Bilingual Students' Articulation and Gesticulation of Mathematical Knowledge During Problem Solving

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Abstract

This research focuses on eliciting bilingual students' problem-solving reasoning by providing mathematical tasks designed to involve the students in the action described in the problem by making connections to contexts familiar to them. Results showed these tasks allowed children to articulate their mathematical reasoning through gestures and speech. Two questions guided this investigation: (a) How do second-grade bilingual students communicate their mathematical reasoning? and (b) What is the role of the mathematical tasks in eliciting this mathematical reasoning? Analysis of seven second-grade Spanish–English bilingual students solving addition and subtraction problems showed that students simultaneously used words and gestures to communicate their mathematical reasoning to others and to regulate their own cognitive activity. In general, the students demonstrated that their developing bilingual proficiency did not constitute an impediment for guiding mathematical tasks to resolution; rather, students imparted mathematical meaning to each task by mutually supporting their verbal and nonverbal behaviors. The paper discusses how researchers and teachers can benefit from parallel attention to bilingual students' verbal and nonverbal communication. Both processes invite reflection on what it means to know and learn mathematics bilingually in early grades.

Introduction

Well-planned problem-solving tasks must include at least four dimensions: (a) a context, (b) an action, (c) a grade-appropriate set of numbers, and (d) a relationship between the set of numbers (Carpenter, Fennema, Franke, Levi, & Empson, 1999). This paper focuses on the co-occurrence of verbal and nonverbal modalities through which students express their mathematical reasoning as facilitated by well-planned problem-solving tasks. This study extends prior research by (a) focusing on bilingual students, (b) using mathematical tasks to purposefully produce words and gestures, and (c) making an important distinction between words and gestures that support communication of mathematical reasoning, and those that support acquisition and regulation of cognitive skills. Most research has investigated gestures as they occurred naturally (Koschmann & LeBaron, 2002), or by using completion mathematical tasks such as 3 + 4 + 5 = __ +5 (Kelly, Singer, Hicks, & Goldin-Meadow, 2002) and with monolingual learners only. Exceptions include Moschkovich's (2002) study with bilinguals, and Church and Goldin-Meadow's (1986) and Church, Schonert-Reichl, Goodman, Kelly, and Ayman-Nolley's (1995) studies with Piagetian conservation tasks.

Unlike prior research, this study focuses on mathematical tasks and offers ideas for teachers of bilingual students who wish to plan mathematics instruction by looking at bilingual students' ways of expressing their mathematical reasoning. The complexity of mathematical ideas extracted from mathematical tasks by the second graders, together with their developing bilingual proficiency in mathematics, requires attention to forms of mathematical communication that transcends language. Also, the various levels of bilingual proficiency that characterize bilingual classrooms constitute a research opportunity to explore the forms that mathematical communication takes when mathematical tasks are carefully planned.

Literature Review

A great deal of bilingual mathematics education scholarship has identified difficulties that bilingual students encounter as they learn mathematics. For example, this previous research has investigated difficulties in understanding math terms or translating word problems into math symbols (Mestre, 1981); the linguistic features of mathematics problems that cause difficulties in understanding and solving algebra problems (Spanos, Rhodes, Dale, & Crandall, 1988); and the syntax and lexical features of word problems that cause difficulties for translation into mathematical equations (Mestre & Gerace, 1986). Similarly, in various ways, studies have suggested the importance of teaching vocabulary to bilingual mathematics students in order to overcome such difficulties (Dale & Cuevas, 1987; MacGregor & Moore, 1992; Olivares, 1996; Rubenstein, 1996). Research on gestures constitutes an alternative within possibilities-oriented research (Díaz Soto, 1992) by emphasizing the existence of learning resources including gestures, instead of impediments. Moschkovich (2002) maintains: “If we focus on a student’s failure to use a technical term, we might miss how a student constructs meaning for mathematical terms or uses multiple resources, such as gestures, objects, or everyday experiences” (p. 193).

Research on gestures also relates to the formative assessment of bilingual students learning mathematics in that both are used during the process of learning and for the purpose of developing learning via informed instruction. Unlike summative assessment (e.g., the Texas Assessment of Knowledge and Skills test in Texas) formative assessment informs and guides instruction. Black, Harrison, Lee, Marshal, and Wiliam (2003) present empirical evidence to demonstrate that students of teachers who relied solely on summative assessment, or what they call *assessment of learning*, achieved at significantly lower levels than those students of teachers who were trained to implement formative assessment, or *assessment for learning*.

Reform mathematics curriculum recognizes the role of communication on the learning of mathematics with understanding (National Council of Teachers of Mathematics, 1989), recommending that mathematics teachers guide instruction and assessment by attending to the students’ communication of problem-solving strategies. For bilingual students, particularly those in the early years of their mathematical education, communication should refer to “the act of capturing a mathematical concept or relationship in some form’” (Pape & Tchoshanov, 2001, p. 119), including nonlinguistic forms.

Research on gestures produced by young learners (Radford, 2003; Vygotsky & Luria, 1994) suggests that gestures may be indicators of the students’ developing relationship to knowledge. Gestures may exemplify what Bourdieu (1977) describes as “the intangible nuances of manner and style which are the imperceptible and yet never unperceived manifestations of the individual’s relationship to knowledge” (p. 338).

Therefore, research on gestures offers at least three benefits for the mathematics education of bilingual students: (a) a possibilities-oriented theoretical perspective, (b) a process-oriented assessment for learning as opposed to the product-driven assessment of learning and (c) an indicator of and a way to develop bilingual students’ relationship to mathematical knowledge. Once recognized as an indicator of bilingual students’ relationship to mathematical knowledge, gestures can provide teachers (and all those who are involved in educating bilingual students) with ways to explore the students’ relationships to mathematical knowledge.

