}) + \left(-\mathbf{T}\frac{\partial \overline{C}^\rho}{\partial z}, \mathbf{T} \cdot \boldsymbol{\nabla}_H \overline{C}^\rho\right)\right\} + O(\alpha^3). \qquad (59)$$

We can find a solution by adopting a generalization of the flux decomposition suggested by Marshall and Shutts (1981). In particular, we associate the diffusive part ($\overline{\rho}\mathsf{K}\boldsymbol{\nabla}\overline{C}^\rho$) of the flux $\overline{\rho \mathbf{u}'_\rho C'_\rho}$ in (58) with the local, irreversible, removal of tracer variance, and the rotational part ($\boldsymbol{\nabla} \times \boldsymbol{\Theta}$) with the advection of tracer variance, $\overline{\phi}$. Doing so puts the left-hand side of (59) to zero so that

$$\boldsymbol{\nabla}\overline{C}^\rho \cdot \mathsf{K}\boldsymbol{\nabla}\overline{C}^\rho = \kappa_C[\overline{\boldsymbol{\nabla} C'_\rho \cdot \boldsymbol{\nabla} C'_\rho}^\rho]. \qquad (60)$$

Note that, since K is positive definite, $\kappa_C[\overline{\boldsymbol{\nabla} C'_\rho \cdot \boldsymbol{\nabla} C'_\rho}]$ must also be positive, as is indeed the case. Since the right-hand side of (59) must also be zero, we can choose $\mathbf{T}$ as before [i.e., as in (51)]. Note that, just as before, the choice of solution given by (51) requires consistency in the sense that the vertical component of the vector inside the $\boldsymbol{\nabla}$ operator on the right-hand side of (59) must also be zero. In a statistically steady state, this condition is again satisfied to $O(\alpha^4)$. [This result uses the steady version of (6). Note that, if the local time derivative term in (6) is not zero, the requirement on the vertical component is satisfied only to $O(\alpha^2)$, so reducing the accuracy of the solution in this case.] Finally, we note that although we have made a very special choice of solution, based on the flux decomposition of Marshall and Shutts (1981), Peterson and Greatbatch (2001) have found evidence to support this decomposition in numerical experiments using a layered model.

Putting these results together, and using (58), we finally obtain a modified solution for $\mathbf{A}$ given by

$$\mathbf{A} = \frac{1}{\overline{C}^\rho_z}\frac{\partial}{\partial z}\left\{\frac{1}{\overline{C}^\rho_z}(\overline{\rho}\,\overline{\mathbf{v}}^\rho \overline{\phi})\right\} - \frac{1}{\overline{C}^\rho_z}\{\overline{\rho \mathbf{v}'_\rho C'_\rho} + (\overline{\rho}\mathsf{K}\boldsymbol{\nabla}\overline{C}^\rho)_H\}, \qquad (61)$$

where "H" denotes horizontal component. Unfortunately (60) does not determine the diffusivity K uniquely, so clearly additional equations are required to close the system. Nevertheless, we have shown how a solution can be found and its relation to both the TRM of MM and the flux decomposition introduced by Marshall and Shutts (1981).

Finally, in this section, we note that the approximate solution found for the vector streamfunction, $\mathbf{A}$, has been derived for a single tracer, C. A question naturally arises as to whether the resulting eddy-induced transport represented by $\mathbf{A}$ is tracer invariant, as assumed in section 2; in particular, is the eddy-induced transport velocity the same for any tracer, C? We note that in addition to the appearance of κ_C (which is tracer dependent) in the expression for K in (60), the expression for $\mathbf{A}$ given in (52) also depends on the details of the mean field, represented by $\overline{C}^\rho_z$, and the strength of the fluctuations, C'_ρ. There is also the difficulty that (52) is only an approximate solution, valid to $O(\alpha^3)$, and it is not clear what happens at finite amplitude. With these thoughts in mind, we now proceed to generalize the analysis in this section to find an exact solution for the eddy-induced mass transport represented by the vector $\mathbf{B}$ in (10) and, hence, for the temporal residual mean velocity itself.

