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Interpolation Spaces between L^1 and BMO on Spaces of Homogeneous Type

LI Gang[1], *LI Wen-ming*[2]

(1. Teaching Department, Hebei Radio and TV University, Hebei 050071, China;
2. Department of Mathematics, Hebei Normal University, Hebei 050016, China)
(E-mail: lwmingg@sina.com)

Abstract: We study the interpolation spaces between L^1 and BMO on spaces of homogeneous type. For $0<\theta<1$, $1\leq q\leq\infty$, we obtain $(L^1,\mathrm{BMO})_{\theta,q}=L_{pq}$, where $\theta=1-\frac{1}{p}$.

Key words: spaces of homogeneous type; BMO; interpolation space.
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1. Introduction and result

The interpolation spaces between function spaces such as L^p spaces, H^p spaces, BMO and other function spaces on R^n have been well developed[1,2]. In [3], we study the interpolation spaces between Hardy spaces H^1 and L^∞ on spaces of homogeneous type using the maximal function characterization obtained in [4]. The purpose of this paper is to study the interpolation spaces between L^1 and BMO on spaces of homogeneous type.

Let (X,ρ,μ) be a space of homogeneous type. In this paper the basic concepts and notations are all same as in [3].

Theorem 1.1[3] *For $0<\theta<1$, $1\leq q\leq\infty$, we have $(H^1,L^\infty)_{\theta,q}=L_{pq}$, where $\theta=1-\frac{1}{p}$.*

Our main result in this paper is the following theorem.

Theorem 1.2 *For $0<\theta<1$, $1\leq q\leq\infty$, we have $(L^1,\mathrm{BMO})_{\theta,q}=L_{pq}$, where $\theta=1-\frac{1}{p}$.*

Since the space BMO is modulo constants, and so are the interpolation spaces $(L^1,\mathrm{BMO})_{\theta,q}$. Therefore, more precisely, we have that for any $F\in(L^1,\mathrm{BMO})_{\theta,q}$, there exists a unique $f\in L_{pq}$ such that

$$C_1\|f\|_{pq}\leq\|F\|_{(L^1,\mathrm{BMO})_{\theta,q}}\leq C_2\|f\|_{pq},$$

where C_1,C_2 are not dependent on f and F.

2. The characterization of $K(t,f;L^1,\mathrm{BMO})$

Lemma 2.1[8] (covering lemma) *Let Ω be an open set of finite measure strictly contained in X and $d(x)=\inf\{\rho(x,y):y\notin\Omega\}$. Given $C\geq1$, let $r(x)=(2AC)^{-1}d(x)$. Then there exists*

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a natural number M that depends on C, and a sequence $\{x_n\}$ such that, denoting $r(x_n)$ by r_n, we have

(i) *The balls $B(x_n,(4A)^{-1}r_n)$ are pairwise disjoint;*

(ii) *$\cup_n B(x_n,r_n)=\Omega$;*

(iii) *For every n, $B(x_n,Cr_n)\subset\Omega$;*

(iv) *For every n, $x\in B(x_n,Cr_n)$ implies that $Cr_n\le d(x)\le 3A^2Cr_n$;*

(v) *For every n, there exists $y_n\notin\Omega$ such that $\rho(x_n,y_n)<3ACr_n$;*

(vi) *For every n, the number of balls $B(x_k,Cr_k)$ whose intersections with $B(x_n,Cr_n)$ are non-empty is at most M.*

Lemma 2.2[4] (partition of the unity) *Let Ω be an open set of finite measure strictly contained in X. Consider the sequence $\{x_n\}$ and $\{r_n\}$ given by Lemma 2.1 for $C=5A$. Then, there exists a sequence $\{\varphi_n(x)\}$ of non-negative functions satisfying*

(i) $\mathrm{supp}\varphi_n\subset B(x_n,2r_n)$;

(ii) *$\varphi_n(x)\ge 1/M$, for $x\in B(x_n,r_n)$;*

(iii) *There exists C such that for every n, $\varphi_n\in\mathcal{M}(x_n,r_n,\beta,\gamma)$ and $\|\varphi_n\|_{\mathcal{M}(x_n,r_n,\beta,\gamma)}\le Cr_n$;*

