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A Pinching Theorem for Riemannian Foliations with Parallel Mean Curvature in a Local-Symmetric Riemannian Manifold

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Abstract We discuss the Riemannian foliations with parallel mean curvature in a local-symmetric Riemannian manifold, and obtain a pinching theorem about it.

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0. Introduction

Geometric notions in the theory of Riemannian submanifolds have their counterparts for foliations on Riemannian manifolds. The harmonic foliations on Riemannian manifolds have been extensively studied in recent years[1,3,5], many harmonic foliations which are not totally geodesic are known. It is known that under some geometric restrictions, harmonicity implies total geodesicness. But the geometric property of Riemannian foliations with parallel mean curvature in spaces is still unknown. The purpose of this paper is to study the Riemannian foliations with parallel mean curvature in a local-symmetric Riemannian manifold. Using the method of Nakagawa and Takagi[5], we calculate divergence of the vector field and obtain a pinching theorem about Ricci curvature of $\mathcal{F}$. We get the following theorem:

Theorem *Let $M^{n+p}(c)$ be a local-symmetric Riemannian manifold with constant sectional curvature $c>0$. And let $\mathcal{F}$ be a Riemannian foliation in $M^{n+p}(c)$ with parallel mean curvature H $(\neq 0)$, $n\geq 2$, and for each leaf of $\mathcal{F}$ the Ricci curvature $\mathrm{Ric}\geq(>)(n-1)c$. Then*

$$\int_{M^{n+p}(c)}\{\frac{3}{2}S^2+[(\sqrt{n}-\frac{2}{n})n^2H^2-nc]S+cn^2H^2\}*1\geq 0.$$

Corollary *With the same conditions as the theorem, for the constant curvature $c\geq 0$, if*

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$\mid -2nH^2+\sqrt{n}n^2H^2-nc \mid \geq \sqrt{6}n \mid H \mid$, and $-\frac{D}{3}<S<\frac{D}{3}$, then the second fundamental form for each leaf of $\mathcal{F}$ is parallel, where $D=\sqrt{(-2nH^2+\sqrt{n}n^2H^2-nc)^2-6cn^2H^2}-\sqrt{6}cn \mid H \mid$.

1. Preliminaries

Let (M,g) be an $(n+p)$-dimensional Riemannian manifold, and let $\mathcal{F}$ be a p-codimension foliation on M with respect to a bundle-like metric. Considering $\mathcal{F}$ as an integrable distribution on M, we denote the orthogonal distribution of $\mathcal{F}$ by $\mathcal{F}^{\perp}$, which is called the normal plane field. For any vector field X on M, we decompose it as $X=X'+X''$, where X' (resp. X'') is tangent (resp. normal)to $\mathcal{F}$. We define two tensors A and h of type (1,2) on M by

$$A(X,Y)=-(\nabla_{Y''}X'')',\quad h(X,Y)=-(\nabla_{Y'}X')'' \tag{1.1}$$

for any vector fields X and Y on M. Where ∇ denotes the Riemannian connection with respect to the Riemannian metric g of $M^{n+p}(c)$.

The restriction of h to each leaf of $\mathcal{F}$ is so-called the second fundamental form of the leaf. We define the second fundamental form B of the normal field $\mathcal{F}^{\perp}$ by Ref. [1]

$$B(X,Y)=\frac{1}{2}\{A(X,Y)+A(X,Y)\} \tag{1.2}$$

for any vector fields X and Y on M. We will use, throughout this paper, the following convention on the range of indices unless otherwise stated.

$$A,B,C,\ldots=1,\ldots,n+p;$$

$$i,j,k,\ldots=1,2,\ldots,n;$$

$$\alpha,\beta,\gamma,\ldots=n+1,\ldots,n+p.$$

Let $e_1,\ldots,e_{n+p}$ be a locally defined orthonormal frame field of M such that, restricting to $\mathcal{F}$, $e_1,\ldots,e_n$ are tangent to $\mathcal{F}$ and $e_{n+1},\ldots,e_{n+p}$ are normal to $\mathcal{F}$. Let $\{\omega_A\}$ be the dual frame field. The structure equations of M are given as follows[1]

$$\mathrm{d}\omega_A=-\omega_{AB}\wedge\omega_B,\omega_{AB}+\omega_{BA}=0, \tag{1.3}$$

$$\mathrm{d}\omega_{AB}+\omega_{AC}\wedge\omega_{CB}=\Omega_{AB},\Omega_{AB}=-\frac{1}{2}\sum R_{ABCD}\omega_C\wedge\omega_D, \tag{1.4}$$

where R_{ABCD} is the curvature tensor of M. Restricting to $\mathcal{F}$:

