

## Inclusion Measures of Convex Bodies

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**Abstract:** In this paper, we get several inequalities of inclusion measures of convex bodies.

**Key words:** convex body; inclusion measure.

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### 1. Introduction

The setting for this paper is  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ . We will denote by convex figure a compact convex subset of  $\mathbb{R}^n$ , and by convex body a convex figure with nonempty interior. Let  $G(n)$  be the group of special motions in  $\mathbb{R}^n$ . Each element,  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , of  $G(n)$  can be represented by

$$g : x \rightarrow ex + b,$$

where  $b \in \mathbb{R}^n$  and  $e$  is an orthogonal matrix of determinant 1. Let  $\mu$  be the Haar measure on  $G(n)$  normalized as follows: Let  $\varphi : \mathbb{R}^n \times SO(n) \rightarrow G(n)$  be defined by  $\varphi(t, e)x = ex + t, x \in \mathbb{R}^n$ , where  $SO(n)$  is the rotation group of  $\mathbb{R}^n$ . If  $\nu$  is the unique invariant probability measure on  $SO(n)$ ,  $\eta$  is the Lebesgue measure on  $\mathbb{R}^n$ , then  $\mu$  is chosen as the pull back measure of  $\eta \otimes \nu$  under  $\varphi^{-1}$ .

The *inclusion measure* of a convex figure  $L$  contained in a convex body  $K$  is defined by

$$m_K(L) = m(L \subseteq K) = \int_{\{g \in G(n) : gL \subseteq K\}} d\mu(g).$$

It gives the measure of the set of copies congruent to convex figure  $L$  contained in a fixed convex body  $K$ . We note that there had been not any breakthroughs on the problem of inclusion measure of convex bodies until the eighties in the twentieth century. D. Ren did some original work on this problem<sup>[4]</sup>. In this paper, we establish the inequalities of inclusion measures as follows: Let  $K_i, i = 1, 2, \dots, s, s > 1, s \in \mathbf{N}$ , be convex bodies and  $L$  be a convex figure in  $\mathbb{R}^n$ , then there holds

$$m_{K_1+K_2+\dots+K_s}(L) > m_{K_1}(L) + m_{K_2}(L) + \dots + m_{K_s}(L).$$

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Specifically, when  $L$  is a line segment, then there holds

$$m_{\alpha_1 K_1 + \alpha_2 K_2 + \dots + \alpha_s K_s}(L) > \alpha_1^n m_{K_1}\left(\frac{\alpha_1}{L}\right) + \alpha_2^n m_{K_2}\left(\frac{\alpha_2}{L}\right) + \dots + \alpha_s^n m_{K_s}\left(\frac{\alpha_s}{L}\right),$$

for  $\alpha_i \geq 1$ . This means that for inclusion measure the linearity does not hold.

## 2. Notations and preliminaries

Let  $K$  be a convex figure in  $\mathbb{R}^n$ . Associated with  $K$  is its *support function*  $h_K$  defined on  $\mathbb{R}^n$  by

$$h_K(x) = \max\{\langle x, y \rangle : y \in K\},$$

where  $\langle x, y \rangle$  is the usual inner product of  $x$  and  $y$  in  $\mathbb{R}^n$ . The function  $h_K$  is positively homogeneous of degree 1. We will be concerned with the restriction of the support function to the unit sphere  $S^{n-1}$ .

The Minkowski addition of two convex figures  $K$  and  $L$  is defined as

$$K + L = \{x + y : x \in K, y \in L\}.$$

The *scalar multiplication*  $\lambda K$  of  $K$ , where  $\lambda \geq 0$ , is defined as

$$\lambda K = \{\lambda x : x \in K\}.$$

For convex figure  $\lambda K + \mu L$ , the support function is

$$h_{\lambda K + \mu L} = \lambda h_K + \mu h_L.$$

Its volume is a homogeneous polynomial in  $\lambda$  and  $\mu$  given by

$$V(\lambda K + \mu L) = \sum_{i=0}^n \binom{n}{i} V_i(K, L) \lambda^{n-i} \mu^i.$$