(iv) *$\sum_n\varphi_n(x)=\chi_\Omega(x)$.*

Theorem 2.3 *There exist constants C_1 and C_2, such that for all $f\in L^1+\mathrm{BMO}$ and for any $t>0$, we have*

$$C_1t(f^\#)^*(t)\le K(t,f;L^1,\mathrm{BMO})\le C_2t(f^\#)^*(t),$$

where $f^\#(x)=\sup_{x\in B}\frac{1}{\mu(B)}\int_B|f(z)-f_B|\mathrm{d}\mu(z)$ is the sharp function and $f_B=\frac{1}{\mu(B)}\int_B f(z)\mathrm{d}\mu(z)$.

Proof Let $f=b+g$, $b\in L^1$, $g\in\mathrm{BMO}$. Then $f^\#\le b^\#+g^\#\le b^\#+\|g\|_{\mathrm{BMO}}$. Therefore,

$$t(f^\#)^*(t)\le t(b^\#)(t)+t\|g\|_{\mathrm{BMO}}\le 2tM(b)^*(t)+t\|g\|_{\mathrm{BMO}}\le C(\|b\|_1+t\|g\|_{\mathrm{BMO}}).$$

Taking the supremum for all $f=b+g$, we get the first inequality .

We now prove the second inequality. Fix $f\in L^1+\mathrm{BMO}$ and $t>0$, and write $\Omega=\{x\in X: f^\#(x)>(f^\#)^*(t)\}$ and $F=\Omega^c$. This set is open and $\mu(\Omega)\le t$. Let $\{\varphi_n(x)\}$ be the partition of unity given by Lemma 2.2, which is associated Ω. Then for every n, let

$$m_n(f)=[\int\varphi_n(z)\mathrm{d}\mu(z)]^{-1}\int f(z)\varphi_n(z)\mathrm{d}\mu(z),$$
$$b(z)=\sum_n b_n(z)=\sum_n[f(z)-m_n(f)]\varphi_n(z),$$
$$g(z)=\sum_n m_n(f)\varphi_n(z)+f(z)\chi_F(z).$$

For $b(x)$, we have

$$\|b\|_1=\sum_n\int|f(z)-[\int\varphi_n(z)\mathrm{d}\mu(z)]^{-1}\int f(x)\varphi_n(x)\mathrm{d}\mu(x)|\varphi_n(z)\mathrm{d}\mu(z)$$
$$\le C\sum_n\int_{B(x_n,2r_n)}|f(z)-f_{B(x_n,2r_n)}|\mathrm{d}\mu(z)$$

$$\leq C\sum_n f^{\#}(t)\mu B(x_n,2r_n) \leq C\mu(\Omega)f^{\#}(t) \leq Ctf^{\#}(t).$$

In the following, we prove that $g \in \mathrm{BMO}$ and $\|g\|_{\mathrm{BMO}} \leq C(f^{\#})^*(t)$. We have only to prove that for any $B(x_0,r)$, there exists constant a such that

$$A(B) = \frac{1}{\mu(B)}\int_B |g(z)-a|\mathrm{d}\mu(z) \leq C(f^{\#})^*(t). \tag{1}$$

For fixed k_0, define $J_0 = \{n : B(x_n,2Ar_n)\cap B(x_{k_0},2Ar_{k_0}) \neq \emptyset\}$. By Lemma 2.1 (vi), J_0 has elements at most M. By Lemma 2.1 (iv), for any $n \in J_0$, we have

$$(3A^2)^{-1}r_{k_0} \leq r_n \leq (3A^2)r_{k_0},$$

and

$$\cup_{n\in J_0}B(x_n,2Ar_n) \subset B(x_{k_0},8A^4r_{k_0}). \tag{2}$$

In fact, for $x \in B(x_n,2Ar_n)$, take $y \in B(x_n,2Ar_n)\cap B(x_{k_0},2Ar_{k_0})$, then

$$\rho(x,x_{k_0}) \leq A[\rho(x,y)+\rho(y,x_{k_0})] \leq A[2Ar_n+2Ar_{k_0}] \leq 8A^4r_{k_0}.$$

So (2) is true. Now we prove (1).

