Drift Velocity of Radiating Quasigeostrophic Vortices

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ABSTRACT

Because of the beta effect, quasigeostrophic monopole vortices propagate westward and excite Rossby waves. This wave radiation depletes the vortex energy, and causes cyclones to drift northward and anticylones southward (in the Northern Hemishpere). In the present work explicit solutions describing such radiating vortices are found by perturbation analysis, assuming the vortex amplitude to be large, and consequently the radiation to be a small perturbation. From these solutions the radiated energy is calculated and then used to obtain a simple expression for the meridional drift. The zonal drift is also modified by the wave radiation, but to calculate this component the complete explicit solution is not necessary; it is enough to consider the ratio of the loss of pseudoenergy to the loss of pseudomomentum.

1. Introduction

Coherent vortices are common in the oceans, with lifetimes up to many months, or even several years (Joyce and Kennelly 1985; McWilliams 1985; Schultz Tokos and Rossby 1991). This is much longer than their rotation time, which is typically 3–6 days. It is therefore tempting to describe them as steadily propagating vortices, and many analytic solutions of this kind have been found. The basic models used include quasigeostrophic and shallow water models with both one layer (Larichev and Reznik 1976; Nycander 1988; Nycander and Sutyrin 1992; Benilov 1996) and two layers (Flierl et al. 1980; Sutyrin and Dewar 1992; Nycander 1994; Pakyari and Nycander 1996).

In general, steady vortices must propagate exactly eastward or westward, and this zonal drift must be outside of the range of possible phase velocities of linear Rossby waves. This ensures that the vortices do not radiate such waves. However, the drift velocity of real oceanic vortices generally has a significant component in the meridional direction. This meridional drift can be a result of Rossby wave radiation. Furthermore, the existence of the Rossby waves that would be created by this process has been established in measurements (Korotaev 1988).

The first work on such radiating vortices is due to Flierl (1984). He used a two-layer model with an outcropping interface to describe a warm core ring. For the upper layer he used the shallow water equations, while the lower layer was quasigeostrophic. To be able to solve the problem by perturbation analysis, he chose an asymptotic region in which the radiation was very weak and the lowest-order translation velocity was determined purely by the upper-layer dynamics. The radiation problem in the lower layer could then be solved with this source velocity regarded as given.

Radiating vortices in the framework of the barotropic vorticity equation (i.e., with a ''rigid lid'') have been studied by Korotaev and Fedotov (1994). A basic difficulty in this case is that the meridional component of the vortex drift is not smaller than the zonal component. Nevertheless, the radiation problem was solved assuming a purely zonal source velocity. It is therefore doubtful whether the expansion procedure is consistent.

McDonald (1998) studied radiating cyclones using the shallow-water equations for one layer. In that case the drift velocity is determined to lowest order by an integral relation for the center-of-mass velocity, involving only the zeroth-order circularly symmetric vortex profile. According to this relation the difference between the drift velocity and $-\beta\hat{x}$ is proportional to the vortex amplitude (Nof 1983; Killworth 1983; Cushman-Roisin et al. 1990; Nycander 1994). (Here β is the meridional derivative of the Coriolis parameter f. In dimensional units $\beta = R_d^2 \partial f/\partial y$, where $R_d = (gH)^{1/2}/f$ is the deformation radius.) McDonald solved the radiation problem with this source velocity regarded as given.

Radiating dipole vortices have been studied by Flierl and Haines (1994). As in the work on monopole vortices mentioned above, the radiation problem was solved by

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perturbation analysis, treating β as a small expansion parameter.

In the present work, the equivalent barotropic vorticity equation is used. The center-of-mass velocity in this case is exactly $-\beta\hat{x}$, that is, equal to the phase velocity of the fastest Rossby waves. However, with this source velocity the wave radiation vanishes. Thus, the radiation depends crucially on the small deviation of the actual drift velocity from $-\beta\hat{x}$, but at the same time this deviation is a result of the wave radiation. This makes the present problem conceptually more difficult than those solved by Flierl (1984) and McDonald (1998), where the radiation problem could be solved with a given velocity.

The problem is solved by perturbation analysis, treating β as a small expansion parameter. This is equivalent to assuming that the swirl speed is much larger than the drift velocity of the vortex (a large amplitude assumption). The drift velocity is then close to $-\beta\hat{x}$, and the wave radiation is a small perturbation.