4. An exact solution for the TRM mass transport

Let us begin with the case in which $\kappa_C = 0$ so that the instantaneous tracer equation is (37). As before, we seek vectors $\boldsymbol{\Theta}$ and $\mathbf{B}$ to satisfy (41), that is,

$$\overline{\rho \mathbf{u}'_\rho C'_\rho} = \boldsymbol{\nabla} \times \boldsymbol{\Theta} + \mathbf{B} \times \boldsymbol{\nabla}\overline{C}^\rho. \qquad (62)$$

Without loss of generality, we can seek a solution for $\mathbf{B}$ that has the property that $\mathbf{B} \cdot \boldsymbol{\nabla}\overline{C}^\rho = 0$. This can be understood because in (62), $\mathbf{B}$ appears only in the form $\mathbf{B} \times \boldsymbol{\nabla}\overline{C}^\rho$ so that the component of $\mathbf{B}$ parallel to $\boldsymbol{\nabla}\overline{C}^\rho$ plays no role. Another way to look at this is that, whereas previously [cf. (11)] we simplified $\mathbf{B}$ by putting the vertical component to zero, here we can put the component parallel to $\boldsymbol{\nabla}\overline{C}^\rho$ to zero. Taking (62) $\times\, \boldsymbol{\nabla}\overline{C}^\rho$ then gives

$$\mathbf{B} = \frac{1}{|\boldsymbol{\nabla}\overline{C}^\rho|^2}\boldsymbol{\nabla}\overline{C}^\rho \times \{(\overline{\rho \mathbf{u}'_\rho C'_\rho}) - \boldsymbol{\nabla} \times \boldsymbol{\Theta}\}. \qquad (63)$$

This is obviously similar in structure to (46). As before, the next step is to solve for $\boldsymbol{\Theta}$. Taking the scalar product of $\boldsymbol{\nabla}\overline{C}^\rho$ with (62) leads, as before, to

$$(\overline{\rho \mathbf{u}'_\rho C'_\rho}) \cdot \boldsymbol{\nabla}\overline{C}^\rho = \boldsymbol{\nabla} \cdot (\boldsymbol{\Theta} \times \boldsymbol{\nabla}\overline{C}^\rho). \qquad (64)$$

We now substitute for $(\overline{\rho \mathbf{u}'_\rho C'_\rho}) \cdot \boldsymbol{\nabla}\overline{C}^\rho$ from the eddy variance equation. However, to obtain an exact solution we need the exact form of that equation. In a statistically steady state, that is

$$\nabla \cdot (\overline{\rho}\,\overline{\mathbf{u}}^{\rho}\overline{\phi}^{\rho} + \overline{\rho\mathbf{u}'_{\rho}\phi}) = -(\overline{\rho\mathbf{u}'_{\rho}C'_{\rho}}) \cdot \nabla\overline{C}^{\rho}. \quad (65)$$

[Note that now we use the density-weighted average of ϕ in the $(\nabla \cdot (\overline{\rho}\,\overline{\mathbf{u}}^{\rho}\,\overline{\phi}^{\rho})$ term.] Combining with (64) gives

$$\nabla \cdot (\overline{\rho}\,\overline{\mathbf{u}}^{\rho}\overline{\phi}^{\rho} + \overline{\rho\mathbf{u}'_{\rho}\phi} + \mathbf{\Theta} \times \nabla\overline{C}^{\rho}) = 0. \quad (66)$$

This can be solved for $\mathbf{\Theta}$, again noting that we can put the component of $\mathbf{\Theta}$ in the direction of $\nabla\overline{C}^{\rho}$ to zero, to obtain

$$\mathbf{\Theta} = -\frac{1}{|\nabla\overline{C}^{\rho}|^2}\nabla\overline{C}^{\rho} \times \{\overline{\rho}\,\overline{\mathbf{u}}^{\rho}\overline{\phi}^{\rho} + \overline{\rho\mathbf{u}'_{\rho}\phi} + \nabla \times \mathbf{G}\}, \quad (67)$$

where $\mathbf{G}$ is a "gauge" vector. In fact, the three-dimensional vector $\mathbf{G}$ corresponds to the two-dimensional vector $\mathbf{F}$ in (49), and the condition (55) corresponds to noting that the component of $\nabla \times \mathbf{G}$ in the direction of $\nabla\overline{C}^{\rho}$ plays no role in the solution for $\mathbf{\Theta}$. The general solution for $\mathbf{B}$ is then