Let $K = \{k : B(x_k,2r_k)\cap B \neq \emptyset\}$. We have

$$A(B) = \frac{1}{\mu(B)}\int_{B\cap\Omega}|g(z)-a|\mathrm{d}\mu(z) + \frac{1}{\mu(B)}\int_{B\cap F}|g(z)-a|\mathrm{d}\mu(z).$$

If $K = \emptyset$, then $B \subset F$. Taking $a = \frac{1}{\mu(B)}\int_B f(z)\mathrm{d}\mu(z)$, we have

$$A(B) = \frac{1}{\mu(B)}\int_B |f(z)-a|\mathrm{d}\mu(z) \leq (f^{\#})^*(t).$$

If $K \neq \emptyset$ and there exists $k_0 \in K$ such that $r \leq \frac{1}{3A^2}r_{k_0}$. Taking $y \in B\cap B(x_{k_0},2r_{k_0})$, for any $x \in B$, we have

$$\rho(x,x_{k_0}) \leq A[\rho(x,y)+\rho(y,x_{k_0})] \leq A[2Ar+r_{k_0}] \leq 2Ar_{k_0}.$$

By Lemma 2.1 (iii), we know that $B \subset \Omega$. Let $a = \frac{1}{\mu(B(x_{k_0},8A^4r_{k_0}))}\int_{B(x_{k_0},8A^4r_{k_0})} f(z)\mathrm{d}\mu(z)$. Then we have

$$\begin{aligned}
A(B) &= \frac{1}{\mu(B)}\int_{B\cap\Omega}|g(z)-a|\mathrm{d}\mu(z)\\
&\leq \frac{1}{\mu(B)}\int_{B\cap\Omega}|\sum_n m_n(f)\varphi_n(z) - \sum_n \varphi_n(z)a|\mathrm{d}\mu(z)\\
&\leq \sum_{k\in K}\frac{1}{\mu(B)}\int_{B\cap B(x_k,2r_k)}|m_n(f)-a|\varphi_n(z)\mathrm{d}\mu(z)\\
&\leq \sum_{k\in K}\frac{\mu(B\cap B(x_k,2r_k))}{\mu(B)}|[\int\varphi_k(z)\mathrm{d}\mu(z)]^{-1}\int f(z)\varphi_k(z)\mathrm{d}\mu(z)-a|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k\in K} \frac{\mu(B\cap B(x_k,2r_k))}{\mu(B)} \frac{C}{\mu(B(x_k,2r_k))} \int_{B(x_k,2r_k)} |f(z)-a|\mathrm{d}\mu(z) \\
&\leq C\frac{1}{\mu(B(x_{k_0},2Ar_{k_0})}\sum_{k\in J_0}\int_{B(x_k,2r_k)}|f(z)-a|\mathrm{d}\mu(z) \\
&\leq C\frac{1}{\mu(B(x_{k_0},8A^4r_{k_0})}\int_{B(x_{k_0},8A^4r_{k_0})}|f(z)-a|\mathrm{d}\mu(z).
\end{aligned}
$$

By Lemma 2.1 (v), we have $B(x_{k_0},8A^4r_{k_0})\cap F\neq\emptyset$, thus $A(B)\leq C(f^{\#})^*(t)$.

If $K\neq\emptyset$ and for all $k\in K$, $r\geq\frac{1}{3A^2}r_k$. It is easy to get $\cup_{k\in K}B(x_k,2r_k)\subset B(x_0,7A^3r)$. Let $a=\frac{1}{\mu(B(x_0,7A^3r)}\int_{B(x_0,7A^3r)}f(z)\mathrm{d}\mu(z)$. Then we have

$$
\begin{aligned}
A(B)&\leq\frac{C}{\mu(B)}\sum_{k\in K}\int_{B(x_k,2r_k)}|f(z)-a|\varphi_n(z)\mathrm{d}\mu(z)+\int_{B\cap F}|f(z)-a|\mathrm{d}\mu(z)\\
&\leq C\frac{1}{\mu(B)}\int|f(z)-a|\sum_{k\in K}\varphi_k(z)\mathrm{d}\mu(z)+\int_{B\cap F}|f(z)-a|\mathrm{d}\mu(z)\\
&\leq C\frac{1}{\mu(B)}\int_{B(x_0,7A^3r)}|f(z)-a|\mathrm{d}\mu(z)\leq C(f^{\#})^*(t).
\end{aligned}
$$

The proof is completed. □

3. The proof of Theorem 1.2

In order to prove Theorem 1.2, we need the following lemmas.