An explicit solution describing the radiating vortex is found, with an arbitrary radial profile in the inner region with trapped fluid. From this explicit solution the radiated energy is calculated. Using the conservation of potential vorticity, the meridional component of the drift velocity can be calculated from the radiated energy and turns out to be of order $\beta^{5/2}$. Interestingly, the zonal component can be found much more easily, by a simple consideration of the ratio between the pseudomomentum loss and the energy loss, and without using the explicit solution. This component is close to $-\beta$, the difference from this value being of order β^2.

The solution is illustrated with a simple example, using a Bessel function zeroth-order profile in the inner region. This corresponds to a linear relation between the potential vorticity and the streamfunction in the moving reference frame, as in the well-known ''rider'' solution (Flierl et al. 1980). Thus this solution can be seen as a generalization of the rider to the weakly nonsteady, radiating case. A notable difference, however, is that the radiating rider found here has a monotonic radial velocity profile. It may therefore not be affected by the instability found by Swenson (1987) for the conventional rider.

The same problem as in the present paper has been treated by Korotaev (1997). However, his solution is given in a more implicit and complicated form, and appears to be different from the present results.

2. Basic equations

The basic model equation used in the present work is the equivalent barotropic vorticity equation:

$$\frac{\partial}{\partial t}(\nabla^2\phi - \phi) + J(\phi, \nabla^2\phi) + \beta\frac{\partial\phi}{\partial x} = 0, \qquad (1)$$

Where ϕ is the streamfunction and J denotes the Jacobian: $J(f, g) \equiv \partial_x f\partial_y g - \partial_y f\partial_x g$. A locally Cartesian coordinate system is used, with the x axis pointing east and the y axis north. Equation (1) has been nondimensionalized using the deformation radius R_d as spatial unit and the inverse Coriolis parameter f^{-1} as time unit.

Equation (1) describes the Lagrangian conservation of potential vorticity (PV), defined as $q \equiv \nabla^2\phi - \phi + \beta y$:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)q = 0, \qquad (2)$$

where $\mathbf{v} = \hat{z} \times \boldsymbol{\nabla}\phi$.

The dispersion relation for Rossby waves is obtained by linearizing and Fourier transforming Eq. (1):

$$\omega = \frac{-\beta k}{1 + k^2 + l^2}, \qquad (3)$$

where k is the wavenumber in the zonal direction and l that in the meridional direction. This relation implies that the phase velocity of Rossby waves propagating in the zonal direction must lie in the interval $-\beta < \omega/k < 0$.

The general equation for steady solutions of Eq. (1) propagating with the velocity $u_v\hat{x}$ says that q must be constant on the streamlines in the moving reference frame:

$$J(\phi + u_v y, \nabla^2\phi - \phi + \beta y) = 0,$$

which gives

$$\nabla^2\phi + (\beta + u_v)y = F(\phi + u_v y). \qquad (4)$$

On closed streamlines the function F is arbitrary, but in the outer region with open streamlines of a localized solution, F can be determined by the condition $\phi \to 0$, $|\mathbf{r}| \to \infty$, which gives

$$\nabla^2\phi = \left(1 + \frac{\beta}{u_v}\right)\phi. \qquad (5)$$

This relation shows that we must have $1 + \beta/u_v > 0$ for the solution to be localized; that is, $u_v > 0$ or $u_v < -\beta$. Thus, as already pointed out, the drift velocity u_v must be outside the region of possible phase velocities of the Rossby waves. Otherwise, the structure will radiate such waves, the radiation field being a solution of Eq. (5).

The relation for the center-of-mass velocity is obtained by multiplying Eq. (1) by $\mathbf{r}$ and integrating over the xy plane, assuming that ϕ decreases sufficiently rapidly as $|\mathbf{r}| \to \infty$:

$$\frac{d}{dt}\frac{\displaystyle\int \mathbf{r}\phi\, dx\, dy}{\displaystyle\int \phi\, dx\, dy} = -\beta\hat{x}. \qquad (6)$$

According to this relation, any steady and sufficiently localized structure satisfying $\int \phi\, dx\, dy \neq 0$, such as a monopole vortex with monotonic radial PV profile, must travel with the velocity $-\beta\hat{x}$. However, from Eq. (5) we see that a vortex moving with this velocity would be poorly localized, and in general have infinite energy. Such a vortex cannot result from an initial condition consisting of a localized vortex with finite energy.