$$\mathbf{B} = \frac{1}{|\nabla\overline{C}^{\rho}|^2}\nabla\overline{C}^{\rho} \times \left\{\overline{(\rho\mathbf{u}'_{\rho}C'_{\rho})} + \nabla \times \left[\frac{1}{|\nabla\overline{C}^{\rho}|^2}\nabla\overline{C}^{\rho} \times (\overline{\rho}\,\overline{\mathbf{u}}^{\rho}\overline{\phi}^{\rho} + \overline{\rho\mathbf{u}'_{\rho}\phi} + \nabla \times \mathbf{G})\right]\right\}. \quad (68)$$

Comparison with (52), which corresponds to putting $\mathbf{G} = 0$, shows the obvious similarities. It is important to note that there is no restriction on the amplitude of the fluctuations in (68). Equation (68) is an exact solution to any order in perturbation amplitude. The nonuniqueness of the solution, noted when discussing (52), is reflected by the appearance of the $\nabla \times \mathbf{G}$ term.

The solution can be extended to include diffusion in the instantaneous tracer equation by invoking the flux decomposition of Marshall and Shutts (1981) exactly as before. Time dependence can also now be included without any loss of accuracy in the solution. Equation (62) is now replaced by (58), as before, and the eddy variance equation becomes (dropping the $\overline{\nabla(C'_{\rho}\nabla C'_{\rho})}^{\rho}$ term, as before)

$$(\overline{\rho}\,\overline{\phi}^{\rho})_t + \nabla \cdot (\overline{\rho}\,\overline{\mathbf{u}}^{\rho}\overline{\phi}^{\rho} + \overline{\rho\mathbf{u}'_{\rho}\phi}) = -(\overline{\rho\mathbf{u}'_{\rho}C'_{\rho}}) \cdot \nabla\overline{C}^{\rho} - \kappa_C\overline{\rho}\,\overline{\nabla C'_{\rho} \cdot \nabla C'_{\rho}}^{\rho}). \quad (69)$$

Following the same flux decomposition as previously, the solution becomes

$$\mathbf{B} = \frac{1}{|\nabla\overline{C}^{\rho}|^2}\nabla\overline{C}^{\rho} \times \left\{\overline{(\rho\mathbf{u}'_{\rho}C'_{\rho})} + \overline{\rho}\mathsf{K}\nabla\overline{C}^{\rho} + \nabla \times \left[\frac{1}{|\nabla\overline{C}^{\rho}|^2}\nabla\overline{C}^{\rho} \times (\overline{\rho}\,\overline{\mathbf{u}}^{\rho}\overline{\phi}^{\rho} + \overline{\rho\mathbf{u}'_{\rho}\phi} + \nabla \times \mathbf{G})\right]\right\} \quad (70)$$

with

$$\nabla\overline{C}^{\rho} \cdot \mathsf{K}\nabla\overline{C}^{\rho} = \overline{\rho}^{-1}(\overline{\rho}\,\overline{\phi}^{\rho})_t + \kappa_C[\overline{\nabla C'_{\rho} \cdot \nabla C'_{\rho}}^{\rho}]. \quad (71)$$

Since K is positive definite, this time we require

$$\overline{\rho}^{-1}(\overline{\rho}\,\overline{\phi}^{\rho})_t + \kappa_C[\overline{\nabla C'_{\rho} \cdot \nabla C'_{\rho}}^{\rho}] \ge 0. \quad (72)$$

In a statistically steady state, this condition is satisfied, as before. The difference from (60) is the inclusion of time dependence. If there is growth in eddy variance, $(\overline{\rho}\,\overline{\phi}^{\rho})_t > 0$, then clearly (72) is satisfied, as one would expect.

