Article ID: 1000-341X(2006)02-0264-05 **Document code**: A

Semicommutative Subrings of Matrix Rings

LIU Zhong-kui

(Department of Mathematics, Northwest Normal University, Lanzhou 730070, China)
(E-mail: liuzk@nwnu.edu.cn)

Abstract: A ring R is called semicommutative if for every $a \in R$, $r_R(a)$ is an ideal of R. It is well-known that the n by n upper triangular matrix ring is not semicommutative for any ring R with identity when $n \geq 2$. We show that a special subring of upper triangular matrix ring over a reduced ring is semicommutative.

Key words: semicommutative ring; Armendariz ring; reduced ring.
MSC(2000): 16S50, 16U80
CLC number: O153.3

All rings considered here are associative with identity 1 ($\neq 0$). For a ring R, the notations $r_R(-)$ and $l_R(-)$ are used for the right and left, respectively, annihilator over R. A ring R is called semicommutative if for every $a \in R$, $r_R(a)$ is an ideal of R. By [1, Lemma 1.2], a ring R is semicommutative if and only if, for any $a, b \in R$, $ab = 0$ implies $aRb = 0$, if and only if any right annihilator over R is an ideal of R, and if and only if any left annihilator over R is an ideal of R. Properties, examples and counterexamples of semicommutative rings are given in [2, 3].

Let S be a ring. Define a subring A_n of the n-by-n full matrix ring $M_n(S)$ over S as follows:

$$A_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in S \right\}.$$

It was proved in [2, Proposition 1.2 and Example 1.3] that if S is a reduced ring, then the ring A_3 is semicommutative but A_n is not semicommutative for $n \geq 4$. Let S be a reduced ring. In this note we will find a semicommutative subring of A_n for any positive integer $n \geq 2$. Our method will be used to give an Armendariz subring of A_n for any positive integer $n \geq 2$.

Received date: 2004-09-13
Foundation item: the National Natural Science Foundation of China (10171082), TRAPOYT, and NWNU-KJCXGC212

Let S be a ring and let

$$R_n = \left\{ A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-2} & a & b \\ & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} & c \\ & & a_1 & \cdots & a_{n-4} & a_{n-3} & a_{n-2} \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & & & a_1 & a_2 & a_3 \\ & & & & & a_1 & a_2 \\ & & & & & & a_1 \end{pmatrix} \mid a_i, a, b, c \in S \right\}.$$

Note that if $a = c$, then the matrix A is called an upper triangular Toeplitz matrix over S[4].

Theorem 1 *If S is a reduced ring, then R_n is semicommutative.*

Proof Suppose that

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{1,n-1} & a_{1n} \\ & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} & a_{2n} \\ & & a_1 & \cdots & a_{n-4} & a_{n-3} & a_{n-2} \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & & & a_1 & a_2 & a_3 \\ & & & & & a_1 & a_2 \\ & & & & & & a_1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-2} & b_{1,n-1} & b_{1n} \\ & b_1 & b_2 & \cdots & b_{n-3} & b_{n-2} & b_{2n} \\ & & b_1 & \cdots & b_{n-4} & b_{n-3} & b_{n-2} \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & & & b_1 & b_2 & b_3 \\ & & & & & b_1 & b_2 \\ & & & & & & b_1 \end{pmatrix}$$

in R_n are such that $AB = 0$. Then

$$a_1 b_1 = 0 \tag{1}$$

$$a_1 b_2 + a_2 b_1 = 0 \tag{2}$$

$$a_1 b_3 + a_2 b_2 + a_3 b_1 = 0 \tag{3}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_1 b_{n-2} + a_2 b_{n-3} + \cdots + a_{n-2} b_1 = 0 \tag{n-2}$$

$$a_1 b_{1,n-1} + a_2 b_{n-2} + \cdots + a_{n-2} b_2 + a_{1,n-1} b_1 = 0 \tag{n-1}$$

$$a_1 b_{1n} + a_2 b_{2n} + a_3 b_{n-2} + \cdots + a_{n-2} b_3 + a_{1,n-1} b_2 + a_{1n} b_1 = 0 \tag{n}$$

$$a_1 b_{2n} + a_2 b_{n-2} + \cdots + a_{n-2} b_2 + a_{2n} b_1 = 0. \tag{n+1}$$

From (1), we see that $b_1 a_1 = 0$ since S is reduced. If we multiply (2) on the right side by a_1, then $a_1 b_2 a_1 + a_2 b_1 a_1 = 0$. Thus $a_1 b_2 a_1 = 0$ and hence $a_1 b_2 = 0$. From (2) it follows that $a_2 b_1 = 0$.