In the following sections, the present study discusses the manifestation of mathematical knowledge in words and gestures and what it means for bilingual students. This discussion is relevant for two reasons: (a) the early age and developing bilingual proficiency of participants, and (b) the early age and developing mathematical proficiency of participants. The supportive role of gestures for the communication of mathematical knowledge and the development of understanding will be discussed. This discussion will help us understand the co-occurrence of articulation and gesticulation in the data analysis section.

Broadening the Manifestation of Mathematical Knowledge

Some researchers question the possibility of representing all aspects of human experience in language, while hypothesizing that humans must encode the information from certain experiences in systems that do not use words (Huttenlocher, 1973, 1976; Alibali & Goldin-Meadow, 1993). This makes the relationship between knowledge and language more complex than it is generally assumed. Not all knowledge reaches explicit linguistic form; in fact, only a small portion of knowing is of this type (Davis, 1997). Habermas (1972) and Gadamer (1962, 1979) question whether it is possible to frame all experience in language. According to Mason (1994), "Words generate more words in explanation, but often draw us away from the experience from which they stem" (p. 176). For Brousseau & Otte (1991), "we obviously know more than we can tell" (p. 17).

In bilingual classrooms, questioning the representation of experience in language is accentuated by the fact that bilingual students possess a rich history of participation in multicultural experiences in various languages. In bilingual mathematics classrooms in which students are in the process of developing both mathematical and bilingual proficiencies, as was the case in this study, consideration of how they attend to mathematical ideas using language and gestures can constitute a way to help them develop these proficiencies. According to Brown (2001), mathematical ideas "are not endowed with a universal meaning but rather derive their meaning through the way in which an individual attends to them" (p. 15). Thus, well-planned problem-solving math tasks, such as the ones used in this research, can serve to examine the students' mathematical reasoning as manifested in words and gestures.

Yet, the emphasis on verbal display of knowledge in school mathematics reflects values that are well established in the mathematics curriculum. For example, Brown (2001) and Gattegno (1988) argue that the most robust areas in the mathematics curriculum are those more easily described linguistically, while other areas such as geometry enter the curriculum only after undergoing linguistic adaptations that destroy the link between geometric knowledge and intuition. To consider the possibility that mathematical knowledge manifests outside of language is to expand our conceptions of knowing and learning mathematics and to build on our view of how bilingual students learn mathematics.

Gestures Support the Communication of Knowledge

The most accepted view on the relationship between language and gestures characterizes these forms of communication as intimately linked and mediated by the human inclination to construct meaning and communicate it to others (Alibali & Goldin-Meadow, 1993; McNeill, 1992).

Kelly et al. (2002) conducted three experiments in which adults were trained to interpret students' gestures as they explained solutions to problems. In considering nonverbal behavior, specifically hand gestures, the researchers suggested that gestures, especially those that accompany speech, reveal information about the cognitive engagement of a student in a task, and that such information is not revealed in the student's speech. For Kendon (1987), gestures "are clearly part and parcel of the individual's openly acknowledged intention to convey meaning" (p. 71).

Other research on the relationship between language and gestures has investigated how gestures support understanding of verbal messages. For example, gestures have been found to enhance messages involving communication of complex geometric shapes (Graham & Argyle, 1975) and to add clarity to an elaborate message (Ekman & Friesen, 1972). According to Feyereisen and de Lannoy (1991), "The primary function of nonverbal behavior would not be to communicate information but to support the speaker's encoding activity" (p. 73).

The implication is that speech and gesture together are better predictors of what students really know in mathematics than either modality in isolation. Further, since research suggests an interdependence between gestures and language (Kendon, 1987; Goldin-Meadow et al., 1993; Roth, 2000), it makes sense to investigate the nature of such interdependence by considering mathematical tasks as the basis for eliciting mathematical reasoning in words and gestures.

Gestures Support the Development of Understanding

Although gestures have been researched in relation to different variables such as affective states, social competence, and cognitive states (Philippot, Feldman, & McGee, 1992), research that associates them with the development of understanding consistently suggests that gestures support cognitive development. Similarly, various researchers concur that gestures help learners in their cognitive development. For example, while Philippot et al. claim that "aspects of nonverbal behavior, mainly gestures, could facilitate the acquisition of verbal abilities" (p. 208), Koschmann and LeBaron (2002) view gestures as "an external manifestation of understanding but also as reflecting a constructive process of connection making" (p. 252).

To research the role of gestures on the development of understanding, some researchers have considered the attention that a person pays to all kinds of input during communication, whether verbal or gestural, concluding that the higher the synchrony between gestures and speech, the better the attention and ability to recall (Feyereisen & de Lannoy, 1991; Woodall & Burgoon, 1981; Goldin-Meadow, Kim, & Singer, 1999). In sum, extensive research recognizes the significant contribution of gestures to the development of understanding.

Gestures as a Representational System

One way that gestures can be viewed simultaneously as aiding in the acquisition and communication of mathematical knowledge is by considering them as a representational system. For example, Radford (2003) argues that young children use gestures to objectify mathematical ideas because they do not exist as physical objects do. In other words, one way young children gain access to and communicate about mathematical ideas is by using gestures to support their articulation of mathematical ideas.