In practice, simulations show that the drift velocity of a monopole vortex is smaller than the center-of-mass velocity, so that it couples to Rossby waves. The difference can be thought of as caused by ‘‘wave drag,’’ and will be calculated in the next section. The resulting radiation field gives a contribution to the center-of-mass, which therefore does not coincide exactly with the position of the vortex itself. Since its energy is gradually depleted by radiation, such a vortex is not exactly steady.

3. Zonal vortex drift calculated from ratio of radiation losses

Equation (1) possesses two quadratic conserved integrals: the energy,

$$E = \frac{1}{2}\int [\phi^2 + (\nabla\phi)^2]\, dx\, dy = -\frac{1}{2}\int \phi\Omega\, dx\, dy, \quad (7)$$

and the pseudomomentum (often called enstrophy),

$$P = -\frac{1}{2\beta}\int \Omega^2\, dxdy. \quad (8)$$

We have here introduced the relative vorticity Ω:

$$\Omega \equiv \nabla^2\phi - \phi. \quad (9)$$

If a vortex drifts in the meridional direction, the relative vorticity changes in the region of trapped fluid because of the conservation of PV. Therefore, both the energy and the pseudomomentum of the vortex change, and these changes should equal the loss of energy and pseudomomentum due to wave radiation. We will now use this equality to determine the zonal drift velocity of the vortex.

If $\delta\Omega$ denotes the change of relative vorticity in a fluid particle due to a meridional displacement δy, we have from Eq. (2)

$$\delta\Omega = -\beta\delta y. \quad (10)$$

The energy variation is in general given by

$$\delta E = -\int \phi\delta\Omega\, dxdy.$$

For a vortex drifting in the meridional direction, the energy change due to a displacement δy is then

$$\delta E_v = -\int_S \phi\delta\Omega\, dx\, dy = \beta\delta y\int_S \phi\, dxdy, \quad (11)$$

where S denotes the region of trapped fluid. Similarly, the change of the vortex pseudomomentum is

$$\delta P_v = -\frac{1}{\beta}\int_S \Omega\delta\Omega\, dx\, dy = \delta y\int_S \Omega\, dx\, dy$$
$$= \delta y\oint_\gamma \mathbf{v}\cdot d\mathbf{r} - \frac{\delta E_v}{\beta}, \quad (12)$$

where γ is the separatrix (the boundary of S). From Eqs. (11) and (12) we obtain

$$\frac{\delta E_v}{\delta P_v} = \frac{\beta}{C/M - 1}, \quad (13)$$

where C is the circulation at the separatrix,

$$C = \oint_\gamma \mathbf{v}\cdot d\mathbf{r}, \quad (14)$$

and M is the vortex mass,

$$M = \int_S \phi\, dxdy. \quad (15)$$

Note that C and M generally have opposite signs.

We can also determine the ratio between the energy and the pseudomomentum carried by the wave field. According to Eqs. (7) and (8) we have for each Fourier component

$$E_{\mathbf{k}} = \frac{-\beta}{1 + k^2 + l^2}P_{\mathbf{k}}.$$

Using the dispersion relation (3), we obtain the general relation

$$\frac{E_{\mathbf{k}}}{P_{\mathbf{k}}} = \frac{\omega}{k}. \quad (16)$$

If the wave field has been excited by an almost steady vortex propagating with the zonal velocity u_v, the resonance criterion requires $\omega/k = u_v$, assuming that the meridional drift is small. (This assumption will be justified later.) The ratio of the radiated energy to the radiated pseudomomentum therefore directly gives this zonal velocity. Setting this ratio equal to that given by Eq. (13), we finally obtain

$$u_v = \frac{\beta}{C/M - 1}. \quad (17)$$

Thus, the zonal velocity is given by the integrated vortex properties (essentially its amplitude). In the next two sections, this relation will be verified by explicit solutions describing a radiating vortex.

4. Explicit solution and meridional drift

In this section, β will be treated as a small expansion parameter, while assuming the vortex amplitude to be of order unity. (Equivalently, one could assume β to be of order unity and the amplitude to be large.) The nondimensional value of β in real oceanic vortices has been estimated by many authors, and is in general of order 10^{-2} (Killworth 1983; Cushman-Roisin et al. 1990; Sutyrin and Dewar 1992).

If β is small, the radiation is a small effect, and the vortex drift $\mathbf{v}_v = u_v\hat{x} + v_v\hat{y}$ is close to $-\beta\hat{x}$. Therefore, $\kappa^2 \equiv -(1 + \beta/u_v)$ is small and can also be used as an expansion parameter. Thus, the wavelength of the radiated Rossby waves is large, as can be seen from Eq. (5). In dimensional units, this means that it is much larger than the Rossby radius.

