Can Turbulence Suppress Double-Diffusively Driven Interleaving Completely?

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ABSTRACT

A linear stability problem is formulated to investigate the effect of turbulence on double-diffusively driven thermohaline interleaving in rotating media. Three cases are considered: (a) intrusions with an alongfront slope in rotating media, (b) intrusions with zero alongfront slope in nonrotating media, (c) intrusions with zero alongfront slope, where the Coriolis force is retained. The physical reason for case c is that the large-scale vertical geostrophic shear in baroclinic fronts will rotate any intrusion with nonzero alongfront slope as long as the alongfront slope vanishes. In all three cases, turbulence works to suppress interleaving so that the growth rate of the fastest growing intrusion decreases with the increase of turbulent diffusivity k^*. However, in cases a and b the growing intrusions exist for any finite value of k^*, while in case c there is a marginal (maximum) value of k^* beyond which growing intrusions do not exist.

1. Introduction

Two papers investigating the influence of turbulent mixing on double-diffusively driven thermohaline interleaving have been published recently in the *Journal of Physical Oceanography* (Walsh and Ruddick 2000; Kuzmina and Zhurbas 2000). The results of these papers seem to contradict each other somewhat. Walsh and Ruddick (2000) in their linear stability analysis showed that growing solutions exist for any finite value of the turbulent diffusivity, suggesting that double-diffusively driven intrusions can exist in the ocean even when double-diffusive fluxes are much weaker than turbulent fluxes. Kuzmina and Zhurbas (2000) in their linear stability analysis of interleaving at baroclinic fronts showed that due to turbulence, growing solutions do not exist for some range of input parameters and found correspondent marginal stability criteria. This paper is just to discuss this contradiction in more detail.

2. Governing equations and growth rate polynomial

Following Stern (1967), Toole and Georgi (1981), and Kuzmina and Rodionov (1992), we write linearized equations of motion for the intrusion-scale field in a form

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$$U'_t - fV' = -\frac{1}{\rho_0}P'_x + \mathrm{Pr}kU'_{zz} + \mathrm{Pr}k^*U'_{zz}, \quad (1)$$

$$V'_t + fU' = -\frac{1}{\rho_0}P'_y + \mathrm{Pr}kV'_{zz} + \mathrm{Pr}k^*V'_{zz}, \quad (2)$$

$$P'_z = -g\rho', \quad (3)$$

$$U'_x + V'_y + W'_z = 0, \quad (4)$$

$$S'_t + W'\overline{S}_z + U'\overline{S}_x = kS'_{zz} + k^*S'_{zz}, \quad (5)$$

$$\rho'_t + W'\overline{\rho}_z = (1-\gamma)k\rho_0\beta S'_{zz} + k^*\rho'_{zz}, \quad (6)$$

where $\rho' = \rho_0(-\alpha T' + \beta S')$, T', S', and P' are density, temperature, salinity, and pressure fluctuations, respectively; U', V', and W' are the x, y, and z velocity components; $\overline{S}_x$ is the mean cross-front gradient of salinity; $\overline{T}_z$, $\overline{S}_z$, and $\overline{\rho}_z = \rho_0(-\alpha\overline{T}_z + \beta\overline{S}_z)$ are the mean vertical gradients of temperature, salinity, and density, respectively; f is the Coriolis parameter; g is the gravitational acceleration; ρ_0 is the reference density; α and β are the coefficients of thermal expansion and haline contraction; $\gamma = \alpha F_T/\beta F_S$ is the nondimensional flux ratio for salt fingering; k and k^* are salt finger and turbulent diffusivities, respectively; and Pr is the Prandtl number. In deriving (6) it is assumed that $\overline{\rho}_x = \rho_0(-\alpha\overline{T}_x + \beta\overline{S}_x) = 0$, where $\overline{\rho}_x$ and $\overline{T}_x$ are the cross-front mean gradients of density and temperature. Equations (1)–(6) with $\mathrm{Pr} = 0$, $k^* = 0$ were first analyzed by Stern (1967), then Toole and Georgi (1981) took into account friction ($\mathrm{Pr} > 0$, $k^* = 0$), and finally Kuzmina and Rodionov

(1992) included into consideration the effect of turbulent mixing (Pr > 0, $k^* > 0$).