According to Alibali and Goldin-Meadow (1993), children solving a problem must decide on how to represent the problem:

> If a child samples a representation which is accessible to both gesture and speech, our model proposes that the child will express the same procedure in both modalities, thus producing a gesture-speech match. If, however, the child samples a representation which is accessible to gesture but *not* to speech, the child will be able to describe the procedure in gesture but will be unable to express the same procedure in speech. In that case, the model proposes that the child will then select another representation (one which *is* accessible to speech) to express in speech and therefore will produce a gesture-speech mismatch. (p. 510)

Alibali and Goldin-Meadow's (1993) model predicts how a child will express or communicate mathematical knowledge to others. They also note that "mathematical thinking, particularly innovative mathematical thinking, is not conducted in words but rather in spatial images—images that may be more easily translated into the global-synthetic representation characteristic of gestures than into a liner-segmented representation characteristic of speech" (p. 515).

This result is of incalculable value for those who work with bilingual students (e.g., teachers, parents, tutors, and researchers) solving mathematics problems in that when gestures are viewed as a type of representational system, they too can provide insights into the student's mathematical reasoning, particularly for bilingual students who are developing their bilingual and mathematical proficiencies.

Classifying Gestures by Function and Modality

Some researchers have classified gestures in terms of their function. The following table summarizes Wiemann and Wiemann's (1975) six functions of nonverbal behavior, which the researcher applied to the analysis of gestures as they are a subset of nonverbal behaviors.

By applying this typology to the analysis of gestures, the researcher found that in addition to support communication of mathematical thinking, participants used certain gestures to regulate their own understanding. This distinction between gestures that support communication of mathematical reasoning and those that regulate mathematical understanding is explained further in the analysis section.

Table 1

Six Functions of Nonverbal Behavior

Function	Description	Example
Repeating	Repeats verbal message.	A person says, "Get out of my house" and points to the door.
Contradicting	Contradicts verbal message.	Before a test, a student says, "I'm not nervous" while trembling.
Substituting	Nonverbal message replaces verbal message.	A hug often takes the place of verbal expression of affection.
Complementing	Elaborates verbal message.	A student adopts an upright posture when talking to a professor.
Accenting	Accents part of verbal message.	Hand and head movements add emphasis to verbal expressions.
Regulating	Regulates verbal messages.	Eye contact and head nods regulate turn taking in conversations.

Method

Participants

The researcher conducted individual interviews in a class of seven second-grade bilingual students using a clinical-interview protocol and video equipment. In the clinical interview method, "the interviewer partly controls what the child does and partly is controlled by the child" (Ginsburg, 1997, p. 39). For example, the interviewer offers a mathematical task to the child to solve and asks an open-ended question such as: "Tell me how you solved this problem" and subsequent questions that respond to what the child said. As Ginsburg states, "In this sense, the child's talk and behavior control what the interviewer does and how the interviewer thinks and theorizes about what the child does. In the clinical interview, control passes back and forth between interviewer and child" (p. 39).

The school is in Austin Independent School District and has a high concentration of language-minority students, for whom a transitional bilingual education program is in place. Two students among the seven participants were proficient bilinguals and five were less proficient bilinguals, from the standpoint of their academic-language proficiency in mathematics, described by the teacher as the ease with which students communicated their mathematical thinking in Spanish and English during problem solving. For the selection of the students, the researcher used a purposive sampling approach by (a) finding an unusually small classroom ($N = 7$) to afford the level of detail required for gesture description and analysis and (b) selecting an early grade (Grade 2) for which mathematical and bilingual proficiency are developing. The researcher resisted classifying students according to existing bilingualism typologies such as English language learners or limited English proficient for at least three reasons: (a) These typologies fail to account for the fact that bilinguals belong to various practices in various languages at various times, and hence their bilingual proficiency is not a monolithic characteristic; (b) Bilingual proficiency requires attention to the context in which this proficiency is demonstrated; and (c) Tests that claim to measure bilingual proficiency tend to be discrete-point (that is, they are designed to test a large number of discrete items) and only measure things such as vocabulary, reading, and writing, ignoring the child's linguistic proficiency in the specific area of mathematics. Therefore, the study uses the definition of bilinguals supplied by Moschkovich (2002): "Rather than defining a bilingual learner as an individual who is proficient in more than one language, I use a situated-sociocultural definition of bilingual learners as those students who participate in multiple-language communities" (pp. 197–198).

Mathematical Problems

On three occasions, students were given two addition problems or two subtraction problems. One problem for each day was taken from the worksheets Technical Education Research Centers (TERC) (1998) regularly assigned to students in class, and the other was created by the researcher. Each researcher-created problem presented (a) a context familiar to the student, (b) an action involving the student, (c) a grade-appropriate set of numbers on which to perform the action, and (d) a mathematical relationship between the numbers (Carpenter et al., 1999). Contexts included fantasy worlds (e.g., a castle with a tower) and activities that children enjoy (e.g., catching fireflies, watching animation movies). Two purposes of these contexts were: (a) to include either direct or implied actions on quantities that can be directly represented via direct modeling or counting (Carpenter et al.) and (b) to involve students in a more physically observable problem-solving process and thus be able to document the parallel production of gestures and words.

The first pair of problems involved comparisons between two sets (24 + ? = 48; 29 + ? = 45). The second pair involved an action performed on one set, resulting in a new set (46 – 30 = ?; 50 – 17 = ?). The third pair presented a static set relation between two sets (26 + 36 = ?; 55 + 49 = ?). Students solved these problems in the language of their choice, which was predominantly Spanish. The English version of all problems appears in Appendix A.