Let us examine the solution given by (70) in more detail. We note that to obtain $\mathbf{B}$, the diffusive part of the eddy flux $(\overline{\rho}\mathsf{K}\nabla\overline{C}^{\rho})$ and the rotational part (given by the $\nabla \times$ term) are removed from the total eddy flux $(\overline{\rho\,\mathbf{u}'_{\rho}\,C'_{\rho}})$, and the remainder is projected on to surfaces of uniform $\overline{C}^{\rho}$. The formula applies for a particular choice of tracer, C, and, as noted at the end of section 3, it is far from clear that the eddy-induced transport velocity associated with $\mathbf{B}$ is tracer invariant. In the case of zonal averaging, Plumb and Mahlman (1987) diagnosed the transport velocity in the vertical plane from the Geophysical Fluid Dynamics Laboratory general circulation/tracer model using two independent tracers with nonzero and nonparallel mean gradients. The diagnosed tracer transport circulation is similar to the residual mean circulation given by the transformed Eulerian mean (Andrews et al. 1987), suggesting that tracer dependence may not be a serious problem in that case. (Under zonal averaging, and for small perturbations about the zonal flow, Plumb and Mahlman note that the transport velocity is indeed tracer invariant. However, it is not clear that tracer invariance holds at large amplitude.) Nevertheless, particularly in the ocean, where the potential temperature and salinity fields have very different mean structures and where the continental boundaries complicate the geometry, lack of tracer invariance could be a problem, requiring careful numerical experimentation to sort out. If $\mathbf{B}$, and hence $\mathbf{u}^{\#}$ in (10), is not tracer invariant, then to write the governing equations in the form of (31)–(33) or, in the Boussinesq case

(34)–(36), will require the choice of a particular tracer to define $\mathbf{u}^{\#}$, and then adjustment of each tracer equation to allow for differences in the tracer transport velocity. This is clearly undesirable and would favor writing the equations in the form (16)–(18), or (19)–(21).

It should be noted that the problem of tracer dependence is not a peculiarity of using density-weighted averaging at fixed height. In fact, the same solution procedure can be used to solve for the eddy-induced transport velocity in the case of isopycnal averaging. In that case, the problem is simpler, because it is in two dimensions. More importantly, most of the effect of the eddies is contained in the thickness-weighted, isopycnal-averaged velocity, $\hat{\mathbf{u}} = \overline{z_\rho \mathbf{u}}/z_\rho$ [see Eq. (32) in Greatbatch 1998]. In the formalism, $\hat{\mathbf{u}}$ plays the role of $\overline{\mathbf{u}}^\rho$ in (10), and a solution with $\mathbf{B} = 0$ is acceptable in the case of isopycnal averaging, as was, in fact, assumed in Gent et al. (1995), Eqs. (6) and (7). On the other hand, when using density-weighted averaging at fixed height, the flux associated with $\mathbf{B}$ is an essential part of the effect of the eddies, and a solution with $\mathbf{B} = 0$ is not an acceptable one. This can be understood from the discussion following Eq. (26), where we noted that the GM90 parameterization is a parameterization for $\mathbf{A}$, the vector streamfunction associated with $\mathbf{B}$. Putting $\mathbf{B} = 0$ actually amounts to saying that the form drag effect of eddies, transferring momentum vertically, is not important, and is unlikely to be the case, in general. It follows that the issue of tracer dependence is probably more serious for the height averaging used in this paper than it is for isopycnal averaging. Tracer dependence, nevertheless, remains an issue for further investigation.

5. Application to the analysis of eddy-resolving Boussinesq ocean models

There is a strong need to analyze eddy-resolving model output in order to verify ideas about eddy-parameterization (e.g., Lee et al. 1997; Killworth 1998; Marshall et al. 1999; Roberts and Marshall 2000; Peterson and Greatbatch 2001). Since almost all eddy-resolving ocean model experiments are carried out using Boussinesq model codes, it may not be immediately obvious how the analysis in this paper, which is carried out using the non-Boussinesq governing equations, can be applied to the output from a Boussinesq ocean model. In this section, we briefly describe the modifications to the analysis that are required in the case of a z-coordinate, Boussinesq ocean model such as the MOM code. All the averaging is done at fixed height, as in the main part of the paper. The analysis can also be modified to apply to the analysis of isopycnal coordinate models, with the thickness (h) playing the role of density (ρ), although we do not discuss this case further here. The equations integrated by a Boussinesq ocean model are (with some possible differences in the dissipation terms) the Boussinesq version of (1)–(3) in which ρ is replaced by the constant reference density, ρ_o, everywhere except in the gravitational acceleration term. Applying Reynolds averaging to the model equations then leads to