The vortex is assumed to be quasi-steady, in the sense that the energy and the structure of the vortex change insignificantly during the characteristic time for the development of the wave field. We also neglect the meridional vortex drift v_v. (However, having obtained the explicit solution, the rate of energy loss can be calculated from the wave field, and this result can then be used to obtain v_v.) These assumptions will be verified afterwards.

We thus start from the general equations for a steady solution moving with the velocity $u_v\hat{x}$, solving Eq. (4) inside a circle with radius r_0, and Eq. (5) outside this circle. We then require ϕ and $\nabla\phi$ to be continuous at $r = r_0$. Furthermore, the solution is constructed so that this circle is a streamline in the moving reference frame.

The separatrix is situated in the outer region where $r > r_0$. Thus, Eq. (5) is used in that part of the region inside the separatrix that is outside the circle $r = r_0$. That is possible since this equation is a special case of Eq. (4).

To zeroth order we neglect β and u_v, but not β/u_v, and get

$$\nabla^2\phi_0 = \begin{cases} F(\phi_0), & r < r_0 \qquad (18) \\ -\kappa^2\phi_0, & r > r_0, \qquad (19) \end{cases}$$

where

$$\kappa^2 \equiv -\left(1 + \frac{\beta}{u_v}\right). \qquad (20)$$

We assume the solution to be circularly symmetric in the inner region to this order, and let the function F be implicitly defined by $\phi_0(r)$ through Eq. (18). The solution of Eq. (19) is taken to be (Flierl 1984)

$$\phi_0 = c_0\left[\frac{Y_0(\kappa r)}{4} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{J_{2n+1}(\kappa r)\cos[(2n+1)\theta]}{2n+1}\right], \quad r > r_0, \qquad (21)$$

where J and Y are Bessel functions of the first and second kind, respectively. The expression (21) is the radiation from a point source and has the asymptotic approximations

$$\phi_0 \approx \begin{cases} \dfrac{c_0}{2\pi}\left(\ln\dfrac{\kappa r}{2} + \gamma\right), & r_0 < r \ll \dfrac{1}{\kappa} \qquad (22) \\ \dfrac{c_0}{\sqrt{2\pi\kappa r}}\sin\left(\kappa r - \dfrac{\pi}{4}\right), & -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}, \quad r \gg \dfrac{1}{\kappa} \qquad (23) \\ 0, & \dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}, \quad r \gg \dfrac{1}{\kappa}, \qquad (24) \end{cases}$$

where $\gamma = 0.577$ is Euler's constant. Equation (24) shows that ϕ_0 satisfies the radiation condition: the radiation field should only exist behind the vortex.

Equation (22) shows that, if $\kappa r_0 \ll 1$, the radiation field (21) can be joined smoothly to the circularly symmetric solution in the inner region. [A similar approximation was made by Flierl and Haines (1994) when matching a radiation field to a dipole vortex.] This is done by choosing $\phi_0(r)$ in the inner region so that the boundary conditions at $r = r_0$ are satisfied. The constant c_0 is then still undetermined. We will see that it is determined by a solvability condition in the first-order problem.

It may seem that the right-hand side of Eq. (19) is small and should therefore instead appear in the first-order equations. However, in this way one would not obtain the radiation field. Keeping the right-hand side in Eq. (19) is a way of smoothly joining the solution near the separatrix to the radiation field far away. In the far region the scaling is different, and the right-hand side cannot be neglected. Near the separatrix, on the other hand, we use the asymptotic expansion (22), which in effect amounts to neglecting the right-hand side of Eq. (19).

The same argument is true for the first-order equations below.

In the first-order equations we include terms proportional to β and u_v:

$$\nabla^2\phi_1 = \begin{cases} (\phi_1 + u_v y)F'(\phi_0) - (\beta + u_v)y, \\ \qquad r < r_0 \qquad (25) \\ -\kappa^2\phi_1, \qquad r > r_0. \qquad (26) \end{cases}$$

