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Algebraic Properties of Toeplitz Operators on Discrete Commutative Groups

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Abstract In this paper, a generalized Toeplitz operator is defined and some of results about the classical Toeplitz operator are generalized. In particular, we obtain the necessary and sufficient condition for the product of two such Toeplitz operators to still be Toeplitz operator and the necessary and sufficient condition for such Toeplitz operator to be normal operator. Finally, a necessary condition for two such Toeplitz operators to be commutative is established.

Keywords discrete commutative group; almost stable subset; Toeplitz operator; hyponormal operator; Hankle operator; analytic Toeplitz operator.

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1. Introduction and preliminaries

Let T denote the unit circle and $\mathrm{d}\theta$ denote the normalized Lebesgue measure on the unit circle. Using the regular sense of product and sum of two functions, we can define the following function spaces on T, the square integrable function space $L^2(T) = \{f(e^{i\theta}) : \int_0^{2\pi} |f(e^{i\theta})|^2 \mathrm{d}\theta < \infty\}$, and the essential bounded function space $L^\infty(T)$, i.e., the function space which consists of all functions f such that $\{x \in T : |f(x)| > M\}$ has measure zero for sufficiently large M. If we define the norms as follows, $\forall f \in L^2(T)$, $\|f\| = (\int_0^{2\pi} |f(e^{i\theta})|^2 \mathrm{d}\theta)^{\frac{1}{2}}$, $\forall g \in L^\infty(T)$, $\|g\| = essup(g)$, i.e., the smallest M which satisfies $\{x \in T : |g(x)| > M\}$ has measure zero, then it is well known that they are Banach spaces, and if $\forall f, g \in L^2(T)$, $\langle f, g\rangle = \int_0^{2\pi} f\overline{g}\mathrm{d}\theta$, then it is also well known that $L^2(T)$ is a Hilbert space under this inner product. For any $n \in \mathcal{Z}$, let x_n denote the following function on T, $x_n(e^{i\theta}) = e^{in\theta}$. Then the classical Hardy space $H^2(T)$ is defined to be $H^2(T) = \{f \in L^2(T) : \int_0^{2\pi} f(e^{i\theta})x_n(e^{i\theta})\mathrm{d}\theta = 0,\ \forall\ n > 0\}$, which is a closed subspace of $L^2(T)$. Let P denote the orthogonal projection from $L^2(T)$ to $H^2(T)$. Then the classical Toeplitz operator and Hankle operator are defined as follows, $\forall \varphi \in L^\infty(T)$, $\forall f \in H^2(T)$, T_φ: $H^2(T) \to H^2(T)$, $T_\varphi(f) = P(\varphi f)$, $H_\varphi : H^2(T) \to H^2(T)^\perp$, $H_\varphi(f) = (I-P)(\varphi f)$, and φ is called the symbol of the operators. There are lots of references concerning the property of the classical Toeplitz operators, such as references [1]–[7]. Since it is very useful in pure and applied science,

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the research on the property of Toeplitz operators and the algebra generated by them keep attracting the interest of mathematicians. Recently, the classical Toeplitz operator is generalized from different points of view by mathematicians. For example, Murphy studied the Toeplitz operators on the discrete commutative partial ordered groups by substituting the ordered group $(\mathcal{Z}, \mathcal{Z}_+)$ with partial groups in [3]. Marco studied the property of the Toeplitz operators on the discrete groups by replacing the integer group $\mathcal{Z}$ with general discrete groups in [4]. In [5], Xu Qingxiang and Chen Xiaoman studied the property of Toeplitz operator on generalized Hardy space by introducing the concepts of finite lifting subset and total finite lifting subset. In this paper, we study the algebraic property of Toeplitz operators on discrete commutative groups by substituting the integer group $\mathcal{Z}$ with general discrete commutative group and substituting $\mathcal{Z}_+$ with almost stable subset. What follows are the preliminaries for this paper. Suppose G is a discrete commutative group, $S \subset G$, if $\forall g \in G$, $S \setminus gS$ is a finite set, then S is called an almost stable subset of G. Under the regular operation on group G, the square integrable function space is denoted as $l^2(G)$. The function δ_g is defined as follows, $\forall h \in G$,

