第27卷第1期 数学研究与评论 Vol.27, No.1
2007年2月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Feb., 2007

**Article ID**: 1000-341X(2007)01-0041-06 **Document code**: A

# Some Sufficient Conditions to Quasi-Convex Functions

*XU Ya-shan*
(School of Mathematical Sciences, Fudan University, Shanghai 200433, China )
(E-mail: yashanxu@gmail.com)

**Abstract**: The concept of the midpoint quasi-convex function is introduced, and some conditions are obtained to ensure that midpoint quasi-convex function is quasi-convex in the measurable function space.

**Key words**: quasi-convex; midpoint quasi-convex; measurability.
**MSC(2000)**: 26A51
**CLC number**: O174.13

## 1. Introduction

Quasi-convex functions play an important role in economics and many economic models are actually quasi-convex functions. The properties of quasi-convex functions have been discussed in e.g. [1].

Let us introduce the concept of quasi-convex function.

**Definition 1.1** *Let $\Omega$ be a convex subset of $I\!R^m$. The function $f$ is quasi-convex on $\Omega$ if the following inequality*

$$f[\lambda x+(1-\lambda)y]\leq \max\{f(x),f(y)\} \tag{1.1}$$

*holds for any $x,y\in\Omega$, and $\lambda\in[0,1]$.*

The definition of the midpoint quasi-convex function is introduced as follows

**Definition 1.2** *Let $\Omega$ be a convex subset of $I\!R^m$. The function $f$ is midpoint quasi-convex on $\Omega$ if the following inequality holds*

$$f(\frac{1}{2}x+\frac{1}{2}y)\leq \max\{f(x),f(y)\} \tag{1.2}$$

*for any $x,y\in\Omega$.*

Compared with the above two concepts, it is clear that a quasi-convex function must be midpoint quasi-convex. But not all midpoint quasi-convex functions are quasi-convex. We can illustrate it by the following counterexample.

**Example 1.1** Let $\Omega=[0,1]$ and

$$f(x)=\begin{cases} 0, & \text{if } x \text{ is rational in } \Omega;\\ 1, & \text{if } x \text{ is irrational in } \Omega.\end{cases} \tag{1.3}$$

**Received date**: 2004-10-18; **Accepted date**: 2006-01-17
**Foundation item**: the National Natural Science Foundation of China (10371024)

Now fix $x, y \in \Omega$. If $\max\{f(x), f(y)\} = 1$, then

$$f(\frac{1}{2}x + \frac{1}{2}y) \leq \max_{z\in\Omega} f(z) = 1 = \max\{f(x), f(y)\};$$

or else, $\max\{f(x), f(y)\} = 0$ holds. Then $x, y$ are all rational. Thus $\frac{1}{2}x + \frac{1}{2}y$ is also rational. Therefore, it follows that

$$f(\frac{1}{2}x + \frac{1}{2}y) = \max_{z\in\Omega} f(z) = 0 \leq \max\{f(x), f(y)\},$$

which together with (1.4) implies that function $f$ is midpoint quasi-convex. But

$$f\Big(\frac{\sqrt{2}}{2}x + (1 - \frac{\sqrt{2}}{2})y\Big) = 1 > 0 = \max\{f(x), f(y)\}.$$

It means that $f$ is not quasi-convex.

Since not all midpoint quasi-convex functions are quasi-convex, it is natural for us to ask that what condition can ensure that a midpoint quasi-convex function is quasi-convex. This paper is devoted to answer this question.

## 2. Main results

The example in the above section shows that quasi-convex function space is just a subset of midpoint quasi-convex function space. But what would happen if the considered function space is restricted?

Let $\Omega$ be a convex subset of $I\!R^m$. Denote by LSC($\Omega$) a set containing all lower semi-continuous function on $\Omega$, and USC($\Omega$) a set containing all upper semi-continuous function on $\Omega$. Now we discuss function $f$ in the framework of LSC($\Omega$) or USC($\Omega$). We has the following result.

**Theorem 2.1** *Let convex set $\Omega \subset I\!R^m$ and $f(\cdot) \in \mathrm{LSC}(\Omega) \cup \mathrm{USC}(\Omega)$. Then the midpoint quasi-convex function $f$ is quasi-convex.*

**Proof** First, assume that $f(\cdot) \in \mathrm{LSC}(\Omega)$ and $f(\cdot)$ is midpoint quasi-convex. Fix $x, y \in \Omega$. It follows by mathematical induction with respect to variable $n$ that

$$f\left(\frac{k}{2^n}x + \frac{2^n - k}{2^n}y\right) \leq \max\{f(x), f(y)\}, \quad n = 0, 1, 2, 3, \cdots, \quad 0 \leq k \leq 2^n.$$