Lemma 3.1 *Let (X,ρ,μ) be a bounded space of homogeneous type, i.e., $\mu(X)<\infty$. Ω is a open subset of X, $\mu(\Omega)\leq\frac{\mu(X)}{2}$. Then there exists a sequence $\{B_k\}$ of balls satisfying*

(i) $\mu(\Omega\cap B_k)\leq\frac{1}{2}\mu(B_k)\leq\mu(\Omega^c\cap B_k)$;

(ii) $\Omega\subset\cup_k B_k$;

(iii) $\mu(\Omega)\leq\sum\limits_k\mu(B_k)\leq C\mu(\Omega)$.

Proof Notice that $\mu(\Omega)\leq\frac{\mu(X)}{2}$ and $\mu(X)<\infty$. We have that for any $x\in X$, there exists a $r_x>0$ such that

$$\frac{1}{4}<\frac{\Omega\cap B(x,r_x)}{\mu(B(x,r_x))}\leq\frac{1}{2}. \tag{3}$$

Clearly, the set $\{r_x:x\in X\}$ is a bounded set and X is covered by the family of balls $\{B(x,\frac{r_x}{5A^2})\}$. We choose $B(x_1,r_1)$ such that $r_1\geq\frac{1}{2}\sup\{r_x:x\in X\}$. Suppose that $B(x_1,r_1)$, $B(x_2,r_2)$, …, $B(x_k,r_k)$ have already been chosen, then we take $B(x_k,r_k)$ such that $B(x_{k+1},\frac{r_{k+1}}{5A^2})$ to be disjoint from $B(x_1,\frac{r_1}{5A^2})$, $B(x_2,\frac{r_2}{5A^2})$, …, $B(x_k,\frac{r_k}{5A^2})$, and

$$r_{k+1}\geq\frac{1}{2}\sup\{r_x:x\in X,B(x,\frac{r_x}{5A^2})\cap B(x_i,\frac{r_i}{5A^2})=\emptyset,\quad i=1,\dots,k\}.$$

In this way we get the sequence $\{B(x_k,r_k)\}$, $k=1,2,\dots$, of balls. This sequence could be finite or infinite. Without loss of generality, we suppose it is infinite. Since $\{B(x_k,\frac{r_k}{5A^2})\}$ are disjoint balls,

$\sum_k \mu(B(x_k, \frac{r_k}{5A^2})) \le \mu(X) < \infty$, and $\lim_{k\to\infty} r_k = 0$. For any $x \in \Omega$, we take the first k with the property that $r_{k+1} < \frac{1}{2}r_x$. Now the ball $B(x, \frac{r_x}{5A^2})$ must intersect one of the k previous balls $B(x_1, \frac{r_1}{5A^2}), B(x_2, \frac{r_2}{5A^2}), \ldots, B(x_k, \frac{r_k}{5A^2})$, say $B(x_{k_0}, \frac{r_{k_0}}{5A^2})$ for some $1 \le k_0 \le k$, and $r_{k_0} \ge \frac{1}{2}r_x$. It is easy to obtain that $B(x, \frac{r_x}{5A^2}) \subset B(x_{k_0}, r_{k_0})$. Thus we prove that $\Omega \subset \cup_k B(x_k, r_k)$, and so $\mu(\Omega) \le \sum\limits_k \mu(B(x_k, r_k))$. Let $B_k = B(x_k, r_k)$, then $\{B_k\}$ satisfy (i), (ii) and the first inequality in (iii). Let $f(x) = \chi_\Omega(x)$. Noticing the first inequality in (3), we have

$$\cup_k B(x_k, \frac{r_k}{5A^2}) \subset \{x : M(f)(x) > \frac{1}{4}\}.$$

Thus

$$\begin{aligned}\sum_k \mu(B_k) &\le C\sum_k \mu(B(x_k, \frac{r_k}{5A^2})) \le C\mu(\cup_k B(x_k, \frac{r_k}{5A^2}))\\ &\le C\mu(\{x : M(f)(x) > \frac{1}{4}\}) \le C\int f(x)\mathrm{d}\mu(x) \le C\mu(\Omega).\end{aligned}$$

This shows that the second inequality of (iii) is true.