Validity

Data for this study consisted of video excerpts in which students were solving problems and communicating their solutions to the researcher. The criterion for the selection of video excerpts was to select only the portions where students were engaged in problem solving. For example, if a student was expressing frustration, this behavior was not considered for the analysis since the focus of the study is on the articulation and gesticulation of mathematical knowledge, not of affective states. With this criterion, the researcher proceeded to transcribe each video clip in order to generate the articulation of mathematical knowledge. By articulation the researcher refers to the process by which participants put their ideas into words.

The interpretive nature of this study required consideration of validity, which was addressed by:

1. Viewing each tape three times, allowing some time between each view, in order to refine description and interpretation of gestures, a process that Bentz and Shaphiro (1998) describe as “the returning to the object of inquiry again and again, each time with an increased understanding and a more complete interpretive account” (p. 110).

2. Asking the classroom teacher to assess the accuracy of the analysis by showing her the videotapes and interpretation of gestures, since the teacher was more familiar with students' idiosyncratic behavior. During this phase, the researcher and the teacher ruled out two instances of nonverbal behavior because of disagreement, and agreed on the rest.

Data Description

Students explained their mathematical reasoning to others and to themselves using both verbal and nonverbal communication. When a student expressed frustration resulting from difficulty executing a strategy (e.g., miscounting, losing track of sets of numbers, inefficient solution strategy, etc.), the researcher intervened by reading the problem again, suggesting that the student consider alternative strategies, or using different modeling materials. This was done because the goal of the research was to document the students' communication of their mathematical thinking, not their frustration. Each problem solver's articulation and gesticulation are summarized in tables. Only the days and problems that simultaneously generated gestures and words are reported. (See Appendix B for details about participants' parallel articulation and gesticulation of mathematical knowledge.)

Results and Discussion

Remarkably, nearly all participants were able to communicate, both verbally and nonverbally, their mathematical reasoning as they solved problems. To do this they extracted the mathematical ideas in each problem and communicated their reasoning about those ideas by mutually supporting their verbal and nonverbal behaviors. Two trends are identified here: speech and gestures that support the students' communication of mathematical knowledge, and speech and gestures that support the students' development of mathematical understanding.

Speech and Gestures Support the Students' Communication of Mathematical Knowledge

Four students (PS1, PS2, PS4, and PS7) executed a sweeping hand movement to indicate the total in a problem, a behavior that *repeats* or adds emphasis to verbal behavior (Weimann & Weimann, 1975). Two students (PS2 and PS7) pointed to numbers for which they did not recall the name, and one student (PS7) made the shape of a square with both hands to request the 100s chart. Both behaviors illustrate the *substitution* function of gestures (Weimann & Weimann). Six students (PS1, PS2, PS3, PS4, PS6, and PS7) simultaneously used nonverbal and verbal behaviors when counting the number of elements in a set by pointing or touching each element while naming the numeral that corresponded to each element counted. This behavior illustrates the complement function of gestures. Altogether, these functions of gestures support the hypothesis that gestures and speech are correlated in meaning (Alibali & Goldin-Meadow, 1993; McNeill, 1992; Graham & Argyle, 1975; Ekman & Friesen, 1972).

Speech and Gestures Support the Students' Development of Mathematical Understanding

Data suggest that students used gestures to regulate their own cognitive activity, or to explain things to themselves (Crowder & Warburton, 1995), rather than to communicate their reasoning to others. For example, PS1's indication of a total number of items with a hand gesture before articulating any verbal explanation exemplifies the primacy of gestures over verbal behavior, a hypothesis that Feyereisen and de Lannoy (1991) frame "apprehending the world as things-of-action" (p. 71). PS1 counted by pointing to each line of 10 tallies, demonstrating understanding of base-10 concepts, namely that groups of 10 can be counted directly, as opposed to one by one (Carpenter et al., 1999). PS3 invented for herself a way of keeping track of counted numbers by extending one finger for each number counted. She also executed a hand gesture to indicate putting away 10 counted numbers. PS5 resorted to finger counting as an embodiment of the skip-counting process that she initially tried through a mental strategy.

While some students used gestures to reenact actions associated with mathematical operations (e.g., PS2 associated the actions *subiendo* [going up] and *bajando* [going down] with the mathematical operations of addition and subtraction, respectively), others exhibited less of these gesture-action associations, for example, when they solved a problem mentally (PS5), when they applied a standard algorithm (PS6), or when they used a method that inhibited the production of gestures, such as a calculator (PS3).

The Role of the Task in Generating Words and Gestures

The mathematical tasks used in this study purposefully involved participants in direct or implied actions performed on grade-appropriate sets of numbers. These action-rich problems generated verbal and nonverbal behaviors that mutually supported the representation of mathematical relations and the communication of the students' mathematical reasoning about these relations. Some nonverbal behaviors, particularly those used to self-regulate cognitive activity, extend the scope of the first research question (How do second-grade bilingual students communicate their mathematical reasoning?), as they seem to be more closely related to students' intrapsychological processes than the communication of such processes. It is also important to note that when students did not produce gestures, it was the consistent result of using a solution method that inhibited gesture production (mental strategy, standard algorithm, use of calculator).

Conclusion

Parallel attention to bilingual students' words and gestures during problem-solving activity offers multiple benefits for researchers and teachers interested in the processes bilingual students use to construct and communicate their mathematical reasoning to others and to themselves: (a) a possibilities-oriented theoretical perspective is a benefit for researchers, (b) a process-oriented assessment for learning as opposed to the product-driven assessment of learning is a benefit for teachers, and (c) an indicator of and a way to develop bilingual students' relationship to mathematical knowledge is a benefit for researchers, teachers, and students.