$$\nabla \cdot \overline{\mathbf{u}} = 0, \tag{73}$$

$$\overline{C}_t + \nabla \cdot (\overline{\mathbf{u}}\,\overline{C}) = -\nabla \cdot (\overline{\mathbf{u}'C'}), \tag{74}$$

$$\overline{\mathbf{u}}_t + \nabla \cdot (\overline{\mathbf{u}}\,\overline{\mathbf{u}}) + \mathbf{f} \times \overline{\mathbf{u}} = -\frac{1}{\rho_o}\nabla \overline{p} - \mathbf{k} g \frac{\overline{\rho}}{\rho_o} - \nabla \cdot (\overline{\mathbf{u}'\mathbf{u}'}). \tag{75}$$

Here $\overline{\mathbf{u}}$ is the Eulerian mean of the model's instantaneous velocity. By analogy with (8), we now require that the tracer transport velocity $\mathbf{u}^{\#}$ satisfy

$$\nabla \cdot \mathbf{u}^{\#} = 0. \tag{76}$$

Now $\overline{\mathbf{u}}$ and $\mathbf{u}^{\#}$ are related by (10) with $\overline{\rho}$ replaced by ρ_o; that is,

$$\mathbf{u}^{\#} = \overline{\mathbf{u}} + \nabla \times \frac{\mathbf{B}}{\rho_o}. \tag{77}$$

The factor ρ_o has been retained so that the vector $\mathbf{B}$ used here corresponds directly to the vector $\mathbf{B}$ in section 2. Introducing the diffusion tensor K, as before, we arrive at either (19)–(21) or (34)–(36), with $\tilde{\mathbf{u}}$ replaced by $\overline{\mathbf{u}}$ and $(\overline{\rho \mathbf{u}'_\rho \mathbf{u}'_\rho})$ replaced by $\rho_o(\overline{\mathbf{u}'\mathbf{u}'})$, i.e.

$$\nabla \cdot \overline{\mathbf{u}} = 0, \tag{78}$$

$$\overline{C}_t + \nabla \cdot (\overline{\mathbf{u}}\,\overline{C}) = \nabla \cdot \left[\mathsf{K}\nabla\overline{C} - \overline{C}\frac{\nabla \times \mathbf{B}}{\rho_o}\right], \tag{79}$$

$$\overline{\mathbf{u}}_t + \nabla \cdot (\overline{\mathbf{u}}\,\overline{\mathbf{u}}) + \mathbf{f} \times \overline{\mathbf{u}} = -\frac{1}{\rho_o}\nabla \overline{p} - \mathbf{k} g \frac{\overline{\rho}}{\rho_o} - \nabla \cdot (\overline{\mathbf{u}'\mathbf{u}'}) \tag{80}$$

or

$$\nabla \cdot \mathbf{u}^{\#} = 0, \tag{81}$$

$$\overline{C}_t + \nabla \cdot (\mathbf{u}^{\#}\overline{C}) = \nabla \cdot (\mathsf{K}\nabla\overline{C}), \tag{82}$$

$$\overline{\mathbf{u}}_t + \nabla \cdot (\mathbf{u}^{\#}\overline{\mathbf{u}}) + \mathbf{f} \times \mathbf{u}^{\#} = -\frac{1}{\rho_o}\nabla \overline{p} - \mathbf{k} g \frac{\overline{\rho}}{\rho_o} + \frac{1}{\rho_o}\nabla \cdot \mathbf{E}, \tag{83}$$

where

$$\mathbf{E} = -(\rho_o \overline{\mathbf{u}'\mathbf{u}'} - \overline{\mathbf{u}}\nabla \times \mathbf{B}) + \hat{\mathbf{k}}\mathbf{f} \times \mathbf{A}. \tag{84}$$