$$\omega_\alpha=0,\ \omega_{ij}=\omega_{\alpha\beta},\ \omega_{\alpha i}=\omega_{i\alpha}, \tag{1.5}$$

$$\omega_{\alpha i}=\sum h^\alpha_{ij}\omega_j+\sum A^j_{\alpha\beta}\omega_\beta,\ h^\alpha_{ij}=h^\alpha_{ji}=h^\beta_{jk}=h^\beta_{ik}, \tag{1.6}$$

$$\mathrm{d}\omega_i=-\omega_{ij}\wedge\omega_j,\omega_{ij}+\omega_{ji}=0, \tag{1.7}$$

$$\mathrm{d}\omega_{ij}=-\omega_{ik}\wedge\omega_{kj}+\frac{1}{2}R_{ijkl}\omega_k\wedge\omega_l, \tag{1.8}$$

$$\mathrm{d}\omega_{\alpha\beta}=-\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}+\frac{1}{2}R_{\alpha\beta kl}\omega_k\wedge\omega_l, \tag{1.9}$$

where R_{ijkl} and $R_{\alpha\beta kl}$ denote the curvature tensor of tangent connection and normal connection of $\mathcal{F}$. If the Riemannian connection ∇ on M is given by $\nabla_{e_A} e_B = \sum \omega_{CB}(e_A)e_C$, then the components h^A_{BC} (resp. A^B_{CD}) of h (resp.A) with respect to $\{e_A\}$ and $\{\omega_A\}$ are given by:

$$h^\alpha_{ij} = \omega_{\alpha i}(e_j),\ A^i_{\alpha\beta} = \omega_{\alpha i}(e_\beta). \tag{1.10}$$

The second fundamental form $\mathcal{H}$ of $\mathcal{F}$ is: $\mathcal{H} = \sum_{a,i,j} h^\alpha_{ij}\omega_i\omega_j e_\alpha$. The length square of $\mathcal{H}$ is: $S = \sum_{a,i,j}(h^\alpha_{ij})^2$. For each α, H^α denotes the matrix (h^α_{ij}). $\xi = \frac{1}{n}\sum_\alpha(\mathrm{tr}H^\alpha)e_\alpha$ is called the mean curvature vector, $H = \|\xi\|$ is called the mean curvature, where tr denotes the trace of the matrix (h^α_{ij}). The foliation $\mathcal{F}$ is said to be harmonic or minimal (resp. totally geodesic) if $\sum h^\alpha_{jj} = 0$ (resp. $h^\alpha_{jj} = 0$). The normal plane field $\mathcal{F}^\perp$ is said to be minimal if $\mathrm{tr}B = \sum A^i_{\alpha\alpha}e_i = 0$. The normal plane field $\mathcal{F}^\perp$ is said to be totally geodesic, if B=0. The metric is bundle-like if and only if $A^i_{\alpha\beta} = -A^i_{\beta\alpha}$, which implies $B = 0$. The foliations $\mathcal{F}$ with bundle-like metric is called Riemannian foliations. For a tensor field $T = (T^{A_1\ldots A_r}_{B_1\ldots B_s})$ on M, we define its 1-order covariant derivatives by[1]

$$T^{A_1\ldots A_r}_{B_1\ldots B_s C}\omega_c = dT^{A_1\ldots A_r}_{B_1\ldots B_s} - \sum T^{A_1\ldots A_{a-1},C,A_{a+1}\ldots A_r}_{B_1\ldots B_s}\omega_{CA_a} - \sum T^{A_1\ldots A_r}_{B_1\ldots B_{b-1},C,B_{b+1}\ldots B_s}\omega_{CB_b}. \tag{1.13}$$

Then we have the definiens of (h^A_{BCD}), (A^A_{BCD}). For details see Ref. [1] or [5].

2. Calculus of the divergence

A vector field $v = \sum \nu_A e_A$ on $M^{n+p}(c)$ is defined by

$$\nu_k = \sum h^\alpha_{ij}h^\alpha_{ijk}, \nu_\alpha = 0. \tag{2.0}$$

By Ref. [1], we know that the divergence of the vector field v is defined by

$$\delta v = \mathrm{div} v = \sum \nu_{AA} = \sum \nu_{kk} + \sum \nu_{\alpha\alpha}. \tag{2.1}$$