However process oriented, these benefits affect the mathematics performance of bilingual students on product-oriented instruments, such as the state-mandated tests (Black et al., 2003). In order to realize these benefits, researchers and teachers must: (a) view speech and gestures as better predictors of students' mathematical knowledge than either modality in isolation, (b) look for how bilingual students attend to the mathematical ideas they extract from the mathematical tasks, because these ideas are not endowed with universal meaning, but their meaning finds expression in the bilingual students' speech and gestures, (c) design mathematical tasks that include actions that students can easily reenact, and quantities that they can easily represent with tangible materials so as to direct the researcher's or the teacher's attention to students' gestures as representational of numerical relationships and the mathematical reasoning about such relationships, and (d) develop understanding of students' representations of how to solve a problem by considering gestures as representational systems capable of providing insights into students' mathematical thinking.

A major argument presented in this paper is that the use of carefully designed mathematical tasks can elicit rich mathematical reasoning that bilingual students use to guide tasks to a resolution. Such mathematical reasoning in turn calls for forms of identification and interpretation that accommodate multiple forms of expression. The argument is justifiable for at least three reasons. First, given the emphasis on communication in the mathematics classroom under the reform mathematics curriculum, attention to students' nonverbal behavior may create access to mathematical meanings that may not be available in linguistic form. Also, the complexity of mathematical ideas, particularly for early-grade students, combined with the students' developing bilingual proficiency, makes it necessary to search for expressions of mathematical reasoning outside language. Finally, given the multi-levels of bilingual proficiency in bilingual classrooms, the exploration of nonverbal communication expands what it means to communicate mathematically and also redefines bilingual education as multi-resourced education.

References

Alibali, M. W., & Goldin-Meadow, S. (1993). Gesture-speech mismatch and mechanisms of learning: What the hands reveal about a child's state of mind. *Cognitive Psychology, 25*, 468–523.

Bentz, V. M., & Shaphiro, J. J. (1998). *Mindful inquiry in social research*. Thousand Oaks, CA: Sage.

Black, P., Harrison, C., Lee, C., Marshal, B., & Wiliam, D. (2003). *Assessment for learning: Putting it into practice*. Buckingham, England: Open University Press.

Bourdieu, P. (1977). *Outline of a theory of practice*. Cambridge, England: Cambridge University Press.

Brousseau, G., & Otte, M. (1991). The fragility of knowledge. In A. J. Bishop, S. Mellin-Olsen, & J. van Dormolen (Eds.), *Mathematical knowledge: Its growth through teaching* (pp. 13–36). The Netherlands: Kluwer Academic Publishers.

Brown, T. (2001). *Mathematics education and language: Interpreting hermeneutics and post-structuralism*. The Netherlands: Kluwer Academic Press.

Carpenter, T., Fennema, E., Franke, M. L., Levi, L., & Empson, S. (1999). *Children's mathematics: Cognitively Guided Instruction*. Portsmouth, NH: Heinemann.

Church, R. B., & Goldin-Meadow, S. (1986). The mismatch between gesture and speech as an index of transitional knowledge. *Cognition, 23,* 43–71.

Church, R. B., Schonert-Reichl, K., Goodman, N., Kelly, S. D., & Ayman-Nolley, S. (1995). The role of gesture and speech communication as a reflection of cognitive understanding. *Journal of Contemporary Legal Issues, 6,* 237–280.

Crowder, E. M., & Warburton, E. (1995, April). *Perspective-taking in classroom science talk.* Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco.

Dale, T., & Cuevas, G. (1987). Integrating language and mathematics learning. In J. Crandall (Ed.), *ESL through content area instruction: Mathematics, science and social studies* (pp. 9–54). Englewood Cliffs, NJ: Prentice Hall.

Davis, B. (1997). Encircling mathematical knowing and mathematical knowledge [Review of the book *Mathematics education and language: Interpreting hermeneutics and post-structuralism*]. *Journal for Research in Mathematics Education, 30*(1), 111–116.

Díaz-Soto, L. (1992). Alternate paradigms in bilingual education research. In R. V. Padilla & A. H. Benavides (Eds.), *Critical perspectives on bilingual education research* (pp. 93–109). Tempe, AZ: Bilingual Press/ Editorial Bilingüe.

Ekman, P. & Friesen, W. V. (1972). Hand movements. *Journal of Communication, 22,* 353–374.

Feyereisen, P., & de Lannoy, J. (1991). *Gestures and speech: Psychological investigations.* New York: Cambridge University Press.

Gadamer, H. G. (1962). *Truth and method.* London: Sheed and Ward.

Gadamer, H. G. (1979). *Philosophical hermeneutics*. Berkeley: University of California Press.

Gattegno, C. (1988). *The science of education*. New York: Educational Solutions.

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice.* New York: Cambridge University Press.

Goldin-Meadow, S., Alibali, M., & Church, R. B. (1993). Transitions in concept acquisition: Using the hand to read the mind. *Psychological Review, 100*(2), 279–297.

Goldin-Meadow, S., Kim, S., & Singer, M. (1999). What the teacher's hands tell the student's mind about math. *Journal of Educational Psychology, 91*(4), 720–730.

Graham, J. A., & Argyle, M. (1975). A cross-cultural study of the communication of extra-verbal meaning by gestures. *International Journal of Psychology, 10*(1), 57–67.

Habermas, J. (1972). *Knowledge and human interests*. London: Heinemann Educational.