Lemma 3.2 *Let f be an integrable function supported on ball B_0. Then*

$$f^{**}(t) - f^*(t) \le C(f^{\#}_{B_0})^*(t), \quad \text{for } 0 < t < \frac{\mu(B_0)}{6},$$

where

$$(f^{\#}_{B_0})^*(t) = \sup_{x\in B, B\subset B_0} \frac{1}{\mu(B)}\int_B |f(z) - f_B|\mathrm{d}\mu(z).$$

Proof Since $|f|^{\#}_{B_0} \le f^{\#}_{B_0}$, without loss of generality, we suppose $f \ge 0$. Let

$$E = \{x \in B_0 : f(x) > f^*(t)\}, \quad F = \{x \in B_0 : f^{\#}_{B_0}(x) > (f^{\#}_{B_0})^*(t)\}.$$

It is easy to see that E and F are open sets and $\mu(E) \le t$, $\mu(F) \le t$. Define $\Omega = E \cup F$, then $\mu(\Omega) \le 2t < \frac{\mu(B_0)}{2}$. By Lemma 3.1, there exists a sequence $\{B_k\}$ of balls satisfying the conditions (i)-(iii).

$$\begin{aligned}t\{f^{**}(t) - f^*(t)\} &= \int_E (f(x) - f^*(t))\mathrm{d}\mu(x)\\ &\le \sum_k \int_{E\cap B_k} (f(x) - f^*(t))\mathrm{d}\mu(x)\\ &\le \sum_k \int_{B_k} |f(x) - f_{B_k}|\mathrm{d}\mu(x) + \sum_k \mu(E\cap B_k)(f_{B_k} - f^*(t))\\ &= \mathrm{I} + \mathrm{II}.\end{aligned}$$

Define $K = \{k : f_{B_k} > f^*(t)\}$, then we have

$$\mathrm{II} \le \sum_{k\in K} \mu(E\cap B_k)(f_{B_k} - f^*(t)) \le \sum_{k\in K} \mu(\Omega\cap B_k)(f_{B_k} - f^*(t)).$$

By Lemma 3.1(i), noticing that $f(x) \le f^*(t)$ for $x \in \Omega^c$, we have

$$\mathrm{II} \le \sum_{k\in K} \int_{\Omega^c \cap B_k} (f_{B_k} - f^*(t)) \le \sum_k \int_{B_k} |f(x) - f_{B_k}| \mathrm{d}\mu(x) \le \mathrm{I}.$$

Thus $t\{f^{**}(t) - f^*(t)\} \le 2\mathrm{I}$. By Lemma 3.1, for any k, $B_k \cap F^c \neq \emptyset$, taking $y_k \in B_k \cap F^c$, we have

$$\mathrm{I} \le \sum_k \mu(B_k) f^{\#}_{B_0}(y_k) \le \sum_k (f^{\#}_{B_0})^*(t)\mu(B_k) \le C\mu(\Omega)(f^{\#}_{B_0})^*(t) \le Ct(f^{\#}_{B_0})^*(t).$$

Lemma 3.3 *Let f be integrable function on B_0, $0 < t \le \frac{\mu(B_0)}{6}$. Then*

$$([f - f_{B_0}]\chi_{B_0})^{**}(t) \le C \int_t^{\mu(B_0)} (f^{\#}_{B_0})^*(s) \frac{\mathrm{d}s}{s}.$$

Proof Let $g = [f - f_{B_0}]\chi_{B_0}$. By Lemma 3.2, we have

$$g^{**}(s) - g^*(s) \le C(g^{\#}_{B_0})^*(s), \quad (0 < s \le \frac{\mu(B_0)}{6}).$$