$$\delta_g(h) = \begin{cases} 1 & g = h \\ 0 & g \neq h \end{cases}.$$

It is easy to see that $\{\delta_g : \ g \in G\}$ is orthonormal basis of $l^2(G)$. The square integrable function space on S is denoted as $l^2(S)$, which is obviously a closed subspace of $l^2(G)$. Let P_s denote the orthogonal projection from $l^2(G)$ to $l^2(S)$. $\forall g \in G$, $\forall h \in G$, $L_g(\delta_h) = \delta_{gh}$, $R_g(\delta_h) = \delta_{hg^{-1}}$, then L_g and R_g are unitary operators on $l^2(G)$. The Von-Neumann algebra generated by $\{L_g : g \in G\}$ is denoted as $W^*(G)$, and the C^*-algebra generated by $\{L_g : g \in G\}$ is denoted as $C_r^*(G)$. The Toeplitz operator on the discrete commutative group G is defined as follows, $\forall T \in W^*(G)$, $T^s : l^2(S) \to l^2(S)$, $T^s(f) = P_s(Tf) \forall f \in l^2(S)$, here T is called the symbol of Toeplitz operator T^s. If $T \in W^*(G)$ and $T(l^2(S)) \subseteq l^2(S)$, then T is called an analytical operator on G, and H^∞ represents the set of all analytical operator on G. If $T \in H^\infty$, then T^s is called an analytical Toeplitz operator on G. In this paper, G^+ represents the set $\{g \in G : gS \subseteq S\}$, and $B(l^2(G))$ represents the Banach space of bounded operators on $l^2(G)$.

2. Lemmas and main results

Lemma 2.1[5] *Suppose G is a discrete commutative group, $S \subset G$ is an almost stable subset and $T \in B(l^2(G))$. Then T is a Toeplitz operator $\Leftrightarrow \forall g \in G^+,\ (L_g^s)^* T L_g^s = T$.*

Lemma 2.2 *Suppose G is a discrete commutative group, $S \subset G$ is a countable almost stable subset, $g \in G^+$, $S \setminus gS = \{x_1, x_2, \dots, x_m\}$, and $P_m = \sum_{i=1}^m \delta_{x_i} \otimes \delta_{x_i}$. Then the identity operator $I = P_m + L_g^s (L_g^s)^*$.*

Proof Let $f \in l^2(S)$. Then $f = \sum_{i=1}^m \alpha_i \delta_{x_i} + \sum_{i=m+1}^\infty \alpha_i \delta_{gs(i)}$, where $s(i) \in S$ such that $gs(i) \in S$. Then we have

$$P_m(f) = \sum_{i=1}^m \sum_{k=1}^m \langle \alpha_i \delta_{x_i},\ \delta_{x_k} \rangle \delta_{x_k} + \sum_{i=m+1}^\infty \sum_{k=1}^m \langle \alpha_i \delta_{gs(i)},\ \delta_{x_k} \rangle \delta_{x_k}$$

$$= \sum_{i=1}^{m} \alpha_i \delta_{x_i}.$$

$$L_g^s (L_g^s)^*(f) = L_g^s \sum_{i=1}^{m} P_s(\alpha_i \delta_{g^{-1}x_i}) + L_g^s \sum_{i=m+1}^{\infty} P_s(\alpha_i \delta_{s(i)})$$
$$= 0 + \sum_{i=m+1}^{\infty} P_s(\alpha_i \delta_{s(i)}) = \sum_{i=m+1}^{\infty} \alpha_i \delta_{s(i)}.$$

Hence

$$P_m(f) + L_g^s (L_g^s)^*(f) = \sum_{i=1}^{m} \alpha_i \delta_{x_i} + \sum_{i=m+1}^{\infty} \alpha_i \delta_{s(i)} = f$$

i.e., $P_m + L_g^s (L_g^s)^* = I$. □

Lemma 2.3 *Suppose G is a discrete commutative group, $S \subset G$ is a countable almost stable subset, $g \in G^+$, $S \setminus gS = \{x_1, x_2, \ldots, x_m\}$, T_1, $T_2 \in W^*(G)$, T_1^s, T_2^s are Toeplitz operators with symbol T_1 and T_2 respectively. If $T_1^s T_2^s$ is still a Toeplitz operator, then*

$$\sum_{i=1}^{m} (L_g^s)^* T_1^s \delta_{x_i} \otimes (L_g^s)^* (T_2^s)^* \delta_{x_i} = 0.$$