Huttenlocher, J. (1973). Language and thought. In G. A. Miller (Ed.), *Communication, language, and meaning: Psychological perspectives* (pp. 172–184). New York: Basic Books.

Huttenlocher, J. (1976). Language and intelligence. In L. B. Resnick (Ed.), *The nature of intelligence* (pp. 261–281). Hillsdale, NJ: Erlbaum.

Kelly, S. D., Singer, M. Hicks, J., & Goldin-Meadow, S. (2002). A helping hand in assessing children's knowledge: Instructing adults to attend to gesture. *Cognition and Instruction, 20*(1), 1–26.

Kendon, A. (1987). On gesture: Its complementary relationship with speech. In A. W. Siegman & S. Feldstein (Eds.), *Nonverbal behavior and communication* (pp. 65–97). Hillsdale, NJ: Lawrence Erlbaum Associates.

Koschmann, T., & LeBaron, C. (2002). Learner articulation as interactional achievement: Studying the conversation of gesture. *Cognition and Instruction, 20*(2), 249–282.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27*(1), 29–63.

MacGregor, M., & Moore, R. (1992). *Teaching mathematics in the multicultural classroom*. Melbourne, Australia: University of Melbourne, Institute of Education.

Mason, J. (1994). Researching from the inside in mathematics education: Locating an I-you relationship. *Proceedings of the Eighteenth Conference of the Group on the Psychology of Mathematics Education, University of Lisbon, 1,* 176–194. Extended version: Centre for Mathematics Education, Open University, Milton Keynes.

McNeill, D. (1992). *Hand and mind*. Chicago: University of Chicago Press.

Mestre, J. (1981). Predicting academic achievement among bilingual Hispanic college technical students. *Educational and Psychological Measurement, 41,* 1255–1264.

Mestre, J. & Gerace, W. (1986). A study of the algebra acquisition of Hispanic and Anglo ninth graders: Research findings relevant to teacher training and classroom practice. *NABE Journal, 10,* 137–167.

Moschkovich, J. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking and Learning, 4*(2 & 3), 189–212.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Olivares, R. A. (1996). Communication in mathematics for students with limited English proficiency. In P. C. Elliot & M. J. Kenney (Eds.), *Communication in mathematics: K–12 and beyond—1996 yearbook* (pp. 219–230). Reston, VA: National Council of Teachers of Mathematics.

Pape, S. J., & Tchoshanov, M. A. (2001). The role of representation(s) in developing mathematical understanding. *Theory Into Practice, 40*(2), 118–127.

Philippot, P., Feldman, R. S., & McGee, G. (1992). Nonverbal behavioral skills in an educational context: Typical and atypical populations. In R. S. Feldman (Ed.), *Applications of nonverbal behavioral theories and research* (pp. 191–213). Hillsdale, NJ: Lawrence Erlbaum Associates.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalizations. *Mathematical Thinking and Learning, 5*(1), 37–70.

Roth, W. M. (2000). From gesture to scientific language. *Journal of Pragmatics, 32*(11), 1683–1714.

Rubenstein, R. N. (1996). Strategies to support the learning of the language of mathematics. In P. C. Elliot & M. J. Kenney (Eds.), *Communication in mathematics: K–12 and beyond—1996 yearbook* (pp. 214–218). Reston, VA: National Council of Teachers of Mathematics.

Spanos, G., Rhodes, N. C., Dale, T. C., & Crandall, J. (1988). Linguistic features of mathematical problem solving: Insights and applications. In R. Cocking & J. Mestre (Eds.), *Linguistic and cultural influences on learning mathematics* (pp. 221–240). Hillsdale, NJ: Lawrence Erlbaum Associates.

Technical Education Research Centers (1998). *Investigation in number, data, and space*. White Plains, NY: Dale Seymour.

Vygotsky, L. S., & Luria, A. (1994). Tool and symbol in child development. In I. R. van der Veer & J. Valsiner (Eds.), *The Vygotsky reader* (pp. 99–174). Oxford, England: Blackwell.

Wiemann, M. O., & Wiemann, J. M. (1975). *Nonverbal communication in the elementary classroom*. Urbana, IL: ERIC Clearinghouse on Reading and Communication Skills. (ERIC Document Reproduction Service No. ED113771)

Woodall, W. G., & Burgoon, J. K. (1981). The effects of nonverbal synchrony on message comprehension and persuasiveness. *Journal of Nonverbal Behavior, 5*, 207–223.

Appendix A

Subtractive Situations
Type of Problem: Comparison
Given on Day 1

1. Jake made 24 cookies. His mom also made cookies. There were 48 cookies altogether. How many cookies did Jake's mom make?
2. You and your best friend are inside a castle. The castle has a high tower with a stair that goes all the way to the top. You and your friend decide to go to the top of the tower, but you get there first. You counted the steps and you know there are 45 steps. When you reach the top, you call your friend on the cell phone and he says: I am on step number 29. How many more steps do I need to go? What should you answer to your friend?

Subtractive Situations, Separate (1) and Compare (2)
Type of Problem: Action Cue
Given on Day 2

1. Kira had 46 pansies in her garden. She picked up 30 to give to her father. How many did she have left?
2. You are collecting fireflies in a glass jar to see them turn on at night. You want to have 50 so you can fill a big jar. You have caught only 17. How many more do you need?

Additive Situations
Type of Problem: Static Set Relation
Given on Day 3

1. One day there were 26 ducks and 36 geese in the lake. How many birds were there in the lake?
2. Today you will see two Pokemon movies with your friends. One movie lasts 55 minutes and the other movie lasts 49 minutes. How many minutes do the two movies last altogether?