The analysis of sections 3 and 4 also has its parallel, Boussinesq version. All that is required is to replace (i) ρ everywhere by ρ_o and (ii) density-weighted averages by Reynolds averages. The factor ρ_o is, once again, included to ensure a direct correspondence between the variables used in the non-Boussinesq and Boussinesq versions, as in Eq. (77). It follows that the exact solution for the vector $\mathbf{B}$ corresponding to equation (68) is now

$$\rho_o^{-1}\mathbf{B} = \frac{1}{|\nabla\overline{C}|^2}\nabla\overline{C} \times \left\{\overline{(\mathbf{u}'C')} + \nabla \times \left[\frac{1}{|\nabla\overline{C}|^2}\nabla\overline{C} \times (\overline{\mathbf{u}}\overline{\phi} + \overline{\mathbf{u}'\phi} + \rho_o^{-1}\nabla \times \mathbf{G})\right]\right\}. \quad (85)$$

In this way, all the formulae developed in sections 2, 3 and 4 can be applied to the output from eddy-resolving Boussinesq ocean models. Obviously, care is nevertheless required in the finite difference application of formulae such as given by (85), an issue that will be model specific and is not addressed here.

6. Summary and conclusions

A framework for developing mesoscale eddy parameterizations based on density-weighted averaging at fixed height has been introduced. The approach is to average the non-Boussinesq equations of motion, and then use the method of McDougall et al. (2001) to show how the averaged non-Boussinesq equations can be approximated for application in Boussinesq ocean models. The formalism has the advantage that it treats the continuity, tracer and momentum equations together as a single entity.

We showed that there are two ways to represent the averaged equations. For application in Boussinesq ocean models, these are Eqs. (19)–(21) and (34)–(36), respectively. The set (19)–(21) includes the advective effect of the eddies in the tracer equation. The GM90 parameterization is commonly used to parameterize the eddy-induced transport velocity $\rho_o^{-1}\nabla \times \mathbf{B}$ in (20). It is also necessary to parameterize the Reynolds stress term, $\rho_o^{-1}\nabla \cdot \overline{(\rho\,\mathbf{u}'_\rho\mathbf{u}'_\rho)}$, in the momentum equation (21). This is commonly done using an eddy viscosity approach, using positive eddy viscosity coefficients. When combined with the GM90 parameterization, the effect is to remove energy from the mean flow. The role eddies can play in driving mean flows (e.g., Holloway 1992; Greatbatch and Li 2000; Greatbatch and Nadiga 2000; Wardle and Marshall 2000) is therefore excluded, and further work is required to find parameterizations to include this effect.

The second approach, corresponding (in the Boussinesq version) to Eqs. (34)–(36), puts the forcing for the eddy-induced transport into the averaged momentum equations and is akin to the approach suggested by Wardle and Marshall (2000). The effect of eddies in either extracting energy from the mean flow, or in driving mean flow, is now contained in a generalized Eliassen–Palm flux divergence term, analogous to that introduced by Gent and McWilliams (1996). The approach requires that two different velocity variables be carried in the averaged momentum equation. These velocities are related by the rotational density flux given by $\nabla \times \mathbf{B}$, which, as we noted above, is commonly parameterized using GM90. There is also the issue of how to parameterize the Eliassen–Palm flux divergence, $\rho_o^{-1}\nabla \cdot \mathbf{E}$ in (36). It is tempting to try and relate $\rho_o^{-1}\nabla \cdot \mathbf{E}$ to the flux of potential vorticity, as suggested by Greatbatch (1998), in the case of isopycnal averaging. However, the Ertel potential vorticity is a fundamentally nonlinear quantity, and it is not easy to see how best to do this in the case of averaging at fixed height. One approach might be to relate this term of the flux of quasigeostrophic potential vorticity, following Wardle and Marshall (2000). As pointed out by these authors, the advantage of using a potential-vorticity-based approach is that the effect of eddies in driving mean flow appears naturally as part of the formalism.