Proof Since $T_1^s T_2^s$ is Toeplitz operator, by Lemma 2.1, we have $(L_g^s)^* T_1^s T_2^s L_g^s = T_1^s T_2^s$. And from Lemma 2.2, we know $I = P_m + L_g^s (L_g^s)^*$. Hence

$$(L_g^s)^* T_1^s (P_m + L_g^s (L_g^s)^*) T_2^s L_g^s = T_1^s T_2^s,$$

i.e.,

$$(L_g^s)^* T_1^s P_m T_2^s L_g^s + (L_g^s)^* T_1^s L_g^s (L_g^s)^* T_2^s L_g^s = T_1^s T_2^s.$$

Since $(L_g^s)^* T_1^s L_g^s = T_1^s$ and $(L_g^s)^* T_2^s L_g^s = T_2^s$,

$$(L_g^s)^* T_1^s P_m T_2^s L_g^s + T_1^s T_2^s = T_1^s T_2^s.$$

Thus $(L_g^s)^* T_1^s P_m T_2^s L_g^s = 0$, i.e., $(L_g^s)^* T_1^s (\sum_{i=1}^{m} \delta_{x_i} \otimes \delta_{x_i}) T_2^s L_g^s = 0$. So

$$\sum_{i=1}^{m} (L_g^s)^* T_1^s \delta_{x_i} \otimes (L_g^s)^* (T_2^s)^* \delta_{x_i} = 0.$$

Theorem 2.4 *Suppose G is a discrete commutative group and $S \subset G$ is a countable almost stable subset. If there exist $g \in G^+$ and $x_0 \in S$ such that $S \setminus gS = \{x_0\}$ and $G \setminus S = \{g^{-n} x_0 : n = 1, 2, \ldots\}$, then $T_1^s T_2^s$ is Toeplitz operator $\Leftrightarrow T_1^* \in H^\infty$ or $T_2 \in H^\infty$.*

Proof $\Rightarrow$. Because $T_1^s T_2^s$ is Toeplitz operator, by Lemma 2.3, we have

$$(L_g^s)^* T_1^s \delta_{x_0} \otimes (L_g^s)^* (T_2^s)^* \delta_{x_0} = 0.$$

So for any $x, y \in S$, we have

$$\langle (L_g^s)^* T_1^s \delta_{x_0},\ \delta_y \rangle \langle \delta_x,\ (L_g^s)^* (T_2^s)^* \delta_{x_0} \rangle = 0.$$

If there exists $x \in S$ such that $\langle \delta_x,\ (L_g^s)^*(T_2^s)^*\delta_{x_0}\rangle \neq 0$, then for any $y \in S$, $\langle (L_g^s)^* T_1^s \delta_{x_0},\ \delta_y\rangle = 0$, i.e.,

$$\begin{aligned}\langle T_1^s\delta_{x_0},\ L_g^s\delta_y\rangle &= \langle T_1^s\delta_{x_0},\ P_sL_g\delta_y\rangle = \langle T_1^s\delta_{x_0},\ P_s\delta_{gy}\rangle \\ &= \langle T_1^s\delta_{x_0},\ \delta_{gy}\rangle = \langle P_sT_1\delta_{x_0},\ \delta_{gy}\rangle \\ &= \langle T_1\delta_{x_0},\ \delta_{gy}\rangle = \langle L_g^*T_1\delta_{x_0},\ \delta_y\rangle \\ &= \langle T_1L_g^*\delta_{x_0},\ \delta_y\rangle = \langle T_1\delta_{g^{-1}x_0},\ \delta_y\rangle = 0.\end{aligned}$$