We first solve Eq. (25) in the inner region. This was done by Nycander (1988), but the procedure is repeated here for convenience. (In that paper, steady nonradiating vortices were found. They satisfy $\int \phi\, dx\, dy = 0$, and their drift velocity is outside the range of Rossby wave phase velocities. The solution in the inner region is the same as in the present case, but in the outer region it is different.) Set

$$\phi_1 = \xi(r)\sin\theta - u_v y, \quad r < r_0. \tag{27}$$

Inserting this into Eq. (25) we obtain

$$\nabla^2\xi - \frac{\xi}{r^2} = F'(\phi_0)\xi + \kappa^2 u_v r. \tag{28}$$

Differentiating Eq. (18) and substituting F' into Eq. (28) we obtain

$$\nabla^2\xi = \frac{\nabla^2\phi_0'}{\phi_0'}\xi + \kappa^2 u_v r. \tag{29}$$

This equation is then solved with the boundary condition

$$\xi(r_0) = 0, \tag{30}$$

which means that we require the circle $r = r_0$ to be a streamline (i.e., we fix the position of the vortex in the y direction). One of the two homogeneous solutions of Eq. (29) is $\xi = \phi_0'$, and we therefore make the ansatz

$$\xi = \eta(r)\phi_0'.$$

This reduces Eq. (29) to a first-order differential equation in η', which can easily be integrated. Using the regularity condition at $r = 0$ we obtain

$$\eta'(r) = \frac{\kappa^2 u_v}{r(\phi_0')^2}\int_0^r s^2\phi_0'(s)\, ds.$$

Integrating once more and using the boundary condition (30), we obtain the final solution for the inner region:

$$\xi(r) = \kappa^2 u_v \phi_0' \int_{r_0}^{r} \frac{ds}{s[\phi_0'(s)]^2}\int_0^s t^2\phi_0'(t)\, dt. \tag{31}$$

The appropriate solution of Eq. (26) for the outer region is

$$\phi_1 = c_1\left[-\frac{\kappa Y_1(\kappa r)\sin\theta}{4} - \frac{\kappa}{\pi}\sum_{n=1}^{\infty}\frac{2n}{4n^2 - 1}J_{2n}(\kappa r)\sin(2n\theta)\right], \quad r > r_0. \tag{32}$$

Apart from the multiplying constant, this expression can be obtained by differentiating Eq. (21) with respect to y. Again, it satisfies the radiation condition, as can be seen from the asymptotic expansion for $\kappa r \gg 1$. The asymptotic expansion for $\kappa r \ll 1$ is

$$\phi_1 \approx c_1\frac{\sin\theta}{2\pi r}, \qquad r_0 < r \ll \frac{1}{\kappa}. \tag{33}$$

Since κr_0 is small, we can use this expression when matching the outer and inner solutions at $r = r_0$. The continuity of ϕ_1 is guaranteed by the requirement that this circle is a streamline, as in Eq. (30), which gives

$$[\phi_1 + u_v y]_{r=r_0} = 0, \tag{34}$$

Inserting Eq. (33) into this condition gives

$$c_1 = -2\pi u_v r_0^2. \tag{35}$$

We also require $\partial\phi_1/\partial r$ to be continuous. Differentiating Eq. (31) we obtain

$$\xi'(r_0) = \frac{\kappa^2 u_v}{r_0\phi_0'(r_0)}\int_0^{r_0} r^2\phi_0'(r)\, dr.$$

Using Eq. (27), this gives an expression for $\partial\phi_1/\partial r$ at $r = r_0$. Setting this equal to the corresponding expression obtained from the outer solution (33) and (35), we obtain

$$2u_v = \frac{\kappa^2 u_v}{r_0\phi_0'(r_0)}\int_0^{r_0} r^2\phi_0'(r)\, dr. \tag{36}$$

A partial integration gives

$$\frac{2\pi r_0\phi_0'(r_0)}{\kappa^2} = \pi r_0^2\phi_0(r_0) - \int_S \phi_0\, dxdy.$$

Neglecting the first term on the right-hand side, which is of order β, and using Eq. (22), we find

$$c_0 = -\kappa^2\int_S \phi_0\, dxdy. \tag{37}$$

Using the definition (20), and identifying c_0 with the circulation C in Eq. (14), we find that Eq. (37) is the same as Eq. (17).

This solvability condition determines c_0 in terms of κ^2 and, thereby, in terms of u_v. The point of view taken in section 3 is the reverse—that it determines u_v in terms of C. This is analogous to the steady monopole vortex solution of the shallow-water equations (Nycander and Sutyrin 1992), which was obtained by a similar perturbation analysis. In that case the corresponding solvability condition gives the center-of-mass velocity, as appropriate for a nonradiating vortex. For nonradiating solutions of the equivalent barotropic vorticity equation (1), the corresponding solvability condition instead gives the necessary condition $\int \phi\, dx\, dy = 0$ for steady and localized solutions (Nycander 1988).