In the second part of the paper, we derived a general solution for the tracer transport velocity, and found a family of solutions that is closely related to the temporal residual mean (TRM) velocity introduced by MM. We started by showing the connection to the approximate TRM solution derived by MM to $O(\alpha^3)$, where α is perturbation amplitude, and then, in section 4, derived an exact solution valid to any order in α. By adopting the flux decomposition suggested by Marshall and Shutts (1981), we extended both MM, and our exact solution, to include diffusion of mean tracer within the TRM framework. The solution, given by either (68) or (70) (the latter including the diffusive contribution to the flux), suggests that the tracer transport velocity could be different for different tracers, depending primarily on the structure of the mean field. This conclusion does not depend on our use of density-weighted averaging at fixed height. In fact, a form of our exact solution [given by (68)] can be derived in two dimensions for the case of isopycnal averaging. It follows that concern that the tracer transport velocity may not be tracer invariant applies quite generally, although we argued at the end of section 4 that the issue of tracer dependence is likely to be more serious in the case of averaging at fixed height, as in this paper, than in the case of isopycnal averaging. The possibility that a different tracer transport velocity may be required for different tracers is suggested by the differing performance of parameterized models on different tracers. For example, the parameterized model of Danabasoglu and McWilliams (1995) is much more successful at simulating the observed potential temperature field than the observed salinity field (although it should be noted that there is also uncertainty in the surface boundary conditions, especially the freshwater flux that is important for salinity). If the tracer transport velocity is indeed tracer dependent, then clearly a particular tracer must be chosen to define $\mathbf{u}^{\#}$ in order to write the governing equations in the form (31)–(33) or (34)–(36), in which eddy effects are primarily transferred to the momentum equation. It would then be necessary to adjust each individual tracer equation to take account of the difference between the tracer transport velocity appropriate to that tracer and $\mathbf{u}^{\#}$.

Finally, in section 5, we showed how, in principle, the ideas and formulae developed in the previous sections can be used to analyze output from eddy-resolving Boussinesq ocean models, even though the basic analysis given in this paper used the non-Boussinesq governing equations.

Acknowledgments. RJG is grateful for funding from NSERC, the Meteorological Service of Canada, MARTEC (a Halifax based company), and the Canadian Institute for Climate Studies. RJG is also grateful to Youyu Lu, Trevor McDougall, K. Andrew Peterson, and Carsten Eden for conversations that have influenced this work.

REFERENCES

Andrews, D. G., J. R. Holton, and C. B. Leovy, 1987: *Middle Atmosphere Dynamics.* Academic Press, 489 pp.

Batchelor, G. K., 1967: *An Introduction to Fluid Dynamics.* Cambridge University Press, Cambridge, 615 pp.

Danabasoglu, G., and J. C. McWilliams, 1995: Sensitivity of the global ocean circulation to parameterizations of mesoscale tracer transports. *J. Climate,* **8,** 2967–2978.

de Szoeke, R. A., and A. F. Bennett, 1993: Microstructure fluxes across density surfaces. *J. Phys. Oceanogr.,* **23,** 2254–2264.

Dukowicz, J. K., and R. D. Smith, 1997: Stochastic theory of compressible turbulent fluid transport. *Phys. Fluids,* **9,** 3523–3529.

——, and R. J. Greatbatch, 1999: The bolus velocity in the stochastic theory of ocean turbulent tracer transport. *J. Phys. Oceanogr.,* **29,** 2232–2239.

Eden, C., and J. Willebrand, 1999: Neutral density revisited. *Deep-Sea Res. II,* **46,** 33–54.

Favre, A., 1965a: Équations des gaz turbulents compressibles I.—Forms générals. *J. Méch.,* **4,** 361–390.

——, 1965b: Équations des gaz turbulents compressibles II.—Méthode des vitesses moyennes; méthode des vitesses macroscopiques pondérées par la masse volumique. *J. Méch.,* **4,** 391–421.

Gent, P. R., and J. C. McWilliams, 1990: Isopycnal mixing in ocean circulation models. *J. Phys. Oceanogr.,* **20,** 150–155.

——, and ——, 1996: Eliassen–Palm fluxes and the momentum equation in non-eddy-resolving ocean circulation models. *J. Phys. Oceanogr.,* **26,** 2539–2546.

——, J. Willebrand, T. J. McDougall, and J. C. McWilliams, 1995: Parameterizing eddy-induced tracer transports in ocean circulation models. *J. Phys. Oceanogr.,* **25,** 463–474.