Since set $\big\{\frac{k}{2^n}, n = 0, 1, 2, 3, \cdots, 0 \leq k \leq 2^n\big\}$ is dense on $[0, 1]$, there exists a sequence

$$\lim_{j\to\infty} \frac{k_j}{2^{n_j}} = \lambda$$

for any given $\lambda \in [0, 1]$. Thus, the following inequality holds from the the lower semi-continuity of function $f$

$$f\big(\lambda x + (1 - \lambda)y\big) \leq \varliminf_{j\to\infty} f\left(\frac{k_j}{2^{n_j}}x + \frac{2^{n_j} - k_j}{2^{n_j}}y\right) \leq \max\{f(x), f(y)\}.$$

Therefore, the midpoint quasi-convex function $f$ is quasi-convex if $f(\cdot) \in \mathrm{LSC}(\Omega)$.

Secondly, assume that $f(\cdot) \in \mathrm{USC}(\Omega)$ and $f(\cdot)$ is midpoint quasi-convex. It follows from the upper semi-continuity of function $f$ that there exists $\delta > 0$ for any $\varepsilon > 0$ such that the following two inequalities hold,

$$f\big((1-\tau)x+\tau y\big) \le f(x)+\varepsilon \le \max\{f(x), f(y)\}+\varepsilon,$$

$$f\big(\tau x+(1-\tau)y\big) \le f(y)+\varepsilon \le \max\{f(x), f(y)\}+\varepsilon$$

for any $0 \le \tau \le \delta$. We denote

$$\Omega_{0,0} = \{(1-\tau)x+\tau y \mid 0 \le \tau \le \delta\}, \quad \text{and} \quad \Omega_{1,0} = \{\tau x+(1-\tau)y \mid 0 \le \tau \le \delta\}.$$

Then it is derived that

$$f(z) \le \max\{f(x), f(y)\}+\varepsilon, \quad \forall\ z \in \Omega_{0,0} \cup \Omega_{1,0}.$$

Set $\{\Omega_{k,n}, 0 \le k \le 2^n, n = 1, 2, \cdots\}$ is defined by following recursion formulas

$$\Omega_{k,(n+1)} = \frac{1}{2}\Omega_{(\frac{k-1}{2}),n} + \frac{1}{2}\Omega_{(\frac{k+1}{2}),n} \equiv \Big\{\frac{1}{2}z_1+\frac{1}{2}z_2 \;\Big|\; z_1 \in \Omega_{(\frac{k-1}{2}),n}, \quad z_2 \in \Omega_{(\frac{k-1}{2}),n}\Big\}, \tag{2.1}$$

for all odd $1 \le k \le 2^n-1$ and

$$\Omega_{k,(n+1)} = \Omega_{(\frac{k}{2}),n}, \tag{2.2}$$

for all even $0 \le k \le 2^n$. It follows by mathematical induction with respect to variable $n$ that

$$f(z) \le \max\{f(x), f(y)\}+\varepsilon, \quad \forall z \in \bigcup_{0 \le k \le 2^n} \Omega_{k,n}. \tag{2.3}$$

As a line segment, $\Omega_{k,n}$ is $\delta$ in length for any $n = 0, 1, 2, 3, \cdots$ and $0 \le k \le 2^n$, which together with the fact that the set $\Big\{\dfrac{k}{2^n}, n = 0, 1, 2, 3, \cdots, 0 \le k \le 2^n\Big\}$ is dense on $[0, 1]$, implies that

$$\bigcup_{\substack{n=0,1,\cdots,\\ 0 \le k \le 2^n}} \Omega_{k,n} = \{\lambda x+(1-\lambda)y \mid 0 \le \lambda \le 1\}. \tag{2.4}$$

Thus, it follows from (2.3)—(2.4) that

$$f(z) \le \max\{f(x), f(y)\}+\varepsilon, \quad \forall\ z \in \{\lambda x+(1-\lambda)y \mid 0 \le \lambda \le 1\}.$$

Let $\varepsilon \to 0$, then we get (1.1). Then the proof is complete. □

From the above Theorem 2.1, it is known that the concept of midpoint quasi-convex function is equivalent to that of quasi-convex in the framework of lower semi-continuous function space or upper semi-continuous function space. Therefore, we could release the above continuity constraint. In what follows, we would discuss in the framework of the space. Denote $\mathrm{BM}(\Omega)$ be a set containing all Lebesgue measurable function on $\Omega$ for a given convex set $\Omega \subset I\!R^m$.

First we consider the case for the dimension of domain $m = 1$. Without loss of generality, assume that $\Omega = [0, 1]$ and the Lebesgue measurable function $f$ satisfies $\max\{f(0), f(1)\} = 0$.

To obtain the main result of this paper, we need the following two lemmas.