It remains to determine the meridional drift. We do this by calculating the energy carried away from the vortex by the Rossby waves. To obtain the general expression for the energy flux we multiply Eq. (1) by ϕ and rewrite the resulting equation in the form

$$\frac{\partial}{\partial t}\left[\frac{(\nabla\phi)^2 + \phi^2}{2}\right] + \nabla\cdot\mathbf{\Phi}_E = 0.$$

The energy flux can be identified as

$$\mathbf{\Phi}_E = -\phi\nabla\frac{\partial\phi}{\partial t} - \phi\nabla^2\phi\hat{z}\times\nabla\phi - \hat{x}\beta\frac{\phi^2}{2}.$$

In the radiation zone far from the vortex we can neglect the cubic advection term, and for a steady solution we set $\partial/\partial t = -u_v\partial/\partial x$. The total power radiated across the lines $y = \pm L$ is then

$$P_E = \int_{-\infty}^{\infty} u_v\left[\phi\frac{\partial^2\phi}{\partial x\partial y}\right]_{y=L} dx - \int_{-\infty}^{\infty} u_v\left[\phi\frac{\partial^2\phi}{\partial x\partial y}\right]_{y=-L} dx. \quad (38)$$

We insert the expression (23)–(24) into Eq. (38). [The contribution from ϕ_1 in Eq. (32) vanishes identically due to symmetry. The physical reason for this is that ϕ_1 merely gives a displacement of the field ϕ_0 in the y direction.] Neglecting terms that vanish in the limit $L \to \infty$ we obtain after some calculations

$$P_E = -\frac{C^2\kappa u_v}{2\pi}.$$

From Eq. (37) we have $\kappa = |C/M|^{1/2}$. Approximating $u_v \cong -\beta$, the final expression for the radiated power is

$$P_E = \frac{\beta C^2}{2\pi}\left|\frac{C}{M}\right|^{1/2}. \quad (39)$$

This should equal the rate of energy decrease due to the meridional drift v_v, which is obtained from Eq. (11):

$$P_E = -\beta v_v M. \quad (40)$$

From Eqs. (39) and (40) we finally obtain the meridional drift:

$$v_v = -\frac{C^2}{2\pi M}\left|\frac{C}{M}\right|^{1/2}. \quad (41)$$

One of the referees of the present article objected that the coefficient c_0 in the zeroth-order solution (22)–(24) turns out to be of first order [cf. Eq. (37)]. However, in my opinion this is not inconsistent since c_0 is, in fact, determined by a solvability condition in the first-order problem. In any case, this objection is irrelevant for the particular solution presented in the next section, which is exact apart from the approximation $\kappa r_0 \ll 1$.

5. Radiating rider

In this section we will give an explicit example of the general solution found in the previous section. We choose the function F relating the PV to the streamfunction ϕ in the moving reference frame to be linear, which means that the radial profile $\phi_0(r)$ in the inner region is given by a Bessel function.

Thus, we again solve Eqs. (4) and (5), with F given by

$$F(\psi) = \mu^2(-\psi + K),$$

where μ and K are constants. Equation (4) can then be written

$$\nabla^2\phi + \mu^2\phi = Ay + \mu^2 K, \quad r < r_0, \quad (42)$$

where

$$A = -(\beta + u_v + \mu^2 u_v) = u_v(\kappa^2 - \mu^2). \quad (43)$$

A solution of Eq. (42), which is regular in the origin, is

$$\phi = k_0 J_0(\mu r) + K + k_1 J_1(\mu r)\sin\theta + \frac{Ay}{\mu^2}, \quad r < r_0, \quad (44)$$

where k_0 and k_1 are constants. This expression is an exact solution of Eq. (4), but it is also a special case of the solution obtained by asymptotic expansion in section 4, with ϕ_0 given by the first two terms and ϕ_1 by the last two terms in Eq. (44).

As the solution of Eq. (5) we use the sum of the expressions (21) and (32). Again, this is an exact solution of Eq. (5), besides being a special case of the expansion in section 4. When matching it to the solution (44) in the inner region, we use the asymptotic expressions (22) and (33), valid for $r_0 < r \ll \kappa^{-1}$. This is the only approximation made in the present section.