Proceeding as is usually done in the linear stability approach, we seek harmonic solutions of the form $\exp[\hat{\lambda}t + i(\hat{r}_x x + \hat{r}_y y + \hat{m}z)]$ to Eqs. (1)–(6), where $\hat{\lambda}$ is the growth rate and ($\hat{r}_x$, $\hat{r}_y$, $\hat{m}$) are (x, y, z) wavenumbers. A solution of this form exists only if the determinant of the coefficient matrix, obtained after substitution, vanishes. After nondimensionalization, the following quartic equation in λ results:

$$\lambda^4 + C_3\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0 = 0, \qquad (7)$$

where

$$C_3 = m^2[1 + 2\,\mathrm{Pr} + 2\zeta(1 + \mathrm{Pr})], \qquad (8)$$

$$C_2 = \omega^2 + m^4[\mathrm{Pr}(2 + \mathrm{Pr})(1 + \zeta)^2 + \zeta(1 + \zeta)(1 + 2\,\mathrm{Pr})] + (r_x^2 + r_y^2), \qquad (9)$$

$$C_1 = \omega^2 m^2(1 + 2\zeta) + m^6(1 + \zeta)^3\,\mathrm{Pr}^2 + m^6\zeta(1 + \zeta)^2\,\mathrm{Pr}(2 + \mathrm{Pr}) + m^2[1 + \mathrm{Pr} + \varepsilon_z + \zeta(1 + \mathrm{Pr})](r_x^2 + r_y^2) - m^2\varepsilon_x r_x, \qquad (10)$$

$$C_0 = m^8\zeta(1 + \zeta)^3\,\mathrm{Pr}^2 + m^4\zeta(1 + \zeta)\omega^2 + m^4(1 + \zeta)(1 + \varepsilon_z + \zeta)\,\mathrm{Pr}(r_x^2 + r_y^2) - m^2\varepsilon_x[m^2(1 + \zeta)\,\mathrm{Pr}r_x + \omega r_y]. \qquad (11)$$

Nondimensional growth rate λ, wavenumbers (r_x, r_y, m), and parameters (ε_z, ε_x, ω, ζ) are defined as

$$\lambda = \frac{\hat{\lambda}}{N}, \quad r_x = \frac{\hat{r}_x}{\hat{m}}, \quad r_y = \frac{\hat{r}_y}{\hat{m}}, \quad m^2 = \frac{k\hat{m}^2}{N},$$

$$\varepsilon_z = \frac{(1 - \gamma)g\beta\overline{S}_z}{N^2}, \quad \varepsilon_x = \frac{\overline{S}_x}{\overline{S}_z}\varepsilon_z, \quad \omega = \frac{f}{N},$$

$$\zeta = \frac{k^*}{k},$$

where $N^2 = -g\overline{\rho}_z/\rho_0$ is the squared Brunt–Väisälä frequency. Note that ε_z can be rewritten as $\varepsilon_z = (1 - \gamma)/(R_\rho - 1)$, where $R_\rho = \alpha\overline{T}_z/\beta\overline{S}_z$ is the density ratio.

It can be easily shown that in the case of no turbulence ($\zeta = 0$) the growth rate polynomial (7)–(11) reduces to Eq. (25) of Toole and Georgi (1981) rewritten for the hydrostatic approximation.

3. Instability models

Since we look for positive roots of (7)–(11), let us first consider the criterion for them to exist. The issue is a bit complicated because it is possible that not only the zero-power coefficient C_0 but also the first-power coefficient C_1 can be negative. If $C_0 < 0$, we are assured of one positive real root (e.g., Stern 1967), and a growing, nonoscillating intrusion exists. In other words, $C_0 < 0$ is a sufficient condition for monotonic instability. Moreover, applying the Descartes's rule of signs (e.g., Korn and Korn 1968), we conclude that polynomial (7)–(11) has one and only one positive root if $C_0 < 0$ and $C_1 > 0$, and no one positive root if $C_0 > 0$ and $C_1 > 0$. Therefore, $C_0 < 0$ becomes the criterion for instability (i.e., the necessary and sufficient condition), provided that $C_1 > 0$ when $C_0 > 0$. In section 3c this criterion will be considered in detail.

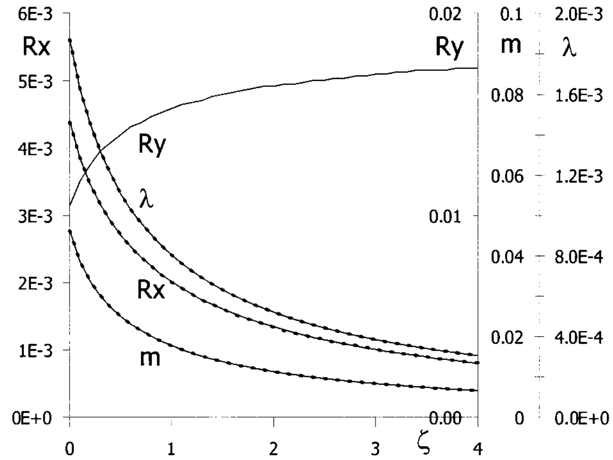

FIG. 1. The growth rate λ, cross-front slope r_x, alongfront slope r_y, and vertical wavenumber m for the fastest-growing intrusion vs the ratio of turbulent diffisivity to salt finger diffusivity ζ (solid). Values of input parameters are Pr = 5, $\varepsilon_x = 0.02$, $\varepsilon_z = 0.4$, $\omega = 0.03$. Dotted curves are ζ dependencies of λ, r_x, m for the 2D case with no rotation ($r_y = 0$, $\omega = 0$).