Since y is arbitrary, and $g \in G^+$, we have

$$\langle T_1\delta_{g^{-1}x_0},\ \delta_{gy}\rangle = \langle T_1\delta_{g^{-1}x_0},\ \delta_{g^2y}\rangle = \cdots = 0.$$

Therefore, we have

$$\langle T_1\delta_{g^{-1}x_0},\ \delta_y\rangle = \langle T_1\delta_{g^{-2}x_0},\ \delta_y\rangle = \langle T_1\delta_{g^{-3}x_0},\ \delta_y\rangle = \cdots = 0.$$

Since $G \setminus S = \{g^{-n}x_0 : n = 1, 2, \ldots\}$, we have $T_1(l^2(s)^\perp) \subseteq l^2(s)^\perp$, thus $T_1^*(l^2(s)) \subseteq l^2(s)$, so $T_1^* \in H^\infty$.

If there exists $y \in S$ such that $\langle (L_g^s)^*T_1^s\delta_{x_0},\ \delta_y\rangle \neq 0$, then for any $x \in S$, we have $\langle \delta_x,\ (L_g^s)^*(T_2^s)^*\delta_{x_0}\rangle = 0$. Similarly we can prove that $T_2 \in H^\infty$.

$\Leftarrow$. For any $h \in G^+,\ x,\ y \in S$, since

$$\begin{aligned}\langle (L_h^s)^*T_1^sT_2^sL_h^s\delta_x,\ \delta_y\rangle &= \langle T_1^sT_2^sP_s\delta_{hx},\ P_s\delta_{hy}\rangle = \langle T_1^sT_2^s\delta_{hx},\ \delta_{hy}\rangle \\ &= \langle T_2^s\delta_{hx},\ (T_1^s)^*\delta_{hy}\rangle = \langle P_sT_2\delta_{hx},\ P_sT_1^*\delta_{hy}\rangle,\end{aligned}$$

if $T_2 \in H^\infty$, then

$$\begin{aligned}\langle P_sT_2\delta_{hx},\ P_sT_1^*\delta_{hy}\rangle &= \langle T_2\delta_{hx},\ P_sT_1^*\delta_{hy}\rangle = \langle T_2\delta_{hx},\ T_1^*\delta_{hy}\rangle \\ &= \langle T_2L_h\delta_x,\ T_1^*L_h\delta_y\rangle = \langle L_hT_2\delta_x,\ L_hT_1^*\delta_y\rangle \\ &= \langle L_h^*L_hT_2\delta_x,\ T_1^*\delta_y\rangle = \langle T_2\delta_x,\ T_1^*\delta_y\rangle \\ &= \langle P_sT_2\delta_x,\ T_1^*\delta_y\rangle = \langle P_sT_2\delta_x,\ P_sT_1^*\delta_y\rangle \\ &= \langle T_2^s\delta_x,\ (T_1^s)^*\delta_y\rangle = \langle T_1^sT_2^s\delta_x,\ \delta_y\rangle,\end{aligned}$$

so $(L_h^s)^*T_1^sT_2^sL_h^s = T_1^sT_2^s$. By Lemma 3.1 we know $T_1^sT_2^s$ is Toeplitz operator.

If $T_1^* \in H^\infty$, similarly we can prove that $T_1^sT_2^s$ is Toeplitz operator. □

Remark 2.5 The classical Toeplitz operator is the special case with $G = Z$, $S = \{0, 1, 2, \ldots\}$, $G^+ = \{0, 1, 2, \ldots\}$, $g = 1 \in G^+$, $x_0 = 0 \in S$.

The following lemma reveals the close relationship between Toeplitz operators and Hankle operators which is parallel to the same result in classical case, we omit the proof.

Lemma 2.6 *Suppose $T_1,\ T_2 \in W^*(G)$. Then $(T_1T_2)^s = T_1^sT_2^s + (H_{T_1}^s)^*H_{T_2}^s$.*

Theorem 2.7 *Suppose G is a discrete commutative group, $S \subset G$ is a countable almost stable subset, and $T \in W^*(G)$. If there exist $g \in G^+$ and $x_0 \in S$ such that $S \setminus gS = \{x_0\},\ G \setminus S = \{g^{-n}x_0 : n = 1, 2, \ldots\}$, then Toeplitz operator T^s is normal operator $\Leftrightarrow$ for any λ, with $|\lambda| = 1$,* *we have* $P_{l^2(G\setminus S)}(Tf) = \lambda P_{l^2(G\setminus S)}(T^*f), \forall f \in l^2(s)$.