The coefficients  $V_i(K, L)$  are called mixed volumes of  $K$  and  $L$ . If  $B$  is the unit ball, then the  $V_i(K, B)$  are called the quermassintegrals of  $K$ . The inradius  $r(K, L)$  of  $K$  with respect to  $L$  is defined by  $r(K, L) = \sup\{\lambda : x \in \mathbb{R}^n \text{ and } x + \lambda L \subset K\}$ . Let  $h_K$  and  $h_L$  be the support functions of  $K$  and  $L$  respectively. First, we assume that  $L$  is a convex body. For a fixed  $\lambda \in [0, r]$ , we consider the function  $h_\lambda = h_K - \lambda h_L$  on the unit sphere, where  $r = r(K, L)$  is the inradius of  $K$  with respect to  $L$ . In general,  $h_\lambda$  is not the support function of a convex body. Denote by  $C(K, L, \lambda)$  the intersection of halfspaces  $\{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_\lambda(u)\}$ ,  $u \in S^{n-1}$ . The boundaries  $\partial C(K, L, \lambda)$  are pairwise disjoint and

$$\bigcup_{0 \leq \lambda \leq r(K, L)} \partial C(K, L, \lambda) = K.$$

The following formula is known<sup>[1-3]</sup>:

$$\frac{d}{d\lambda} V(C(K, L, \lambda)) = -nV_1(C(K, L, \lambda), L). \quad (2.1)$$

By integrating both sides of (2.1), we get

$$V(K) - V(C(K, L, \lambda)) = n \int_0^\lambda V_1(C(K, L, \sigma), L) d\sigma, \quad (2.2)$$

and

$$V(C(K, L, \lambda)) = n \int_\lambda^{r(K, L)} V_1(C(K, L, \sigma), L) d\sigma. \quad (2.3)$$

By a limit process, (2.2) and (2.3) are seen to hold for any convex figure  $L$ . Lemma 1 below implies that  $C(K, L, \lambda) = \Phi$  if  $\lambda > r(K, L)$ , and thus (2.2) and (2.3) hold for any  $\lambda \geq 0$ . This lemma contained in [8] is useful in our paper.

### 3. Main results

**Lemma 1**<sup>[8]</sup> *The intersection  $C(K, L, \lambda)$  of halfspaces  $\{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_\lambda(u)\}$  is equal to the set  $\{x \in \mathbb{R}^n : x + \lambda L \subseteq K\}$ , i.e.,*

$$C(K, L, \lambda) = \{x \in \mathbb{R}^n : x + \lambda L \subseteq K\}, \quad \lambda \geq 0.$$

**Lemma 2**<sup>[8]</sup> *If  $K$  is a convex body and  $L$  is a convex figure in  $\mathbb{R}^n$ , then the inclusion measure of  $L$  contained in  $K$  is*

$$m_K(L) = \int_{SO(n)} V(C(K, eL, 1)) d\nu(e),$$

where  $\nu$  is the unique invariant probability measure on  $SO(n)$ .

**Lemma 3**<sup>[7]</sup> *Let  $K$  be a convex body and  $L$  be a line segment of length  $l$  in  $\mathbb{R}^n$ . Then*

$$m_{sK}(L) = s^n m_K\left(\frac{L}{s}\right), \quad \text{for any } s > 0.$$

**Lemma 4** *If  $K$  is a convex body and  $L$  a convex figure in  $\mathbb{R}^n$ , then*

$$C(\alpha K, L, \lambda) = \alpha C\left(K, \frac{1}{\alpha} L, \lambda\right) = \alpha C\left(K, L, \frac{\lambda}{\alpha}\right), \quad \alpha > 0, \lambda \geq 0.$$

*In addition, if  $0 < \alpha < 1$ , then  $C(\alpha K, L, \lambda) \subset \alpha C(K, L, \lambda)$ ; if  $\alpha > 1$ , then  $C(\alpha K, L, \lambda) \supset \alpha C(K, L, \lambda)$ .*

**Proof** From Lemma 1, we can get

$$\begin{aligned} C(\alpha K, L, \lambda) &= \{x \in \mathbb{R}^n : x + \lambda L \subseteq \alpha K\} = \{x \in \mathbb{R}^n : \frac{1}{\alpha}(x + \lambda L) \subseteq K\} \\ &= \{x \in \mathbb{R}^n : \frac{1}{\alpha}x \in K - \frac{\lambda}{\alpha}L\} = \{\alpha x \in \mathbb{R}^n : x \in K - \frac{\lambda}{\alpha}L\} \\ &= \alpha C\left(K, \frac{1}{\alpha}L, \lambda\right) = \alpha C\left(K, L, \frac{\lambda}{\alpha}\right). \end{aligned}$$

Furthermore, if  $0 < \alpha < 1$ , then

$$C\left(K, L, \frac{\lambda}{\alpha}\right) \subset C(K, L, \lambda).$$

So

$$\alpha C(K, L, \frac{\lambda}{\alpha}) \subset \alpha C(K, L, \lambda).$$

Hence,

$$C(\alpha K, L, \lambda) \subset \alpha C(K, L, \lambda).$$

Similarly, if  $\alpha > 1$ , then

$$C(K, L, \lambda) \subset C(K, L, \frac{\lambda}{\alpha}).$$

So

$$\alpha C(K, L, \lambda) \subset \alpha C(K, L, \frac{\lambda}{\alpha}).$$

Hence,

$$\alpha C(K, L, \lambda) \subset C(\alpha K, L, \lambda).$$

In the following, we write  $C(K) = C(K, L, \lambda)$ , then we can get the following results.