Continuing in this manner, we can show that $a_ib_j = 0$ when $i+j = 2, \cdots, n-1$. Hence $b_ja_i = 0$. Multiplying (n-1) on the right side by a_1, we obtain $0 = a_1b_{1,n-1}a_1 + a_2b_{n-2}a_1 + \cdots + a_{n-2}b_2a_1 + a_{1,n-1}b_1a_1 = a_1b_{1,n-1}a_1$. Thus $a_1b_{1,n-1} = 0$. Hence

$$a_2b_{n-2} + \cdots + a_{n-2}b_2 + a_{1,n-1}b_1 = 0. \tag{*}$$

Multiplying (*) on the right side by a_2, we obtain $0 = a_2b_{n-2}a_2 + \cdots + a_{n-2}b_2a_2 + a_{1,n-1}b_1a_2 = a_2b_{n-2}a_2$. Thus $a_2b_{n-2} = 0$. Continuing in this manner, we can show that $a_ib_j = 0$ when $i+j = n$ and $a_1b_{1,n-1} = 0$, $a_{1,n-1}b_1 = 0$. Similarly, from (n+1), it follows that $a_1b_{2n} = 0$ and $a_{2n}b_1 = 0$. Now multiplying (n) on the right side by a_1, we have $0 = a_1b_{1n}a_1 + a_2b_{2n}a_1 + a_3b_{n-2}a_1 + \cdots + a_{n-2}b_3a_1 + a_{1,n-1}b_2a_1 + a_{1n}b_1a_1 = a_1b_{1n}a_1$. Thus $a_1b_{1n} = 0$. Hence

$$a_2b_{2n} + a_3b_{n-2} + \cdots + a_{n-2}b_3 + a_{1,n-1}b_2 + a_{1n}b_1 = 0. \tag{**}$$

If we multiply (**) on the right side by a_2, then $0 = a_2b_{2n}a_2 + a_3b_{n-2}a_2 + \cdots + a_{n-2}b_3a_2 + a_{1,n-1}b_2a_2 + a_{1n}b_1a_2 = a_2b_{2n}a_2$. Thus $a_2b_{2n} = 0$. Continuing in this manner, we can show that $a_ib_j = 0$ when $i+j = n+1$, $a_{1,n-1}b_2 = 0$ and $a_{1n}b_1 = 0$. Since S is a reduced ring, it is semicommutative. So for any $a, b \in S$, $ab = 0$ implies that $aSb = 0$. Now for every

$$C = \begin{pmatrix} r_1 & r_2 & r_3 & \cdots & r_{n-2} & r_{1,n-1} & r_{1n} \\ & r_1 & r_2 & \cdots & r_{n-3} & r_{n-2} & r_{2n} \\ & & r_1 & \cdots & r_{n-4} & r_{n-3} & r_{n-2} \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & & & r_1 & r_2 & r_3 \\ & & & & & r_1 & r_2 \\ & & & & & & r_1 \end{pmatrix} \in R_n,$$

it is easy to see that $ACB = 0$. Hence R_n is semicommutative.

Corollary 2 (Proposition 1.2)[2] *Let S be a reduced ring. Then*

$$R_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} | a, b, c, d \in S \right\}$$

is a semicommutative ring.

According to [5], a ring R is called an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. (The converse is always true.) The name "Armendariz ring"was chosen because E. Armendariz [6, Lemma 1] had noted that a reduced ring satisfies this condition. Properties, examples and counterexamples of Armendariz rings are given in E.Armendariz[6], M.B.Rege and S.Chhawchharia[5], D.D.Anderson and V.Camillo[7], C.Huh, Y.Lee and A.Smoktunowicz[3], N.K.Kim and Y.Lee[8], and T.K.Lee and T.L.Wong[9]. Generalizations of Armendariz rings have been investigated in [9-12].

Note that from [8, Proposition 2 and Example 3], R_3 is an Armendariz ring when S is a reduced ring and R_n is not Armendariz for any ring S when $n \geq 4$. By analogy with the proof of

Theorem 1 we have the following result on Armendariz rings. Note that this result also follows from [13, Theorem 1.4].