To examine if the instability can be fully suppressed by turbulence, let us analyze (11) for three different cases.

a. 3D interleaving: Intrusions with alongfront slope in rotating media

If intrusions in rotating media ($\omega \neq 0$) are allowed to have alongfront slope ($r_y \neq 0$), growing solutions (i.e., $\lambda > 0$) will exist for any (large) value of turbulent diffusivity. To prove it, we are to show that there is a locus of points (r_x, r_y, m^2) where $C_0 < 0$ for any $\zeta > 0$. We take any positive values for ζ, Pr, ω, ε_z, ε_x, r_x, and r_y, and decrease the vertical wavenumber m^2 in (11) until C_0 is negative. Doing that we are assured that C_0 will inevitably become negative because the term $-m^2\varepsilon_x\omega r_y$ is the major term of (11) when $m^2 \to 0$.

Numerically calculated ζ dependencies of r_x, r_y, m, and λ for the fastest-growing intrusion at Pr = 5, $\varepsilon_z = 0.4$, $\varepsilon_x = 0.02$, and $\omega = 0.03$ are shown in Fig. 1 (solid curves). Values of Pr, ε_z, and ε_x were chosen the same as those of Walsh and Ruddick (2000) for comparison. The growth rate λ, vertical wavenumber m, and cross-front slope r_x for the fastest-growing intrusion are decreasing monotonically with ζ to vanish at $\zeta \to \infty$. The alongfront slope for the fastest-growing intrusion r_y is increasing monotonically with ζ approaching asymptotically a finite value r_y^∞ at $\zeta \to \infty$. If ε_z is not too large, r_y^∞ can be approximated as $r_y^\infty \approx (5/3)\omega(1 + \mathrm{Pr})^{-1/2}$.

b. 2D interleaving: Intrusions with zero alongfront slope in nonrotating media

Let us consider a partial case of (7)–(11) when $r_y = 0$ (intrusions with zero alongfront slope only) and $\omega = 0$ (Earth's rotation unimportant). Like the previous case, in this case the growing solutions exist for any finite value of ζ. To prove it, we proceed as follows. First, we substitute $r_y = 0$ and $\omega = 0$ into (11), and consider a partial case $r_x = m^2$. Then, we decrease $m^2 \equiv r_x$ in (11) until C_0 is negative. We are guaranteed to satisfy condition $C_0 < 0$, because the term $-m^4\varepsilon_x(1 + \zeta)\mathrm{Pr}r_x$ is the major term of (11) with $r_y = 0$ and $\omega = 0$ in the limit $m^2 \equiv r_x \to 0$. This very result was obtained by Walsh and Ruddick (2000), who considered a similar problem of 2D double-diffusively driven thermohaline interleaving at $r_y = 0$, $\omega = 0$ in the presence of turbulence, provided that the flux ratio γ is a function of the density ratio R_ρ (e.g., Schmitt 1979).

The ζ dependencies of r_x, m, and λ for the fastest-growing intrusion at the same values of parameters Pr, ε_x, and ε_z but for the 2D case are shown in Fig. 1 by dots. One cannot miss the full coincidence of respective solid and dotted (r_x, m, λ) curves. This coincidence was first pointed by McDougall (1985). He showed that the fastest-growing intrusions (in nonturbulent case) move directly across the front with zero velocity component in the alongfront direction so that the only effect of rotation is to introduce an alongfront tilt to the intrusions: the wavenumbers, velocity components, and the growth rate are independent of the rotation rate. Therefore, McDougall's finding is valid for double-diffusively driven intrusions even in the presence of turbulence. In general, the ζ dependencies of r_x, m, and λ in Fig. 1 are similar to that obtained by Walsh and Ruddick (2000). Some quantitative differences between Walsh and Ruddick's results and our results presented in Fig. 1 are likely due to the fact that we do not consider the R_ρ dependence of γ [in contrast to Walsh and Ruddick (2000)].

c. 2D interleaving: Intrusions with zero alongfront slope where the Coriolis force is retained