From the definition of g^{**} and Newton-Leibniz formula, we have

$$g^{**}(t) - g^{**}(u) = \int_t^u (g^{**}(s) - g^*(s)) \frac{\mathrm{d}s}{s}.$$

Thus for $0 < t \le u \le \frac{\mu(B_0)}{6}$, we have

$$g^{**}(t) - g^{**}(u) \le C \int_t^u (g^{\#}_{B_0})^*(s) \frac{\mathrm{d}s}{s}.$$

Taking $u = \frac{\mu(B_0)}{6}$, and noticing that

$$g^{**}(\frac{\mu(B_0)}{6}) \le \frac{6}{\mu(B_0)} \int_0^{\mu(B_0)} g^*(s)\mathrm{d}s = \frac{6}{\mu(B_0)} \int_{B_0} |g(z)| \mathrm{d}\mu(z) = 6|g|_{B_0},$$

we have

$$g^{**}(t) \le C\{\int_t^{\mu(B_0)} (g^{\#}_{B_0})^*(s) \frac{\mathrm{d}s}{s} + |g|_{B_0}\}.$$

By the definition of g, we have $|g|_{B_0} \le f^{\#}_{B_0}(y)$ for any $y \in B_0$. Thus

$$|g|_{B_0} \le \frac{5}{6\mu(B_0)} \int_t^{\mu(B_0)} (f^{\#}_{B_0})^*(s) \frac{\mathrm{d}s}{s}.$$

Lemma 3.4 *Let $f \in L_{\mathrm{loc}}(X)$, $1 \le q$. If*

$$\int_1^\infty (f^{\#})^*(s) \frac{\mathrm{d}s}{s} < \infty,$$

then the limit $f_\infty = \lim_{\mu(B)\to\infty} f_B$ exists.

Proof For fixed $x_0 \in X$, define $B_k = B(x_0, 2^k)$. By Lemma 3.3, we have

$$f_{B_{k+1}} - f_{B_k} = [(f_{B_{k+1}} - f_{B_k})\chi_{B_k}]^{**}(\frac{\mu(B_k)}{6})$$

$$
\begin{aligned}
&\le [(f-f_{B_k})\chi_{B_k}]^{**}(\frac{\mu(B_k)}{6}) + [(f-f_{B_{k+1}})\chi_{B_k}]^{**}(\frac{\mu(B_k)}{6}) \\
&\le C(\int_{\frac{\mu(B_k)}{6}}^{\mu(B_k)} + \int_{\frac{\mu(B_k)}{6}}^{\mu(B_{k+1})}) f^{**}(s)\frac{\mathrm{d}s}{s}, \\
&\le C(\int_{\frac{\mu(B_k)}{6}}^{\mu(B_k)} + \int_{\frac{\mu(B_k)}{6}}^{\mu(B_{k+1})}) [f^{**}(s)]^q\frac{\mathrm{d}s}{s},
\end{aligned}
$$

thus $\{f_{B_k}\}$ is a Cauchy sequence and $\lim_{k\to\infty} f_{B_k}$ exists.

For any $\epsilon > 0$ and M large enough such that

$$
C\int_{\frac{M}{2}}^{\infty} (f^{\#})^*(s)\frac{\mathrm{d}s}{s} < \frac{\epsilon}{3},
$$

and for any ball B satisfying $\mu(B) > M$, we choose B_k such that $\mu(B_k) > M$ and $|f_{B_k} - f_\infty| < \frac{\epsilon}{3}$. Taking ball B' contains B and B_k, we have

$$
\begin{aligned}
|f_B - f_\infty| &\le |f_B - f_{B'}| + |f_{B'} - f_{B_k}| + |f_{B_k} - f_\infty| \\
&\le C(\int_{\mu(B)}^{\infty} + \int_{\mu(B_k)}^{\infty})(f^{\#})^*(s)\frac{\mathrm{d}s}{s} + \frac{\epsilon}{3} < \epsilon.
\end{aligned}
$$

Lemma 3.5 *Let $f \in L_{\rm loc}(X)$, $1 \le q$. If*

$$
\int_1^{\infty} (f^{\#})^*(s)\frac{\mathrm{d}s}{s} < \infty,
$$

then

$$
(f-f_\infty)^{**}(t) \le C\int_t^{\infty} (f^{\#})^*(s)\frac{\mathrm{d}s}{s}.
$$