Appendix B

Problem Solver 1 (PS1)

Articulation (Day 1)	**Gesticulation (Day 1)**
In Problem 1, PS1 drew a set of 24 circles for the cookies and then another set of 24 circles, indicating that the solution was 24 by . . .	. . . executing a sweeping hand movement over the second set of 24 circles. This hand gesture preceded a verbal explanation.
In both problems, PS1 said each counting number while . . .	. . . moving the tip of the pencil from one circle to the next.
Articulation (Day 2)	**Gesticulation (Day 2)**
PS1 explains Problem 1: "*Puse filas de 10 para que se me hiciera más fácil, y aq- . . . y al último nomás son 6 y entonces 10 + 10 + 10 son 30, y esas son las que le dió a su papá y aquí son 10 + 6 = 16* [I put rows of 10 so it would be easier for me, and . . . and at the end there's only 6 and then 10 + 10 +10 is 30, and those are the ones she gave to her father and here it's 10 + 6 = 16]."	He pointed to each line of 10 circles as he said "10 + 10 + 10."
Articulation (Day 3)	**Gesticulation (Day 3)**
In both problems and as he represented the two sets of numbers with circles, PS1 named each circle with a number while . . .	. . . tapping on each circle with the tip of his pencil.

Problem Solver 2 (PS2)

Through words and gestures, PS2 expressed the important problem-solving ability of flexibly considering addition or subtraction as options for solving Problem 2, Day 1. The use of gestures also helped her connect a specific action (e.g., *subiendo*, *bajando*) to the mathematical operations of addition and subtraction.

Articulation (Day 1)	**Gesticulation (Day 1)**
Reseacher: "*Imagínate la escalera, cuando cuentas del 45 hacia atrás, ¿qué estás haciendo en la escalera?* [Imagine the stairs, when you count backwards from 45, what are you doing on the stairs?]" PS2: "*Como cuentando . . . las escaleras[escalones] . . . como . . .* [Like counting . . .the stairs (steps) . . . like . . .]"	She touched her forearm with the eraser of the pencil repeatedly as if counting the steps.
Researcher: "*¿Hacia dónde vas en la escalera?* [Which direction are you going on the stairs?]" PS2: "*Como de arriba, de arriba . . . como . . . a abajo* [Like from the top . . . the top . . . like . . . to the bottom]."	She looked up and moved the pencil up in the air.
Researcher: "*Y si cuentas del 29 al 45 ¿qué estás haciendo?* [What if you count from 29 to 45, what are you doing?]" PS2: "*Subiendo* [Going up]."	She moved her hand up and pointed up with the pencil.

In Problem 1, Day 2, PS2 experienced frustration caused by not knowing all the numbers in sequential counting, losing track of numbers while counting, and producing miscounts as evidenced by the incorrect answer of 26. In a moment when she refocused on the problem, she imparted meaning to the operations of addition and subtraction by using words and gestures:

Articulation (Day 2)	**Gesticulation (Day 2)**
"Oh! es el total . . . para atrás es quítale . . . [Oh, it's the total . . . going backwards is like take away . . .]"	She pointed to the right with her thumb.
y para enfrente es más . . . [and going forward is like add . . .]	She pointed to the left, this time with four fingers together.
Researcher: "*¿Cuál es la respuesta?* [What is the answer?]" PS2: "*Pues el número que cae es la respuesta, éste cayó* . . . [Well, the number you land on is the answer, I landed on this one . . .]"	She pointed at 26 on the chart, as a result of experiencing difficulty naming this number.
Articulation (Day 3)	**Gesticulation (Day 3)**
Researcher: How many birds were there in the lake? PS2: "*¿En el lago? ¿Cuántos* . . . *en total?* [On the lake? How many . . . in total?]	She slid the tip of the pencil on the table as she said the word "total."
To confirm PS2's understanding of the problem, the researcher asked her why she decided to add. PS2: "*Porque* . . . *uh* . . . *sumé en el problema porque dice en total* . . . [Because . . . uh . . . I added in the problem because it says in total . . .]"	She executed a flat hand movement from left to right as she said the word "total."
In Problem 2, PS2 asked: "*¿Dice cuánto duran las dos películas en total?* [Does it ask how long the two movies last in total?]"	She executed a flat hand movement from left to right as she said the word "total."

Problem Solver 3 (PS3)

In Problem 2, Day 1, PS3 located 29 on the 100s chart and counted from 29 until she reached 45. She said the answer was 16, which was correct. However, she checked her answer by adding 45 + 29, which gave her 74. The researcher challenged her incorrect answer of 74 by asking her to explain her first answer of 16 and also by reminding her that there were only 45 steps on the stairs. PS3 decided to revise her work on the 100s chart by inventing a new way:

Articulation (Day 1)	**Gesticulation (Day 1)**
For every number counted verbally from the chart, starting from 29 . . .	. . . she put out one finger, so 30 was 1, 31 was 2, and so on.
"*Aquí van 10* [So far there're 10]."	She shook off her hands after counting 10 numbers, as if she were putting away groups of 10.
Articulation (Day 2)	**Gesticulation (Day 2)**
In problem 1, PS3 used the calculator to subtract one set from the other: "46 . . . take away . . . 30 . . . 16!"	None.

Interestingly, the use of the calculator seemed to inhibit gesture production in a student who had expressed meaning through gestures the previous day when she was not using the calculator.

Articulation (Day 3)	**Gesticulation (Day 3)**
She solved Problem 1 mentally.	Verbal explanation was not accompanied by gestures.
She solved Problem 1 mentally.	Verbal explanation was not accompanied by gestures.