If an intrusion with nonzero alongfront slope was considered in the baroclinic front, the large-scale vertical geostrophic shear would stretch/rotate the intrusion kinematically until the alongfront slope vanished. For this reason, Kuzmina and Rodionov (1992), May and Kelley (1997), and Kuzmina and Zhurbas (2000) in their analysis of interleaving at baroclinic fronts considered intrusions with zero alongfront slope ($r_y = 0$). However, treating intrusions with zero alongfront slope at baroclinic fronts we do not see any reason to suggest the Coriolis force is unimportant. The above does explain the physical meaning for considering the case $r_y = 0$, $\omega \neq 0$. Substituting $r_y = 0$ into (11), we obtain

$$C_0 = m^4(1 + \zeta)[m^4\zeta(1 + \zeta)^2\,\mathrm{Pr}^2 + \zeta\omega^2 + (1 + \varepsilon_z + \zeta)\,\mathrm{Pr}r_x^2 - \varepsilon_x\,\mathrm{Pr}r_x]. \quad (11')$$

Analyzing (11′), it is worthwhile to address the term $\zeta\omega^2$. It is the term that makes all the difference. Being positive at $\omega, \zeta \neq 0$ this term vanishes if turbulence and/or rotation vanish. Therefore, this term describes a specific stabilization of the process by coupled effect of turbulence and rotation. As a result, growing intrusions do not exist if the turbulent diffusivity and/or the rotation rate are large enough. To prove it, we rewrite (11′) in a form

$$C_0 = m^4(1 + \zeta)\left\{m^4\zeta(1 + \zeta)^2\,\mathrm{Pr} + (1 + \varepsilon_z + \zeta) \times \mathrm{Pr}\left[r_x - \frac{\varepsilon_x}{2(1 + \varepsilon_z + \zeta)}\right]^2 - \frac{\varepsilon_x^2\,\mathrm{Pr}}{4(1 + \varepsilon_z + \zeta)} + \zeta\omega^2\right\}. \quad (11'')$$

It is easily seen from (11″) that the sufficient condition for growing intrusions to exist is

$$\zeta\omega^2 < \frac{\varepsilon_x^2\,\mathrm{Pr}}{4(1 + \varepsilon_z + \zeta)}. \quad (12)$$

Indeed, if we take for the wavenumbers $r_x = 0.5\varepsilon_x/(1 + \varepsilon_z + \zeta)$ and $m^2 \to 0$, the right side of (11″) will be negative in view of (12). Note that if $\zeta \ll 1$, Eq. (12) reduces to Eq. (23.2) of Kuzmina and Zhurbas (2000).

To examine when the sufficient condition (12) is the criterion for instability, let us present (10) with $r_y = 0$ in a form

$$\begin{aligned} C_1 &= m^2[m^4\,\mathrm{Pr}(1 + \zeta)^2(2\zeta + 2\zeta\,\mathrm{Pr} + \mathrm{Pr}) \\ &\quad + (1 + 2\zeta)\omega^2 \\ &\quad + (1 + \varepsilon_z + \zeta + \mathrm{Pr} + \zeta\,\mathrm{Pr})r_x^2 - \varepsilon_x r_x] \\ &= \frac{m^2}{\mathrm{Pr}}[Am^4 + Br_x^2 + C + \zeta\omega^2(\mathrm{Pr} - 1) + \zeta\omega^2 \\ &\quad + (1 + \varepsilon_z + \zeta)\,\mathrm{Pr}r_x^2 - \varepsilon_x\,\mathrm{Pr}r_x], \end{aligned} \quad (10')$$

where A, B, and C are some positive functions of parameters ζ, ω, and Pr. Equation (10′) shows that $C_1 > 0$ if $\zeta\omega^2 > \varepsilon_x^2\mathrm{Pr}/[4(1 + \varepsilon_z + \zeta)]$ (i.e., $C_0 > 0$) and $\mathrm{Pr} \geq 1$. Therefore, Eq. (12) becomes the criterion for instability, provided that $\mathrm{Pr} \geq 1$.

The ζ dependencies of r_x, m, and λ for the fastest-growing intrusion at $\mathrm{Pr} = 5$, $\varepsilon_x = 0.02$, $\varepsilon_z = 0.4$, and $\omega = 0.03$ for the case of no alongfront slope intrusions ($r_y = 0$) are shown in Fig. 2. In accordance with (12), the growth rate λ for the fastest-growing intrusion vanishes when the ratio of turbulent diffusivity to double-diffusive diffusivity ζ approaches from below a marginal value ζ_{mar}.