**Lemma 2.2** *Let $f(\cdot)$ be measurable on $[0,1]$ and $\max\{f(0), f(1)\} = 0$. If $f(\cdot)$ is mid-point quasi-convex on $[0,1]$, then the following equation holds*

$$\frac{m([a,b] \cap E)}{b-a} = C_0, \tag{2.5}$$

*for any $0 \le a < b \le 1$, where $E = \{\lambda \in [0,1] \mid f(\lambda) > 0\}$, $C_0 = m(E)$, and $m(\cdot)$ is Lebesgue measurable.*

**Proof** Fix any $x_0, x_1 \in [0,1] \setminus E$. Then $f(x_0), f(x_1) \le 0$. Now let

$$x_\lambda = (1-\lambda)x_0 + \lambda x_1, \quad \lambda \in [0,1].$$

It follows from the definition of midpoint quasi-convexity of $f$ that

$$f(x_\lambda) \le \max\{f(x_0),\ f(x_{2\lambda})\}, \qquad \forall\ \lambda \in [0, \frac{1}{2}]. \tag{2.6}$$

Due to $f(x_0) \le 0$, we have

$$f(x_{2\lambda}) > 0, \qquad \text{if } f(x_\lambda) > 0. \tag{2.7}$$

Thus, the following holds

$$\left\{x_{2\lambda} \mid x_\lambda \in E,\ \lambda \in \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right]\right\} \subset E \bigcap \left\{x_\mu \mid \mu \in \left[\frac{1}{2^k}, \frac{1}{2^{k-1}}\right]\right\}. \tag{2.8}$$

for any $k = 1, 2, 3, \cdots$. Therefore,

$$m\left(\left\{x_{2\lambda} \mid x_\lambda \in E,\ \lambda \in \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right]\right\}\right) \le m\left(E \bigcap \left\{x_\mu \mid \mu \in \left[\frac{1}{2^k}, \frac{1}{2^{k-1}}\right]\right\}\right).$$

Since $m(A+z) = m(A)$ and $m(2A) = 2m(A)$ hold for any measurable set $A \subset I\!R^1$ and $z \in I\!R^1$,

$$2m\left(E \bigcap \left\{x_\lambda \mid \lambda \in \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right]\right\}\right) \le m\left(E \bigcap \left\{x_\mu \mid \mu \in \left[\frac{1}{2^k}, \frac{1}{2^{k-1}}\right]\right\}\right).$$

Sum up the both sides respectively with respect to $k$, then we have

$$m\left(E \bigcap \{x_\lambda \mid \lambda \in [0, 1/2]\}\right) \le m\left(E \bigcap \{x_\lambda \mid \lambda \in [1/2, 1]\}\right). \tag{2.9}$$

If interchange $x_0$ with $x_1$, similarly we have

$$m\left(E \bigcap \{x_\lambda \mid \lambda \in [1/2, 1]\}\right) \le m\left(E \bigcap \{x_\lambda \mid \lambda \in [0, 1/2]\}\right),$$

which together with (2.9) implies that

$$m\left(E \bigcap \{x_\lambda \mid \lambda \in [0, 1/2]\}\right) = m\left(E \bigcap \{x_\lambda \mid \lambda \in [1/2, 1]\}\right) = \frac{1}{2} m(E \bigcap [x_0, x_1]) \tag{2.10}$$

holds for any $x_0, x_1 \in [0,1] \setminus E$.

If take $x_0 = 0$ and $x_1 = 1$, then the above equality gives

$$\frac{m\big(E\bigcap[0,1/2]\big)}{1/2} = \frac{m\big(E\bigcap[1/2,1]\big)}{1/2} = m(E) = C_0. \tag{2.11}$$

Note that

$$k/2^n \in [0,1] \setminus E, \quad \forall\, n = 0,1,2\cdots, \;\; 0 \le k \le 2^n.$$

It follows from (2.10)—(2.11) by mathematical induction with respect to variable $n \ge 1$ that

$$2^n m\bigg(E\bigcap\Big[\frac{k}{2^n}, \frac{k+1}{2^n}\Big]\bigg) = m(E) = C_0, \quad 0 \le k \le 2^n.$$