Proof For any $\epsilon > 0$ and ball B, we have $|f_B - f_\infty| < \epsilon$ as long as $\mu(B)$ large enough. Thus

$$
\begin{aligned}
[(f-f_\infty)\chi_B]^{**}(t) &\le [(f-f_B)\chi_B]^{**}(t) + |f_B - f_\infty| \\
&\le C\int_t^{\infty} (f^{\#})^*(s)\frac{\mathrm{d}s}{s} + \epsilon.
\end{aligned}
$$

Let $B \nearrow X$. Since $[(f-f_\infty)\chi_B]^{**}(t) \nearrow (f-f_\infty)^{**}(t)$ and ϵ is arbitrary, we know Lemma 3.5 is true.

Proof of Theorem 1.2 Let $F \in (L^1, \mathrm{BMO})_{\theta,q}$. When $q < \infty$, we have

$$
\int_0^{\infty} [t^{-\theta}K(t,F;L^1,\mathrm{BMO})]^q\frac{\mathrm{d}t}{t} < \infty.
$$

By Theorem 2.3,

$$
\int_1^{\infty} (F^{\#})^*(t)\frac{\mathrm{d}t}{t} = \int_1^{\infty} t^{-\frac{1}{p}}t^{\frac{1}{p}}(F^{\#})^*(t)\frac{\mathrm{d}t}{t} \le C\int_1^{\infty} t^{\frac{q}{p}}[(F^{\#})^*(t)]^q\frac{\mathrm{d}t}{t} < \infty.
$$

From Lemma 3.4, for every f in the equivalent class F, $f_\infty = \lim_{\mu(B)\to\infty} f_B$ exists. We choose f such that $f_\infty = 0$. By Lemma 3.5, we have

$$
f^{**}(t) \le C\int_t^{\infty} (F^{\#})^*(s)\frac{\mathrm{d}s}{s}, \quad 0 < t < \infty.
$$

Noticing $0 < \theta < 1$ and using the Hardy inequality, we obtain $f \in L_{pq}$ and prove the first inequality. When $q = \infty$, the proof is easy and omitted here.

Conversely, if $f \in L_{pq}$ and F belongs to the equivalent class of f, then using $F^{\#}(x) \le 2M(f)(x)$, we have $(F^{\#})^*(t) \le Cf^{**}(t)$. Using the Hardy inequality again, we get the second inequality of the Theorem 1.2. This completes the proof of Theorem 1.2. □

Corollary 3.6 *For* $0 < \theta < 1$, $1 \le q \le \infty$, *we have* $(H^1, \mathrm{BMO})_{\theta,q} = L_{pq}$, *where* $\theta = 1 - \frac{1}{p}$.

Proof By Theorems 1.1 and 1.2, we have

$$(H^1, L^\infty)_{\theta,q} = L_{pq} = (L^1, \mathrm{BMO})_{\theta,q}, \quad 0 < \theta < 1, 1 \le q \le \infty,$$

where $\theta = 1 - \frac{1}{p}$. Since $L^\infty \hookrightarrow \mathrm{BMO}$ and $H^1 \hookrightarrow L^1$,

$$(H^1, L^\infty)_{\theta,q} \subset (H^1, \mathrm{BMO})_{\theta,q} \subset (L^1, \mathrm{BMO})_{\theta,q}.$$

This proves the corollary.

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齐型空间上 L^1 与 BMO 的内插空间

李刚[1], 李文明[2]
(1. 河北广播电视大学教务处，河北 石家庄 050071;
2. 河北师范大学数学与信息科学学院，河北 石家庄 050016)

摘要: 本文讨论齐型空间上 L^1 与 BMO 的内插空间, 得到下列结果: 对于 $0 < \theta < 1$, $1 \le q \le \infty$, 有 $(L^1, \mathrm{BMO})_{\theta,q} = L_{pq}$, 其中 $\theta = 1 - \frac{1}{p}$.

关键词: 齐型空间; BMO; 内插空间.