**Lemma 5** Let  $K_i, i = 1, 2, \dots, s, s \in \mathbf{N}$ , be convex bodies and  $L$  be a convex figure in  $\mathbb{R}^n$ . Then for  $s > 1$ , there holds

$$\sum_{i=1}^s C(K_i) \subset C(\sum_{i=1}^s K_i).$$

**Proof** For any  $x \in \sum_{i=1}^s C(K_i)$ ,  $x$  can be expressed as

$$x = x_1 + x_2 + \dots + x_s, \quad x_i \in C(K_i), \quad i = 1, 2, \dots, s.$$

From the definition of  $C(K_i)$ , we can get

$$\langle x_1, u \rangle \leq h_{K_1}(u) - \lambda h_L(u),$$

$$\langle x_2, u \rangle \leq h_{K_2}(u) - \lambda h_L(u),$$

.....

$$\langle x_s, u \rangle \leq h_{K_s}(u) - \lambda h_L(u),$$

for  $u \in S^{n-1}$ . So

$$\langle \frac{x_1 + x_2 + \dots + x_s}{s}, u \rangle \leq \frac{h_{K_1}(u) + h_{K_2}(u) + \dots + h_{K_s}(u)}{s} - \lambda h_L(u).$$

That is,

$$\langle \frac{x}{s}, u \rangle \leq \frac{h_{K_1+K_2+\dots+K_s}(u)}{s} - \lambda h_L(u),$$

$$\frac{1}{s} \langle x, u \rangle \leq h_{\frac{K_1+K_2+\dots+K_s}{s}}(u) - \lambda h_L(u).$$

Hence,

$$\frac{\sum_{i=1}^s C(K_i)}{s} \subseteq C(\frac{\sum_{i=1}^s K_i}{s}).$$

Since  $0 < \frac{1}{s} < 1$ , from Lemma 4, we can get

$$\frac{\sum_{i=1}^s C(K_i)}{s} \subseteq C\left(\frac{\sum_{i=1}^s K_i}{s}\right) \subset \frac{1}{s}C\left(\sum_{i=1}^s K_i\right).$$

So

$$\sum_{i=1}^s C(K_i) \subset C\left(\sum_{i=1}^s K_i\right).$$

**Theorem 1** Let  $K_i, i = 1, 2, \dots, s, s > 1, s \in \mathbf{N}$ , be convex bodies and  $L$  be a convex figure in  $\mathbb{R}^n$ . Then

$$m_{K_1+K_2+\dots+K_s}(L) > m_{K_1}(L) + m_{K_2}(L) + \dots + m_{K_s}(L).$$

**Proof** Firstly, we consider the case when  $s = 2$ . From Lemma 2, we can get

$$m_{K_1+K_2}(L) = \int_{SO(n)} V(C(K_1 + K_2, eL, \lambda))d\nu(e).$$

From Lemma 5 and Brunn-Minkowski inequality, there holds

$$V^{\frac{1}{n}}(C(K_1 + K_2)) \geq V^{\frac{1}{n}}(C(K_1) + C(K_2)) \geq V^{\frac{1}{n}}(C(K_1)) + V^{\frac{1}{n}}(C(K_2)).$$

So,

$$V(C(K_1 + K_2)) > V(C(K_1)) + V(C(K_2)).$$

Therefore,

$$\begin{aligned} m_{K_1+K_2}(L) &= \int_{SO(n)} V(C(K_1 + K_2, eL, \lambda))d\nu(e) \\ &> \int_{SO(n)} V(C(K_1, eL, \lambda))d\nu(e) + \int_{SO(n)} V(C(K_2, eL, \lambda))d\nu(e) \\ &= m_{K_1}(L) + m_{K_2}(L). \end{aligned}$$

By the finite induction, it can be easily gotten

$$m_{K_1+K_2+\dots+K_s}(L) > m_{K_1}(L) + m_{K_2}(L) + \dots + m_{K_s}(L).$$

**Corollary 1** Let  $K_i, i = 1, 2, \dots, s, s > 1, s \in \mathbf{N}$ , be convex bodies and  $L$  be a convex figure in  $\mathbb{R}^n$ . Then

$$m_{\alpha_1 K_1 + \alpha_2 K_2 + \dots + \alpha_s K_s}(L) > \alpha_1^n m_{K_1}(L) + \alpha_2^n m_{K_2}(L) + \dots + \alpha_s^n m_{K_s}(L),$$

for  $\alpha_i \geq 1, i = 1, 2, \dots, s$ .