The constants K, μ, k_1, and c_1 are determined from the matching conditions between the outer and inner solutions. From Eq. (22) and the θ-independent terms in Eq. (44) we obtain

$$K = \frac{c_0}{2\pi}\left(\ln\frac{\kappa r_0}{2} + \gamma\right) - k_0 J_0(\mu r_0). \quad (45)$$

We determine μ from the requirement that the radial derivative of the θ-independent part should be continuous:

$$\mu r_0 J_1(\mu r_0) = -\frac{c_0}{2\pi k_0}. \quad (46)$$

Continuity of the terms proportional to $\sin\theta$ is imposed by using the condition (34). This means that the circle $r = r_0$ is a streamline. (If we only required continuity, we would end up with one more undetermined constant that determined the position of the vortex in the y direction.) From the last two terms in Eq. (44) we obtain

$$k_1 J_1(\mu r_0) + \frac{A r_0}{\mu^2} + u_v r_0 = 0.$$

Using Eq. (43) this can be written

$$k_1 = -\frac{\kappa^2 u_v r_0}{\mu^2 J_1(\mu r_0)}. \quad (47)$$

Similarly, by using Eq. (33) we find that c_1 is again given by Eq. (35).

All solution parameters have now been determined, except c_0 and k_0. Requiring the radial derivative of the terms in Eqs. (33) and (44) that are proportional to $\sin\theta$ to be continuous gives a solvability condition:

$$k_1\mu J_1'(\mu r_0) + \frac{A}{\mu^2} = -\frac{c_1}{2\pi r_0^2}.$$

Substituting k_1, A, and c_1 from Eqs. (47), (43), and (35), and using a suitable recursion relation for the derivative of Bessel functions this can be written

$$\kappa^2 = \frac{2\mu J_1(\mu r_0)}{r_0 J_2(\mu r_0)}. \qquad (48)$$

Since the matching of the outer and inner solutions requires that $\kappa r_0 \ll 1$, Eq. (48) implies that μr_0 must be close to the first root x_1 of J_1, $\mu r_0 \approx x_1 = 3.83$. This means that the swirl velocity at $r = r_0$ is small, or order β, and from Eq. (46) we see that c_0/k_0 must be small.

The solvability condition (48) is a special case of Eq. (36). To see this one uses the following general relation for Bessel functions,

$$r_0^2 J_2(\mu r_0) = \mu \int_0^{r_0} r^2 J_1(\mu r)\, dr, \qquad (49)$$

and sets

$$\phi_0(r) = k_0 J_0(\mu r) + K.$$

We finally rewrite Eq. (48) into an equation that relates c_0 and κ, as in section 4. Equation (49) gives

$$k_0\pi r_0^2 J_2(\mu r_0) = -\pi \int_0^{r_0} \frac{\partial}{\partial r}[k_0 J_0(\mu r) + K] r^2\, dr$$

$$= -\frac{c_0 r_0^2}{2}\left(\ln\frac{\kappa r_0}{2} + \gamma\right) + \int_{r<r_0} \phi_0\, dxdy.$$

Using this expression in Eq. (48), and substituting J_1 from Eq. (46), we obtain

$$\kappa^2 = c_0\left[\frac{c_0 r_0^2}{2}\left(\ln\frac{\kappa r_0}{2} + \gamma\right) - \int_{r<r_0} \phi_0\, dxdy\right]^{-1}. \qquad (50)$$

For $c_0/k_0 \ll 1$ this reduces to

$$\kappa^2 = -\frac{c_0}{\displaystyle\int_{r<r_0} \phi_0\, dxdy} \approx -\frac{C}{M}.$$

Using the definition of κ we again obtain Eq. (17), giving u_v in terms of C and M and completing the solution.

The constant k_0 in the present solution (essentially the vortex amplitude) is undetermined and can be chosen arbitrarily. Formally, the same is true of one of the constants c_0 or u_v [the other one then being given by eq. (17)], but in reality this is not so.

We have already noted that the requirement $\kappa r_0 \ll 1$ means that c_0/k_0 must be small. But this ratio cannot be too small, if the separatrix is to be situated in the region $r > r_0$, as assumed above. To see this, we calculate the position of the stagnation point on the separatrix from the streamfunction $\psi = \phi + u_v y$ in the moving refence frame. Using the inner asymptotic of the outer solution we have

$$\psi = \frac{c_0}{2\pi}\left(\ln\frac{\kappa r}{2} + \gamma\right) + u_v\left(r - \frac{r_0^2}{r}\right)\sin\theta,$$

$$r_0 < r \ll \frac{1}{\kappa}. \qquad (51)$$

From the condition $\partial\psi/\partial\theta = 0$ we find that $\sin\theta = \pm 1$ at the stagnation point, depending on the sign of c_0. Assuming that $c_0 > 0$ (corresponding to the upper sign) we then obtain the distance r_s to the stagnation point from the condition $\partial\psi/\partial r = 0$:

$$r_s = -\frac{c_0}{4\pi u_v} \pm \sqrt{\left(\frac{c_0}{4\pi u_v}\right)^2 - r_0^2}. \qquad (52)$$