Gill, A. E., 1982: *Atmosphere–Ocean Dynamics.* Academic Press, 662 pp.

Greatbatch, R. J., 1998: Exploring the relationship between eddy-induced transport velocity, vertical momentum transfer, and the isopycnal flux of potential vorticity. *J. Phys. Oceanogr.,* **28,** 422–432.

——, and G. Li, 2000: Alongslope mean flow and an associated upslope bolus flux of tracer in a parameterization of mesoscale turbulence. *Deep-Sea Res. I,* **47,** 709–735.

——, and B. Nadiga, 2000: Four-gyre circulation in a barotropic model with double-gyre wind forcing. *J. Phys. Oceanogr.,* **30,** 1461–1471.

——, Y. Lu, and Y. Cai, 2001: Relaxing the Boussinesq approximation in ocean circulation models. *J. Atmos. Oceanic Technol.,* in press.

Griffies, S. M., 1998: The Gent and McWilliams skew flux. *J. Phys. Oceanogr.,* **28,** 831–841.

Holloway, G., 1992: Representing topographic stress for large-scale ocean models. *J. Phys. Oceanogr.,* **22,** 1033–1046.

Jackett, D. R., and T. J. McDougall, 1997: A neutral density variable for the world's oceans. *J. Phys. Oceanogr.,* **27,** 237–263.

Killworth, P. D., 1998: On the parameterization of eddy transfer. Part II: Tests with a channel model. *J. Mar. Res.,* **56,** 349–374.

Lee, M.-M., D. P. Marshall, and R. G. Williams, 1997: On the eddy transfer of tracers: Advective or diffusive? *J. Mar. Res.,* **55,** 483–505.

Marshall, D. P., R. G. Williams, and M.-M. Lee, 1999: The relation between eddy-induced transport and isopycnic gradients of potential vorticity. *J. Phys. Oceanogr.,* **29,** 1571–1578.

Marshall, J. C., and G. Shutts, 1981: A note on rotational and divergent eddy fluxes. *J. Phys. Oceanogr.,* **11,** 1677–1680.

McDougall, T. J., 1987: Neutral surfaces. *J. Phys. Oceanogr.,* **17,** 1950–1964.

——, and C. J. R. Garrett, 1992: Scalar conservation equations in a turbulent ocean. *Deep-Sea Res.,* **39,** 1953–1966.

——, and P. C. McIntosh, 1996: The temporal-residual-mean velocity. Part I: Derivation and the scalar conservation equations. *J. Phys. Oceanogr.,* **26,** 2653–2665.

——, and ——, 2001: The temporal-residual-mean velocity. Part II: Isopycnal interpretation and the tracer and momentum equations. *J. Phys. Oceanogr.,* **31,** 1222–1246.

Pacanowski, R. P., and S. M. Griffies, cited 1999: MOM 3.0 manual. NOAA/Geophysical Fluid Dynamics Laboratory, Princeton, NJ. [Available online at http://www.gfdl.gov.]

Peterson, K. A., and R. J. Greatbatch, 2001: Vorticity fluxes in shallow water ocean models. *Atmos.–Ocean,* **39,** 1–14.

Plumb, R. A., and J. D. Mahlman, 1987: The zonally averaged transport characteristics of the GFDL general circulation/transport model. *J. Atmos. Sci.,* **44,** 298–327.

Redi, M. H., 1982: Oceanic isopycnal mixing by coordinate rotation. *J. Phys. Oceanogr.,* **12,** 1154–1158.

Roberts, M. J., and D. P. Marshall, 2000: On the validity of downgradient eddy closures in ocean models. *J. Geophys. Res.,* **105,** 28 613–28 628.

Smith, R. D., 1999: The primitive equations in the stochastic theory of adiabatic stratified turbulence. *J. Phys. Oceanogr.,* **29,** 1865–1880.

Tung, K. K., 1986: Nongeostrophic theory of zonally averaged circulation. Part 1: Formulation. *J. Atmos. Sci.,* **43,** 2600–2618.

Wardle, R., and J. Marshall, 2000: Representation of eddies in primitive equation models by a PV flux. *J. Phys. Oceanogr.,* **30,** 2481–2503.