Proof $\Rightarrow$. Because T^s is normal operator, we have $(T^s)^*T^s = T^s(T^s)^*$, so

$$(L_g^s)^*(T^s)^*T^sL_g^s = (L_g^s)^*T^s(T^s)^*L_g^s.$$

Since $I = P_m + L_g^s(L_g^s)^*$, we get

$$(L_g^s)^*(T^s)^*(P_m + L_g^s(L_g^s)^*)T^sL_g^s = (L_g^s)^*T^s(P_m + L_g^s(L_g^s)^*)(T^s)^*L_g^s,$$

i.e.,

$$\begin{aligned}&(L_g^s)^*(T^s)^*P_mT^sL_g^s + (L_g^s)^*(T^s)^*L_g^s(L_g^s)^*T^sL_g^s\\ &\quad= (L_g^s)^*T^sP_m(T^s)^*L_g^s + (L_g^s)^*T^sL_g^s(L_g^s)^*(T^s)^*L_g^s.\end{aligned}$$

So we have

$$(L_g^s)^*(T^s)^*P_mT^sL_g^s = (L_g^s)^*T^sP_m(T^s)^*L_g^s.$$

$P_m = \delta_{x_0} \otimes \delta_{x_0}$ leads to

$$(L_g^s)^*(T^s)^*\delta_{x_0} \otimes \delta_{x_0}T^sL_g^s = (L_g^s)^*T^s\delta_{x_0} \otimes \delta_{x_0}(T^s)^*L_g^s,$$

i.e.,

$$(L_g^s)^*(T^s)^*\delta_{x_0} \otimes (L_g^s)^*(T^s)^*\delta_{x_0} = (L_g^s)^*T^s\delta_{x_0} \otimes (L_g^s)^*T^s\delta_{x_0}.$$

So $(L_g^s)^*(T^s)^*\delta_{x_0} = \lambda(L_g^s)^*T^s\delta_{x_0}$, here $|\lambda| = 1$ is any constant. So $\forall\ x \in S$, we have

$$\langle(L_g^s)^*(T^s)^*\delta_{x_0},\ \delta_x\rangle = \lambda\langle(L_g^s)^*T^s\delta_{x_0},\ \delta_x\rangle,$$

$$\langle(T^s)^*\delta_{x_0},\ L_g^s\delta_x\rangle = \lambda\langle T^s\delta_{x_0},\ L_g^s\delta_x\rangle,$$

$$\langle P_sT^*\delta_{x_0},\ P_sL_g\delta_x\rangle = \lambda\langle P_sT\delta_{x_0},\ P_sL_g\delta_x\rangle,$$

$$\langle P_sT^*\delta_{x_0},\ P_s\delta_{gx}\rangle = \lambda\langle P_sT\delta_{x_0},\ P_s\delta_{gx}\rangle.$$

Since $g \in G^+$, $gx \in S$, then the above equality becomes $\langle T^*\delta_{x_0},\ \delta_{gx}\rangle = \lambda\langle T\delta_{x_0},\ \delta_{gx}\rangle$, i.e., $\langle L_g^*T^*\delta_{x_0},\ \delta_x\rangle = \lambda\langle L_g^*T\delta_{x_0},\ \delta_x\rangle$. The fact that G is commutative group implies that $W^*(G)$ is commutative algebra. So $\langle T^*L_g^*\delta_{x_0},\ \delta_x\rangle = \lambda\langle TL_g^*\delta_{x_0},\ \delta_x\rangle$, thus $\langle\delta_{g^{-1}x_0},\ T\delta_x\rangle = \lambda\langle\delta_{g^{-1}x_0},\ T^*\delta_x\rangle$. Since x is arbitrary and $g \in G^+$, we know that for any n, $\langle\delta_{g^{-n}x_0},\ T\delta_x\rangle = \lambda\langle\delta_{g^{-n}x_0},\ T^*\delta_x\rangle$. $G \setminus S = \{g^{-n}x_0 : n = 1, 2, 3, \ldots\}$ means that

$$\forall f \in l^2(s),\ P_{l^2(G\setminus S)}(Tf) = \lambda P_{l^2(G\setminus S)}(T^*f).$$