Since $\mathcal{F}$ is a Riemannian foliations, i.e., $A^i_{\alpha\beta} = -A^i_{\beta\alpha}$, we have

$$A^i_{\alpha\alpha} = 0. \tag{2.2}$$

Taking exterior differentiation of (2.0) and giving attention to (2.2), we have

$$\begin{aligned}\nu_{kk} =& \sum h^\alpha_{ijk}h^\alpha_{ijk} + \sum h^\alpha_{ij}h^\alpha_{ijkk} + \\ & \sum h^\alpha_{ij}h^\beta_{ij}h^\beta_{mk}h^\alpha_{mk} + \sum h^\alpha_{ij}h^\alpha_{jl}h^\beta_{lk}h^\beta_{kj} + \sum h^\alpha_{ij}h^\alpha_{jl}h^\beta_{lk}h^\beta_{ki} + \sum h^\alpha_{ij}h^\alpha_{ij\beta}h^\beta_{kk}, \\ \nu_{\alpha\alpha} =& \sum_{k,\alpha}\nu_k A^k_{\alpha\alpha} = 0.\end{aligned} \tag{2.3}$$

By similar calculation to Refs. [1], [5], we have

$$\begin{aligned}h^\alpha_{ij}h^\alpha_{ijkk} =& \sum h^\alpha_{ij}h^l_{kkij} - 2\sum h^\alpha_{ij}h^\beta_{ij}h^\alpha_{lk}h^\beta_{lk} + 4\sum h^\alpha_{ij}h^\beta_{jl}h^\alpha_{lk}h^\beta_{ki} - 2\sum h^\alpha_{ij}h^\alpha_{jl}h^\beta_{lk}h^\beta_{ki} - \\ & 2\sum h^\alpha_{ij}h^\beta_{jk}h^\beta_{kl}h^\alpha_{li} + 2\sum h^\alpha_{ij}h^\beta_{jl}h^\alpha_{li}h^\beta_{kk} - \sum h^\alpha_{ij}h^\alpha_{ij\beta}h^\alpha_{kk} + \sum h^\alpha_{ij}h^\alpha_{kk\beta}h^\beta_{ij} - \\ & 4\sum_\alpha(\sum_k h^\alpha_{kk})^2 + 4n\sum_{\alpha,i,j}(h^\alpha_{ij})^2.\end{aligned} \tag{2.4}$$

Now assume e_α is the mean curvature vector. Hence $\sum_k h^\beta_{kk} = 0\ (\beta \neq \alpha)$. From (1.11) we have

$$\begin{cases} \sum_{k,c} h^\alpha_{kkc}\omega_c = ndH, \\ \sum_{k,c} h^\beta_{kkc}\omega_c = nH\omega_{\beta\alpha}\ (\beta \neq \alpha). \end{cases} \tag{2.5}$$

Here H is the mean curvature. The vector e_α is parallel in the normal bundle. This is equivalent to[2]

$$\begin{cases} \omega_{\alpha\beta} = 0, \\ H^\alpha H^\beta = H^\beta H^\alpha. \end{cases} \tag{2.6}$$

From (2.5) and (2.6) we have

$$\begin{cases} \sum_{k,c} h^\alpha_{kkc}\omega_c = ndH, \\ h^\beta_{kkc} = 0\ (\beta \neq \alpha). \end{cases} \tag{2.7}$$

We take exterior differentiation of (2.7):

$$\begin{cases} h^\alpha_{kkij} = -\sum h^\beta_{kk}h^\beta_{li}h^\alpha_{lj} - 2\sum h^\alpha_{kl}h^\beta_{li}h^\beta_{kj} - \sum h^\alpha_{kk\beta}h^\beta_{ij}, \\ h^\beta_{kkij} = -\sum h^\alpha_{kk}h^\alpha_{li}h^\beta_{lj} - 2\sum h^\beta_{kl}h^\alpha_{li}h^\alpha_{kj}\ (\beta \neq \alpha). \end{cases} \tag{2.8}$$

From (2.4) and (2.8) we have

$$\begin{aligned} h^\alpha_{ij}h^\alpha_{ijkk} = & -2\sum h^\alpha_{ij}h^\beta_{ij}h^\alpha_{lk}h^\beta_{lk} + 2\sum h^\alpha_{ij}h^\beta_{jl}h^\alpha_{lk}h^\beta_{ki} - 2\sum h^\alpha_{ij}h^\alpha_{jl}h^\beta_{lk}h^\beta_{ki} - \\ & 2\sum h^\alpha_{ij}h^\beta_{jk}h^\beta_{kl}h^\alpha_{li} + \sum h^\alpha_{ij}h^\beta_{jl}h^\alpha_{li}h^\beta_{kk} - \sum h^\alpha_{ij}h^\alpha_{ij\beta}h^\beta_{kk} - \\ & c\sum_\alpha(\sum_k h^\alpha_{kk})^2 + cn\sum_{\alpha,i,j}(h^\alpha_{ij})^2. \end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.3), we have