It follows from the continuity of the Lebesgue measure that

$$\frac{m\big(E\bigcap[a,b]\big)}{b-a} = C_0$$

holds for any $0 \le a < b \le 1$. □

**Lemma 2.3** *Let $G$ be a measurable subset of $I\!R^1$ and $0 \ne m(G) < \infty$. If $0 < \alpha < 1$, then there exists $(a,b)$ such that*

$$\frac{m\big(G \cap [a,b]\big)}{b-a} \ge \alpha.$$

**Proof** By the measurability of the set $G$, there exist a sequence of open intervals $\{(a_i,b_i) \mid i = 1,2,\cdots\}$ such that

$$G \subseteq \bigcup_{i=1}^{\infty}(a_i,b_i); \quad m(G) > \alpha \sum_{i=1}^{\infty}[b_i - a_i],$$

for any given $0 < \alpha < 1$[2]. It follows that

$$\begin{aligned}\alpha &< \frac{m(G)}{\sum\limits_i [b_i - a_i]} = \sum_j \frac{m(G \cap (a_j,b_j))}{\sum\limits_i [b_i - a_i]} = \sum_j \left\{ \frac{m(G \cap (a_j,b_j))}{b_j - a_j} \cdot \frac{b_j - a_j}{\sum\limits_i [b_i - a_i]} \right\} \\ &\le \left[\sup_j \frac{m(G \cap (a_j,b_j))}{b_j - a_j}\right] \sum_j \frac{b_j - a_j}{\sum\limits_i [b_i - a_i]} = \sup_j \frac{m(G \cap (a_j,b_j))}{b_j - a_j}.\end{aligned}$$

Therefore, there exists a $j$ such that $\dfrac{m(G \cap [a_j,b_j])}{b_j - a_j} \ge \alpha$. □

Now, we present the following main result.

**Theorem 2.4** *Let $f(\cdot)$ be measurable on $[0,1]$. If $f(\cdot)$ is midpoint quasi-convex on $[0,1]$, then $f$ is either a quasi-convex function or a constant function almost everywhere.*

**Proof** Without loss of generality, we still assume that $\max\{f(0), f(1)\} \le 0$. It follows by Lemmas 2.2 and 2.3 that

$$C_0 = 0, \quad \text{or} \quad C_0 = 1.$$

(1) If $C_0 = 0$, we claim that $E = \emptyset$. To see this, suppose there exists an $x_0 \in E$, i.e., $f(x_0) > 0$. If $x_0 \in [0, \frac{1}{2}]$, it follows from the fact $f(z) \leq 0$ almost everywhere on $[0, x_0]$ and

$$0 < f(x_0) \leq \max\{f(z), f(2x_0 - z)\},$$

that

$$f(x) \geq 0 \quad \forall\, x \in [x_0, 2x_0],$$

which contradicts the fact $C_0 = 0$. Thus $[0, 1/2] \cap E = \emptyset$. Similarly, $[1/2, 1] \cap E = \emptyset$. Therefore, $E = \emptyset$, that is, $f$ is a quasi-convex function.

(2) If $C = 1$, we claim that there exists $\beta > 0$ such that $f(\cdot)$ equals $\beta$ almost everywhere on $[0, 1]$. To see this, fix a $\tau > 0$ and write

$$f_\tau(\cdot) = f(\cdot) - \tau, \quad \text{and} \quad E_\tau = \{x \in [0,1] \mid f_\tau(x) > 0\}.$$

Since $\max\{f_\tau(0), f_\tau(1)\} \leq 0$ holds, it follows by Lemmas 2.2 and 2.3 that $m(E_\tau) = 0$ or 1. Note function $m(E_\tau)$ is monotone increasing with respect to $\tau$. Hence there exists $\beta$ such that:

$$m(E_\tau) = \begin{cases} 0, & \tau < \beta, \\ 1, & \tau \geq \beta. \end{cases}$$

Therefore, $f(\cdot) = \beta$ almost everywhere on $[0, 1]$. We obtain the result. □

**Remark** The above result can be extended to high dimensional Euclidean space by the same method. Precisely, let convex set $\Omega \subset I\!R^n$ and suppose $f(\cdot)$ is measurable on $\Omega \times [0, 1]$. If $f(\cdot)$ is mid-point quasi-convex and satisfies $f(x, 0) \leq 0$, $f(x, 1) \leq 0$, for all $x \in \Omega$, then one of the following two results holds, either $f(x, y) \leq 0, \forall (x, y) \in \Omega \times [0, 1]$ holds or $f$ is a constant function almost everywhere.

## References:


[1] AUBIN J P. *Optima and Equilibria. An Introduction to Nonlinear Analysis* [M]. Translated from the French by Stephen Wilson, Springer-Verlag, Berlin, 1993.

[2] MUKHERJEA A, POTHOVEN K. *Real and Functional Analysis. Part A. Real Analysis* [M]. Second edition. Mathematical Concepts and Methods in Science and Engineering, 27. Plenum Press, New York, 1984.

[3] YANG Xin-min. *Quasi-convexity of upper semicontious functions* [J]. Chinese Journal of Operations Research, 1999, **3**(1): 48–51.


# 拟凸函数的若干充分条件


许 亚 善

(复旦大学数学科学学院，上海 200433 )



**摘要**: 本文提出了中点拟凸函数的概念，在可测函数空间中，给出了中点拟凸函数拟凸的若干个充分条件.



**关键词**: 拟凸函数；中点拟凸；可测函数.