$\Leftarrow$. Because $P_{l^2(G\setminus S)} = I - P_s$, and $\forall\ \lambda,\ |\lambda| = 1,\ P_{l^2(G\setminus S)}(Tf) = \lambda P_{l^2(G\setminus S)}(T^*f)$, we get that $\forall\ \lambda,\ |\lambda| = 1,\ H_T^s(f) = \lambda H_{T^*}^s(f)$. Especially, let $\lambda = 1$. We get $H_T^s(f) = H_{T^*}^s(f)$, i.e., $H_T^s = H_{T^*}^s$. By Lemma 2.4, we have

$$\begin{aligned}(TT^*)^s &= T^s(T^s)^* + (H_T^s)^*H_T^s = T^s(T^s)^* + (H_T^s)^2,\\ (T^*T)^s &= (T^s)^*T^s + H_T^sH_T^s = (T^s)^*T^s + (H_T^s)^2.\end{aligned}$$

Since $TT^* = T^*T$, $(T^s)^*T^s = T^s(T^s)^*$. So T^s is normal operator.

From the above proof we get the following corollary.

Corollary 2.8 *Suppose G is a discrete commutative group and $S \subset G$ is a countable almost stable subset. If there exist $g \in G^+$ and $x_0 \in S$ such that $S \setminus gS = \{x_0\},\ G \setminus S = \{g^{-n}x_0 : n = 1, 2, \ldots\}$ $T \in W^*(G)$, then Toeplitz operator T^s is normal operator $\Leftrightarrow \forall\ \lambda, |\lambda| = 1,\ H^s_T = \lambda H^s_{T^*}$.*

Theorem 2.9 *Suppose G is a discrete commutative group, $S \subset G$ is a countable almost stable subset and there exist $g \in G^+$ and $x_0 \in S$ such that $S \setminus gS = \{x_0\},\ G \setminus S = \{g^{-n}x_0 : n = 1, 2, \ldots\}$, and $T_1,\ T_2 \in W^*(G)$ with $T_1,\ T_2$ not both analytical or co-analytical. If T_1^s and T_2^s are commutative, then there exist constants $C_1,\ C_2$ such that*

$$((T_2^s)^* - C_1(T_1^s)^*)\delta_{x_0} \ \text{ and } \ (T_2^s - C_2T_1^s)\delta_{x_0} \in \mathrm{Ker}(L_g^s)^*.$$

Proof Since $T_1^sT_2^s = T_2^sT_1^s$, we have $(L_g^s)^*T_1^sT_2^sL_g^s = (L_g^s)^*T_2^sT_1^sL_g^s$. By Lemma 2.2 we know that $I = P_m + L_g^s(L_g^s)^* = \delta_{x_0} \otimes \delta_{x_0} + L_g^s(L_g^s)^*$, so

$$(L_g^s)^*T_1^s(\delta_{x_0} \otimes \delta_{x_0} + L_g^s(L_g^s)^*)T_2^sL_g^s = (L_g^s)^*T_2^s(\delta_{x_0} \otimes \delta_{x_0} + L_g^s(L_g^s)^*)T_1^sL_g^s,$$
$$(L_g^s)^*T_1^s(\delta_{x_0} \otimes \delta_{x_0})T_2^sL_g^s + (L_g^s)^*T_1^sL_g^s(L_g^s)^*T_2^sL_g^s = (L_g^s)^*T_2^s(\delta_{x_0} \otimes \delta_{x_0})T_1^sL_g^s +$$
$$(L_g^s)^*T_2^sL_g^s(L_g^s)^*T_1^sL_g^s.$$