$$\begin{aligned} \nu_{kk} = & \sum h^\alpha_{ijk}h^\alpha_{ijk} + 2\sum h^\alpha_{ij}h^\beta_{jl}h^\alpha_{lk}h^\beta_{ki} - 2\sum h^\alpha_{ij}h^\alpha_{jl}h^\beta_{lk}h^\beta_{ki} - \\ & \sum h^\alpha_{ij}h^\beta_{ij}h^\alpha_{lk}h^\beta_{lk} + \sum h^\alpha_{ij}h^\beta_{jl}h^\alpha_{li}h^\beta_{kk} - \\ & c\sum_\alpha(\sum_k h^\alpha_{kk})^2 + cn\sum_{\alpha,i,j}(h^\alpha_{ij})^2. \end{aligned} \tag{2.10}$$

Let H^α denote the matrix (h^α_{ij}). From (2.1)–(2.3), (2.10) it follows

$$\begin{aligned} \delta v = & \sum h^\alpha_{ijk}h^\alpha_{ijk} + \sum_{\alpha,\beta}\mathrm{tr}(H^\alpha H^\beta - H^\beta H^\alpha)(H^\alpha H^\beta - H^\beta H^\alpha) - \\ & \sum_{\alpha,\beta}[\mathrm{tr}(H^\alpha H^\beta)]^2 + \sum_{\alpha,\beta}[\mathrm{tr}(H^\alpha)^2 H^\beta]\mathrm{tr}(H^\beta) - \\ & c\sum_\alpha(\mathrm{tr}(H^\alpha))^2 + cn\sum_\alpha \mathrm{tr}(H^\alpha)^2. \end{aligned} \tag{2.11}$$

3. Proof of the main theorems

From Ref. [2], we have:

Lemma *Let M^n be a submanifold in $N^{n+p}(c)$. Suppose that the mean curvature of M^n, $H \neq 0$, and the Ricci curvature* Ric $\geq (>)(n-1)c$. *Then the second fundamental form about the mean* *curvature of M is semi-definite.*

Theorem *Let $M^{n+p}(c)$ be a local-symmetric Riemannian manifold with constant sectional curvature $c > 0$. And let $\mathcal{F}$ be a Riemannian foliation in $M^{n+p}(c)$ with parallel mean curvature $H(\neq 0)$, $n \geq 2$, and for each leaf of $\mathcal{F}$ the Ricci curvature* Ric $\geq (>)(n-1)c$. *Then*

$$\int_{M^{n+p}(c)} \{\frac{3}{2}S^2 + [(\sqrt{n} - \frac{2}{n})n^2H^2 - nc]S + cn^2H^2\} * 1 \geq 0.$$

Proof Assume $\mathcal{F}$ is a foliation with parallel mean curvature $H(\neq 0)$. Let $e_{n+1} = \xi/\|\xi\|$ and denote $H_\alpha = (h_{ij}^\alpha)$. Then from (2.11), we have

$$-\delta v + \sum h_{ijk}^\alpha h_{ijk}^\alpha = N(H^\alpha H^\beta - H^\beta H^\alpha) + \sum_{\alpha,\beta}[\operatorname{tr}(H^\alpha H^\beta)]^2 -$$
$$\sum_{\alpha,\beta} \operatorname{tr}[(H^\alpha)^2 H^\beta]\operatorname{tr}(H^\beta) + c\sum_\alpha (\operatorname{tr}H^\alpha)^2 - ncS. \tag{3.1}$$

Let

$$\Delta_1 = N(H^\alpha H^\beta - H^\beta H^\alpha) + \sum_{\alpha,\beta}[\operatorname{tr}(H^\alpha H^\beta)]^2, \tag{3.2}$$

$$\Delta_2 = -\sum_{\alpha,\beta} \operatorname{tr}[(H^\alpha)^2 H^\beta]\operatorname{tr}(H^\beta) + c\sum_\alpha (\operatorname{tr}H^\alpha)^2 - ncS. \tag{3.3}$$

As e_{n+1} is the mean curvature vector, we know that in (3.3), β in the first item and α in the second item must be $n+1$ so that the item is not zero. Then, we have

$$\Delta_2 = -nH\sum_{\alpha,\beta} \operatorname{tr}[(H^\alpha)^2 H^{n+1}] + cn^2H^2 - ncS. \tag{3.4}$$