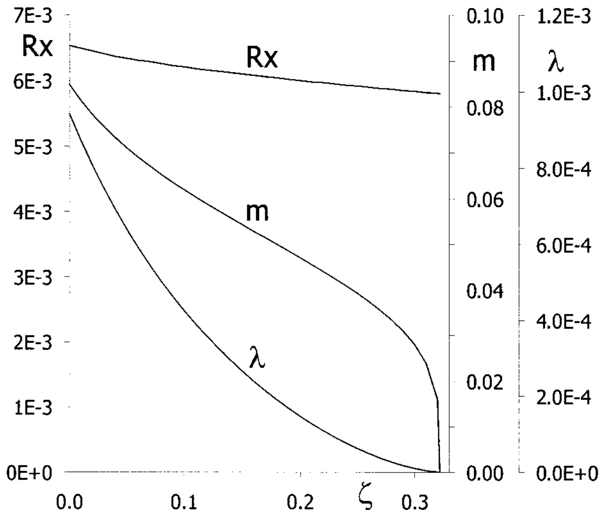

FIG. 2. The same as in Fig. 1 but for the case $r_y = 0$, $\omega \neq 0$.

$$\zeta_{\text{mar}} = 0.5[-(1 + \varepsilon_z) + \sqrt{(1 + \varepsilon_z)^2 + \varepsilon_x^2 \text{Pr}\omega^{-2}}]. \quad (13)$$

Note that $\zeta_{\text{mar}} \to \infty$ when $\omega \to 0$. This is just in accordance with results of Walsh and Ruddick (2000), who found that in nonrotating media the double-diffusively driven intrusions could grow at any (large) value of turbulent diffusivity.

In view of (11″), respective marginal values of the cross-front slope and vertical wavenumber are

$$(r_x)_{\text{mar}} = \frac{\varepsilon_x}{2(1 + \varepsilon_z + \zeta_{\text{mar}})}, \qquad m_{\text{mar}} = 0. \quad (14)$$

Substituting Pr = 5, $\varepsilon_x = 0.02$, $\varepsilon_z = 0.4$, and $\omega = 0.03$ into (13) and (14), we obtain $\zeta_{\text{mar}} = 0.322$, $(r_x)_{\text{mar}} = 5.81 \times 10^{-3}$ in accordance with Fig. 2. The obtained value of ζ_{mar} seems surprisingly small because intrusions are observed in the ocean to grow in many areas where turbulent diffusivity is likely comparable with or larger than the double-diffusive diffusivity. We can suggest at least three possibilities to explain this discrepancy between observations and the theory.

First, the values of parameters Pr and ε_x that we chose may be underestimated. For example, if we choose for Pr and ε_x greater but still reasonable values of Pr = 10 and $\varepsilon_x = 0.05$, we will get $\zeta_{\text{mar}} = 2.03$.

Second, David Walsh (2000, personal communication) suggests that the discrepancy may be resolved if one takes into account the R_ρ dependence of double-diffusive diffusivity. Walsh and Ruddick (1995) found that the main effect of an R_ρ dependent diffusivity on interleaving could be reproduced by replacing the finger diffusivity k by an ‘‘effective’’ diffusivity $k_{\text{eff}} = k + (\gamma - R_\rho)dk/dR_\rho$. The effective diffusivity k_{eff} is larger than k if dk/dR_ρ is negative, and k_{eff} may be substantially larger than k if Schmitt's (1981) formulation for $k(R_\rho)$ is correct. Thus it seems possible that k^*/k_{eff}, the physically valid analog of ζ, may be quite small even when $\zeta = k^*/k$ is of $O(1)$.

Third, growing intrusions at $\zeta > \zeta_{\text{mar}}$ may be related to a specific type of the baroclinic instability developed in baroclinic fronts in the presence of double-diffusive convection (Kuzmina and Rodionov 1992; May and Kelley 1997; Kuzmina and Zhurbas 2000).

4. Conclusions

Coming back to the question in the title, the answer depends on the approach we used. If intrusions in rotating media are allowed to have nonzero alongfront slope or if intrusions with zero alongfront slope are considered in nonrotating media, growing intrusions exist for any finite value of the turbulent diffusivity—the result obtained by Walsh and Ruddick (2000). If intrusions with zero alongfront slope are considered while the Coriolis force is retained [an approach applied to interleaving at baroclinic fronts by Kuzmina and Rodionov (1992), May and Kelley (1997), and Kuzmina and Zhurbas (2000)], there is a marginal (maximum) value of the turbulent diffusivity beyond which growing intrusions do not exist. However, more study is needed to decide which approach is more relevant to ocean intrusions.

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