Problem Solver 4 (PS4)

Articulation (Day 1)	Gesticulation (Day 1)
In Problem 1, PS4 tried counting the sets mentally and with tally marks, producing miscounts in both cases.	Fast coordination of verbal counting with the nonverbal counting of tapping with the pencil on the tally marks.
In Problem 2, he represented the sets with tally marks. No miscounts produced.	Slower coordination of verbal and nonverbal (tapping) counting. Also counted the sets twice.

In both problems, PS4 represented the two sets of numbers with tally marks. When he finished Problem 2, he explained:

Articulation (Day 2)	Gesticulation (Day 2)
"*Porque hice 50, y aquí hice los 17, y los encerré así para que no me confundiera . . . y conté estos y me salieron 33* [Because I made 50, and right here I made the 17, and I circled them like this so I won't get confused . . . and I counted these and it came out 33]."	He executed a sweeping hand movement over the entire set of marks, as he said "50."
Verbal counting of each tally mark.	He tapped on tally marks with the tip of the pencil.
Articulation (Day 3)	**Gesticulation (Day 3)**
In both problems, PS4 drew tally marks. His verbal behavior occurred as he named each tally mark with a numeral.	His nonverbal behavior occurred simultaneously as he pointed to each tally mark.

Problem Solver 5 (PS 5)

Articulation (Day 1)	**Gesticulation (Day 1)**
For Problem 2, PS5 explained: "*Estoy contando en mi mente . . . de cinco en cinco para llegar más rápido* [I'm counting in my head . . . by fives so I can get there faster]." This mental strategy gave him a wrong answer. As a result, he sat back, moved both hands under the table, and looked down at his finger counting. Getting the correct answer, he exclaimed with a smile: "*¡16!*"	Hand gestures in the form of finger counting allowed him to create a clearer representation than the initial mental strategy allowed.
Articulation (Day 2)	**Gesticulation (Day 2)**
PS5 solved Problem 1 mentally and explained: "*Porque 30 . . . 46, quítale 36, al 4 le quitas 3 y se hace 1, y queda 6 y se hace 16* [Because 30 . . . 46, take away 36, you take 3 away from 4 and it becomes 1, and I still have 6, so it makes 16]."	None.
Articulation (Day 3)	**Gesticulation (Day 3)**
PS5 solved Problem 1 mentally, but before solving it, he asked: "*¿Pero tienen que ser estos y estos juntos en total?* [But it has to be these and these altogether, in total?]"	He pointed to the number of ducks and number of geese as he said the Spanish demonstrative "*estos*" twice.

Problem Solver 6 (PS6)

Articulation (Day 1)	**Gesticulation (Day 1)**
In Problem 1, PS6 explained: "*Conté primero . . . puse 24* [cubes], *y luego seguí contando hasta llegar a . . . 48* [First I counted . . . I put 24 (cubes), and then I kept counting until I got to . . . 48]."	She touched the cubes while saying the number for each cube counted.
Articulation (Day 2)	**Gesticulation (Day 2)**
PS6 solved Problem 1 using the standard subtraction algorithm. She explained: "*Le quité . . . a 4 le quité 3 y, al 6 le quité cero, y me salió 16* [I took away . . . I took 3 away from 4 and, I took zero away from 6, and it came out 16]."	None.
Articulation (Day 3)	**Gesticulation (Day 3)**
In both problems, PS6 used the standard addition algorithm.	She supported the use of the algorithm with finger counting.

Problem Solver 7 (PS7)

Articulation (Day 1)	**Gesticulation (Day 1)**
In Problem 1, PS7 represented the sets with tally marks and explained: "*Le puse . . . 24, y . . . des- . . . de aquí le seguí hasta 40 . . . digo 48, y . . . ¿Le, le debo poner el total?* [I put . . . 24, and . . . from here I continued to 40 . . . I mean 48, and . . . Should I put the total?]"	He moved his hand across the rows of tally marks repeatedly as he asked this question.

Problem Solver 7 (PS7), cont.,

Articulation (Day 1)	**Gesticulation (Day 1)**
For Problem 2, he explained: "*De aquí, hasta 29, 29 y luego* . . . [From here to 29, 29 and then . . .] (Counted the 29 marks) . . . *hasta aquí, y luego le comencé a contar aquí* [to here, and then I started counting here]."	He moved about in his chair to count the remaining marks.
"*conté estos y eran 16 para subir hasta* . . . *hasta* . . . [I counted these and it was 16 to go up to . . . to . . .]"	He tapped with the pencil on the last tally mark that corresponded to 45.
Articulation (Day 2)	**Gesticulation (Day 2)**
In problem 2, PS7 represented the sets of numbers with small circles and explained: "*Primero puse los 17, y conté uh . . . los 50 . . . y puse esta rayita para saber . . . donde* (unintelligible) [I first put 17, and counted uh . . . the 50 . . . and I put this little line to know . . . where (unintelligible)]"	He executed a sweeping hand movement over the whole set as he said: "*conté, uh . . . los 50 . . .*" [I counted, uh . . . the 50 . . .]
Articulation (Day 3)	**Gesticulation (Day 3)**
In Problem 2, PS7 represented the sets of numbers with tally marks. As he moved into larger numbers, he experienced difficulty naming the numbers sequentially. He solved this inconvenience in the following manner: "I need a . . . the 100s chart."	He made the shape of a square with his hands as he was requesting the 100s chart. He used the 100s chart not for counting, but as a reference to know the next number name in the sequence.