Diagonalizing H^α for fixed α gives $h_{ij}^\alpha = \lambda_i \delta_{ij}$. By Schwarz inequality, we have

$$\lambda_i^2(h_{ij}^{n+1} - 2H) \leq \sqrt{\sum_i \lambda_i^4 \sum_j (h_{jj}^{n+1} - 2H)^2}$$
$$\leq \sqrt{(\sum_i \lambda_i^2)^2[\sum (h_{jj}^{n+1})^2 - 4\sum h_{jj}^{n+1} + 4nH^2]} = (\sum_i \lambda_i^2)\sqrt{\sum (h_{jj}^{n+1})^2}. \tag{3.5}$$

That is,

$$-nH\operatorname{tr}[(H^\alpha)^2 H^{n+1}] = -nH\lambda_i^2(h_{jj}^{n+1} - 2H) - 2nH^2\sum_i \lambda_i^2$$
$$\leq n \mid H \mid (\sum_i \lambda_i^2)\sqrt{\sum (h_{jj}^{n+1})^2} - 2nH^2\sum_i \lambda_i^2$$
$$\leq [\sqrt{n}\sum (h_{jj}^{n+1})^2 - 2nH^2]\sum_{i,j}(h_{ij}^\alpha)^2. \tag{3.6}$$

By the Lemma we know that the second fundamental form about the mean curvature for each leaf of $\mathcal{F}$ is semi-definited, which guarantees that $\forall j$ $(j = 1, \ldots, n)$, $h_{jj}^{n+1} \geq 0$. So we have $\sum (h_{jj}^{n+1})^2 \leq (\sum h_{jj}^{n+1})^2 = n^2H^2$. Then (3.6) becomes

$$-nH\operatorname{tr}[(H^\alpha)^2 H^{n+1}] \leq [n^2\sqrt{n}H^2 - 2nH^2]S, \tag{3.7}$$

where S is the length square of the second fundamental form of $\mathcal{F}$. By (3.7) we know that

$$\Delta_2 \le [n^2\sqrt{n}H^2 - 2nH^2]S - ncS + cn^2H^2. \tag{3.8}$$

Considering (3.3), and by Ref. [4] we know

$$\Delta_1 \le \frac{3}{2}S^2. \tag{3.9}$$

Applying (3.1), (3.8) and (3.9), we obtain

$$-\delta v + \sum h^{\alpha}_{ijk}h^{\alpha}_{ijk} \le \frac{3}{2}S^2 + [n^2\sqrt{n}H^2 - 2nH^2 - nc]S + cn^2H^2.$$

Integrating the inequality above, and by Green's theorem, we have

$$\begin{aligned} 0 &\le \int_{M^{n+p}(c)} \sum h^{\alpha}_{ijk}h^{\alpha}_{ijk} * 1 \\ &\le \int_{M^{n+p}(c)} \{\frac{3}{2}S^2 + [n^2\sqrt{n}H^2 - 2nH^2 - nc]S + cn^2H^2\} * 1. \end{aligned} \tag{3.10}$$

The Theorem is proved. □

Corollary *With the same conditions as in the theorem, for the constant curvature* $c \ge 0$, *if* $\mid -2nH^2 + \sqrt{n}n^2H^2 - nc \mid \ge \sqrt{6}n \mid H \mid$, *and* $-\frac{D}{3} < S < \frac{D}{3}$, *then the second fundamental form for each leaf of* $\mathcal{F}$ *is parallel, where*

$$D = \sqrt{(-2nH^2 + \sqrt{n}n^2H^2 - nc)^2 - 6cn^2H^2} - \sqrt{6c}n \mid H \mid .$$

Proof Considering function $\varphi = \frac{3}{2}S^2 + [n^2\sqrt{n}H^2 - 2nH^2 - nc]S + cn^2H^2$, discriminant of which is $\widetilde{\Delta} = (-2nH^2 + \sqrt{n}n^2H^2 - nc)^2 - 6cn^2H^2$, and for the factor $\widetilde{\Delta} \ge 0$, we know that there are two different real roots for the equation $\varphi(S) = 0$:

$$S_1 = \frac{(\sqrt{n} - \frac{2}{n})n^2H^2 - nc + \sqrt{\widetilde{\Delta}}}{-3}, \ S_2 = \frac{-[(\sqrt{n} - \frac{2}{n})n^2H^2 - nc] + \sqrt{\widetilde{\Delta}}}{3}.$$

When $S_1 < S < S_2$, $\varphi \le 0$. That is, $\varphi \le 0$ when $-\frac{D}{3} \le S_1 < S < S_2 \le \frac{D}{3}$. By the Theorem, we deduce that $\sum h^{\alpha}_{ijk}h^{\alpha}_{ijk} = 0$ under this condition, i.e., $h^{\alpha}_{ijk} = 0$. The Corollary obviously holds.

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