Only the upper sign is relevant here. For the solution to be real we must have

$$c_0 > -4\pi u_v r_0 \simeq 4\pi\beta r_0. \qquad (53)$$

If this condition is not satisfied, the separatrix coincides with the circle $r = r_0$, and there are two stagnation points on this circle. (In this case the condition $\partial\psi/\partial r = 0$ instead determines the value of θ at the stagnation points.) This is the structure of a dipole vortex, rather than the monopole studied in the present paper. Thus, the condition (53) guarantees that the separatrix is that of a monopole vortex and that it is situated outside $r = r_0$. Note also that Eq. (53) implies that β/k_0 must be small.

Another requirement is that c_0 must not be so large that the separatrix is in the radiation zone. If $c_0 \gg 4\pi\beta r_0$, Eq. (52) simplifies to

$$r_s \approx \frac{c_0}{2\pi\beta}.$$

This should be smaller than one wavelength, $\kappa r_s < 1$. Inserting κ from Eqs. (48) and (46) and assuming r_0 to be of order unity we obtain the order-of-magnitude estimate

$$\frac{c_0}{k_0} < \left(\frac{\beta}{k_0}\right)^{2/3}.$$

6. Discussion

In section 4 we used perturbation analysis to obtain an explicit solution of Eq. (1) describing a quasigeostrophic radiating vortex. The zeroth-order radial profile ϕ_0 is arbitrary in the inner region $r < r_0$, and given by Eq. (21) in the outer region, where the separatrix is situated. The first-order solution is given by Eqs. (27) and (31) in the inner region and by Eq. (32) in the outer region.

As an illustration, a particular case of this solution, the ‘‘radiating rider,’’ was given in section 5. In this solution the potential vorticity (PV) is a linear function of the streamfunction in the moving reference frame in the inner region, similarly to the ''rider'' solution found by Flierl et al. (1980). However, in contrast to the conventional rider, the present solution has a monotonic radial velocity profile and is therefore likely to be more stable.

The vortex drift is close to $-\beta\hat{x}$, the deviation from this value being caused by the radiation. The meridional component of the drift is neglected in the explicit solution, but can be calculated from the energy radiation and the conservation of PV [cf. Eq. (41)]. The zonal component is obtained as a solvability condition in the explicit solution, but can also be obtained without the explicit solution, from a simple consideration of the ratio between the energy radiation and the pseudomomentum radiation (cf. section 3).

It remains to check that the solution is consistent with the assumptions about small parameters made in the calculations. We first estimate the circulation C at the separatrix, defined in Eq. (14). At the separatrix, the swirl velocity due to the vortex is of the same magnitude as the drift velocity of the vortex. (At the stagnation point, they are, by definition, equal.) This gives the relation $C \sim 2\pi a u_v \sim 2\pi\beta a$, where a is the radius of the separatrix (which is, strictly speaking, not a circle). Since a is assumed to be of order unity, we get $C \sim \beta$. Also assuming that the amplitude ϕ_0 is of order unity, and using Eq. (17), we find that $u_v + \beta \sim \beta^2$; that is, the deviation of the zonal drift from $-\beta$ is much smaller than the drift itself. This guarantees that the wavenumber κ of the Rossby waves is small, $\kappa^2 \sim \beta$, as was assumed in the explicit solution in section 4.

From Eq. (41) we have $\upsilon_v \sim \beta^{5/2}$. (Thus, the meridional drift υ_v is even smaller than the deviation of the zonal drift from $-\beta$.) We use this to estimate the energy loss during the time it takes for the vortex to travel one wavelength λ. This time is $T_\lambda \sim \lambda/u_v \sim (\kappa\beta)^{-1} \sim \beta^{-3/2}$, and the energy lost is $E_\lambda \sim P_E T_\lambda \sim \beta^2$, using Eq. (39). This is much smaller than the vortex energy E, which is of order unity. Hence, only a small fraction of the vortex energy is lost during the time T_λ, which is the characteristic time for the development of the wave field. This justifies the quasi-steady assumption and the neglect of υ_v in the explicit solution.

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