Corollary 3 *If S is a reduced ring, then R_n is an Armendariz ring.*

Proof Let $f(x)=\sum_{i=0}^{p}A_ix^i$, $g(x)=\sum_{j=0}^{q}B_jx^j\in R_n[x]$ be such that $f(x)g(x)=0$. Suppose that

$$A_i=\begin{pmatrix} a_1^i & a_2^i & a_3^i & \cdots & a_{n-2}^i & a_{1,n-1}^i & a_{1n}^i \\ & a_1^i & a_2^i & \cdots & a_{n-3}^i & a_{n-2}^i & a_{2n}^i \\ & & a_1^i & \cdots & a_{n-4}^i & a_{n-3}^i & a_{n-2}^i \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & & & a_1^i & a_2^i & a_3^i \\ & & & & & a_1^i & a_2^i \\ & & & & & & a_1^i \end{pmatrix},\quad i=0,1,\cdots,p,$$

$$B_j=\begin{pmatrix} b_1^j & b_2^j & b_3^j & \cdots & b_{n-2}^j & b_{1,n-1}^j & b_{1n}^j \\ & b_1^j & b_2^j & \cdots & b_{n-3}^j & b_{n-2}^j & b_{2n}^j \\ & & b_1^j & \cdots & b_{n-4}^j & b_{n-3}^j & b_{n-2}^j \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & & & b_1^j & b_2^j & b_3^j \\ & & & & & b_1^j & b_2^j \\ & & & & & & b_1^j \end{pmatrix},\quad j=0,1,\cdots,q.$$

Let $f_1=\sum_{i=0}^{p}a_1^ix^i$, $f_2=\sum_{i=0}^{p}a_2^ix^i$, $\cdots$, $f_{n-2}=\sum_{i=0}^{p}a_{n-2}^ix^i$, $f_{1,n-1}=\sum_{i=0}^{p}a_{1,n-1}^i\ x^i$, $f_{1n}=\sum_{i=0}^{p}a_{1n}^ix^i$, $f_{2n}=\sum_{i=0}^{p}a_{2n}^ix^i$, $g_1=\sum_{j=0}^{q}b_1^jx^j$, $g_2=\sum_{j=0}^{q}b_2^jx^j$, $\cdots$, $g_{n-2}=\sum_{j=0}^{q}b_{n-2}^jx^j$, $g_{1,n-1}=\sum_{j=0}^{q}b_{1,n-1}^jx^j$, $g_{1n}=\sum_{j=0}^{q}b_{1n}^jx^j$, $g_{2n}=\sum_{j=0}^{q}b_{2n}^jx^j$. Note that $S[x]$ is a reduced ring since S is reduced. So as in the proof of Theorem 1, we obtain that $f_ig_j=0$ when $i+j=2,3,\cdots,n+1$ and $f_1g_{1,n-1}=0$, $f_{1,n-1}g_1=0$, $f_1g_{2n}=0$, $f_{2n}g_1=0$, $f_1g_{1n}=0$, $f_2g_{2n}=0$, $f_{1,n-1}g_2=0$, $f_{1n}g_1=0$. Since reduced rings are Armendariz, it follows that each coefficient of f_i annihilates each coefficient of g_j, $i+j=2,3,\cdots,n+1$, each coefficient of f_1 annihilates each coefficient of $g_{1,n-1}$, etc. Now it is easy to see that $A_iB_j=0$. Thus R_n is an Armendariz ring.

Note that every subring of an Armendariz ring is Armendariz. Thus the ring consisting of all upper triangular Toeplitz matrix over S is Armendariz when S is reduced.

Corollary 4 (Theorem 5)[7] *If R is a reduced ring, then $R[x]/(x^n)$ is an Armendariz ring, where (x^n) is the ideal generated by x^n.*

Proof It follows from the fact that the ring $R[x]/(x^n)$ is isomorphic to the ring of all upper triangular Toeplitz matrix over R.

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矩阵环的半交换子环

刘 仲 奎

(西北师范大学数学系，甘肃 兰州 730070)

摘要: 称环 R 是半交换的，如果对任意 $a \in R, r_R(a)$ 是 R 的理想. 若 $n \geq 2$, 则任意具有单位元的环 R 上的 n 阶上三角矩阵环不是半交换环. 我们证明了 reduced 环上的上三角矩阵环的一类特殊子环是半交换环.

关键词: 半交换环; Armendariz 环; reduced 环.