Because $(L_g^s)^*T_1^sL_g^s = T_1^s$ and $(L_g^s)^*T_2^sL_g^s = T_2^s$,

$$(L_g^s)^*T_1^s\delta_{x_0} \otimes (L_g^s)^*(T_2^s)^*\delta_{x_0} + T_1^sT_2^s = (L_g^s)^*T_2^s\delta_{x_0} \otimes (L_g^s)^*(T_1^s)^*\delta_{x_0} + T_2^sT_1^s,$$

i.e.,

$$(L_g^s)^*T_1^s\delta_{x_0} \otimes (L_g^s)^*(T_2^s)^*\delta_{x_0} = (L_g^s)^*T_2^s\delta_{x_0} \otimes (L_g^s)^*(T_1^s)^*\delta_{x_0}. \tag{2.1}$$

Now that $T_1,\ T_2$ are not both analytical or co-analytical, without loss of generality, suppose T_1^s is not analytical and T_2^s is not co-analytical. Then

$$(L_g^s)^*(T_2^s)^*\delta_{x_0} \neq 0,\ (L_g^s)^*(T_1^s)^*\delta_{x_0} \neq 0.$$

In fact, if $(L_g^s)^*(T_2^s)^*\delta_{x_0} = 0$, then for any $x \in S$ we have $\langle (L_g^s)^*(T_2^s)^*\delta_{x_0},\ \delta_x\rangle = 0$, i.e., $\langle T_2^s\delta_{x_0},\ L_g^s\delta_x\rangle = 0$, thus $\langle T_2\delta_{x_0},\ L_g\delta_x\rangle = 0$, so $\langle T_2\delta_{g^{-1}x_0},\ \delta_x\rangle = 0$. Since x is arbitrary and $g \in G^+$, we get

$$\langle T_2\delta_{g^{-n}x_0},\ \delta_x\rangle = 0,\ \forall\ n \in Z^+.$$

So $T_2(l^2(s)^\perp) \subseteq l^2(s)^\perp$, thus $T_2^*(l^2(s)) \subseteq l^2(s)$, i.e., T_2 is co-analytical, resulting in a contradiction. Similarly we can prove that $(L_g^s)^*(T_1^s)^*\delta_{x_0} \neq 0$. From Equation 2.1, we have

$$\begin{aligned}&\langle (L_g^s)^*(T_1^s)^*\delta_{x_0},\ (L_g^s)^*(T_1^s)^*\delta_{x_0}\rangle (L_g^s)^*T_2^s\delta_{x_0}\\ &= \langle (L_g^s)^*(T_1^s)^*\delta_{x_0},\ (L_g^s)^*(T_2^s)^*\delta_{x_0}\rangle (L_g^s)^*T_1^s\delta_{x_0},\end{aligned}$$

i.e., $\|(L_g^s)^*(T_1^s)^*\delta_{x_0}\|^2(L_g^s)^*T_2^s\delta_{x_0} - \langle (L_g^s)^*(T_1^s)^*\delta_{x_0},\ (L_g^s)^*(T_2^s)^*\delta_{x_0}\rangle (L_g^s)^*T_1^s\delta_{x_0} = 0$. So

$$(L_g^s)^*\left(\|(L_g^s)^*(T_1^s)^*\delta_{x_0}\|^2T_2^s\delta_{x_0} - \langle (L_g^s)^*(T_1^s)^*\delta_{x_0},\ (L_g^s)^*(T_2^s)^*\delta_{x_0}\rangle T_1^s\delta_{x_0}\right) = 0.$$

Let $C_1 = \frac{\langle (L_g^s)^*(T_1^s)^*\delta_{x_0},\ (L_g^s)^*(T_2^s)^*\delta_{x_0}\rangle}{\|(L_g^s)^*(T_1^s)^*\delta_{x_0}\|^2}$. Then we have $(T_2^s - C_1T_1^s)\delta_{x_0} \in \mathrm{Ker}(L_g^s)^*$. Similarly, we can prove that there exists C_2 such that $((T_2^s)^* - C_2(T_1^s)^*)\delta_{x_0} \in \mathrm{Ker}(L_g^s)^*$. □

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