**Proof** Firstly, we consider the case when  $s = 2$ . From Lemmas 2 and 4, we can get

$$\begin{aligned} m_{\alpha_1 K_1}(L) &= \int_{SO(n)} V(C(\alpha_1 K_1, eL, \lambda))d\nu(e) \\ &\geq \alpha_1^n \int_{SO(n)} V(C(K_1, eL, \lambda))d\nu(e) = \alpha_1^n m_{K_1}(L). \end{aligned}$$

Similarly, we can get

$$m_{\alpha_2 K_2}(L) \geq \alpha_2^n m_{K_2}(L).$$

By the finite induction and Theorem 1, we have

$$m_{\alpha_1 K_1 + \alpha_2 K_2 + \cdots + \alpha_s K_s}(L) > \alpha_1^n m_{K_1}(L) + \alpha_2^n m_{K_2}(L) + \cdots + \alpha_s^n m_{K_s}(L),$$

for  $\alpha_i \geq 1, i = 1, 2, \cdots, s$ .

**Corollary 2** Let  $K_i, i = 1, 2, \cdots, s, s > 1, s \in \mathbf{N}$ , be convex bodies and  $L$  be a line segment of length  $l$ . Then

$$m_{\alpha_1 K_1 + \alpha_2 K_2 + \cdots + \alpha_s K_s}(L) > \alpha_1^n m_{K_1}\left(\frac{\alpha_1}{L}\right) + \alpha_2^n m_{K_2}\left(\frac{\alpha_2}{L}\right) + \cdots + \alpha_s^n m_{K_s}\left(\frac{\alpha_s}{L}\right),$$

for  $\alpha_i > 0, i = 1, 2, \cdots, s$ .

**Proof** From Theorem 1 and Lemma 3, we have

$$\begin{aligned} m_{\alpha_1 K_1 + \alpha_2 K_2 + \cdots + \alpha_s K_s}(L) &> m_{\alpha_1 K_1}(L) + m_{\alpha_2 K_2}(L) + \cdots + m_{\alpha_s K_s}(L) \\ &= \alpha_1^n m_{K_1}\left(\frac{\alpha_1}{L}\right) + \alpha_2^n m_{K_2}\left(\frac{\alpha_2}{L}\right) + \cdots + \alpha_s^n m_{K_s}\left(\frac{\alpha_s}{L}\right), \end{aligned}$$

for  $\alpha_i > 0, i = 1, 2, \cdots, s$ .

## References:

- [1] ALEXANDROV A D. *On the theory of mixed volumes of convex bodies* [J]. Mat. Sbornik, 1938, **45**(3): 28–46.
- [2] BURAGO Y D, ZALGALLER V A. *Geometric Inequalities* [M]. Springer, Berlin, 1980.
- [3] LEICHTWEISS K. *Konvexe Mengen* [M]. Springer, Berlin, 1980.
- [4] REN De-lin. *An Introduction to Integral Geometry* [M]. Shanghai: Science and Technology Press, 1988.
- [5] SANTALÓ L A. *Integral Geometry and Geometric Probability* [M]. Reading, MA: Addison-Wesley, 1976.
- [6] SCHNEIDER R. *Convex Bodies: The Brunn-Minkowski Theory* [M]. Cambridge University Press, Cambridge, 1993.
- [7] XIONG Ge, LI De-yi, ZHENG Lu-zhou. *Intersection bodies and inclusion measures of convex bodies* [J]. J. Math. (Wuhan), 2003, **23**: 6–10.
- [8] ZHANG Gao-yong. *Geometric inequalities and inclusion measures of convex bodies* [J]. Mathematika, 1994, **41**: 95–116.
- [9] ZHANG Gao-yong. *Dual kinematic formulas* [J]. Trans. Amer. Math. Soc., 1999, **351**: 985–995.

## 凸体的包含测度

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**摘要:** 凸体的包含测度理论是积分几何中一个十分重要的课题, 它提供了一种研究凸体性质的全新方法. 本文获得了关于凸体包含测度的几个不等式, 从而证明了包含测度对于凸体没有线性.

**关键词:** 凸体; 包含测度.