第26卷第2期 数 学 研 究 与 评 论 Vol.26, No.2
2006年5月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION May, 2006

Article ID: 1000-341X(2006)02-0253-11 **Document code**: A

Split Graphs with Completely Regular Endomorphism Monoids

LI Wei-min

(Department of Mathematics, Shanghai Jiaotong University, Shanghai 200240, China)
(E-mail: liyangc@online.sh.cn)

Abstract: In this paper, split graphs with complete endomorphism-regularity are characterized explicitly. Hopefully, the main idea of the proofs can also be used for other classes of graphs.

Key words: endomorphism; regularity; split graph.
MSC(2000): 05C25
CLC number: O157.5

0. Introduction

This paper is a continuation of [7], in which split graphs with a regular endomorphism monoid are characterized explicitly (Theorems 2.13 and 3.3 in [7]). In this paper, we will further characterize split graphs with a complete regular endomorphism monoid (Theorems 2.14 and 2.18). The complete regularity of the full transformation semigroup $\mathcal{T}(X)$ is also investigated as a special case (Corollary 2.19).

It seems to be generally agreed that one of the most important semigroups associated with a graph is the endomorphism monoid. Quite a few research papers have been devoted to this theme. For a survey see [6] and [11]. The researches in these lines are motivated by the applications of semigroup theory to graph theory. Just as pointed out in [3], the class of regular semigroups is much more extensive than the class of groups, which is certainly a class of regular semigroups, and the most coherent part of semigroup theory at the present time is the part concerned with the structure of regular semigroups of various kinds. Since regularity and complete regularity are among those concepts of basic importance in semigroup theory, it seems reasonable to make some investigations to them in the endomorphism monoid of a graph.

The question, for which graph G is the endomorphism monoid of G regular, was posed in [10]. The characterization of all graphs with a regular or completely regular endomorphism monoid seems difficult. In [9], a regular endomorphism of a graph is characterized by means of idempotents. In [12], connected bipartite graphs with a regular endomorphism monoid are found. Split graphs may be regarded as the graphs between bipartite graphs and their complements[1,4]. Hopefully, the main idea (Proposition 2.4) to derive the characterization of a split graph with a

Received date: 2003-08-20

completely regular endomorphism monoid can also be used for the investigation of other classes of graphs.

1. Basic notions

The graph we consider in this paper are finite undirected graphs without loops and multiple edges. If G is a graph, we denote by $V(G)$ (sometimes simply G) and $E(G)$ its vertex set and edge set, respectively. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Moreover, if for any $x, y \in V(H)$, $\{x, y\} \in E(G)$ implies $\{x, y\} \in E(H)$, then H is called an *induced subgraph* of G. We denote by K_n a complete graph with n vertices, and $\overline{K}_n$ an empty graph with n vertices. A complete subgraph of G is called a *clique* of G. Let G be a graph and let H be a clique of G. If H has the maximal order of all the cliques of G, i.e. $|V(A)| \leq |V(H)|$ for any clique A of G, then H is called a maximal clique of G. A *stable set* of G is a set of pairwise non-adjacent vertices and a *complete set* of G is a set which induced a clique. Let $v \in G$. Denote $N(v) := \{x \in G | \{x, v\} \in E(G)\}$, called the *neighborhood* of v in G, and $d(v) := |N(v)|$, called the *degree* of v (in G).

A graph $G(V, E)$ is called a *split graph* if its vertex-set can be partitioned into two disjoint (non-empty) sets K and S, i.e. $V = K \cup S$ $(K, S \neq \emptyset)$, such that S is a stable set and K is a complete set (cf. [1,4]), that is, the subgraph of G induced by K is K_n if $|K| = n$ and the subgraph induced by S is $\overline{K}_m$ if $|S| = m$. (Note: the partition is not necessarily unique, and the clique induced by vertex-set K is also denoted by K.) A *complete split graph* is a split graph such that every vertex of S is adjacent to every vertex of K.

Let G and H be graphs. A *homomorphism* $f : G \to H$ is a vertex-mapping $V(G) \to V(H)$ which preserves adjacency, i.e. such that for any $a, b \in V(G)$, $\{a, b\} \in E(G)$ implies $\{f(a), f(b)\} \in E(H)$. Moreover, if f is bijective and its inverse mapping is also a homomorphism, then we call f an *isomorphism* from G to H, and in this case we say that G is *isomorphic* to H (under f), denoted by $G \cong H$. A homomorphism from G to itself is called an *endomorphism* of G. A bijective endomorphism of G is called an *automorphism* of G. An endomorphism f is said to be *half-strong* (cf. [5]) if $\{f(a), f(b)\} \in E(G)$ implies that there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(G)$. By $End(G)$, $hEnd(G)$ and $Aut(G)$ we denote the sets of endomorphisms, half-strong endomorphisms and automorphisms of G respectively. It is well-known that $End(G)$ is a monoid (a *monoid* is a semigroup with an identity element) and $Aut(G)$ is a group with respect to the composition of mappings. Let $f \in End(G)$ and let $a \in G$. Denote $f^{-1}(a) := \{x \in V(G) | f(x) = a\}$. If A is a subgraph of graph G, we denote by f_A the restriction of f on A and $f(A) := \{f(x) | x \in V(A)\}$.

Let $f \in End(G)$. A subgraph of G is called the *endomorphic image* of G under f, denoted by I_f, if $V(I_f) = f(G)$, and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(G)$, where $a, b, c, d \in V(G)$ (cf. [9] for the reasonableness of this definition).

Let $G(V, E)$ be a graph. Let $\rho \subseteq V \times V$ be an equivalence relation on V. Denote by $[a]_\rho$

the equivalence class of $a \in V$ under ρ. A graph, denoted by G/ρ, is called the *factor graph* of G under ρ, if $V(G/\rho) = V/\rho$ and $\{[a]_\rho, [b]_\rho\} \in E(G/\rho)$ if and only if there exist $c \in [a]_\rho, d \in [b]_\rho$ such that $\{c, d\} \in E(G)$. Let f be an endomorphism of G, by ρ_f denote the equivalence relation on $V(G)$ induced by f, i.e. for $a, b \in V(G)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. The graph G/ρ_f is simply called *the factor graph of* f. Define a mapping $i_f : V(G/\rho_f) \to V(I_f)$ with $i_f([x]_{\rho_f}) = f(x)$ for $x \in V(G)$. We now quote some concerned statements which will be used later.

Proposition 1.1[8] *Let G be a graph and let $f \in \mathrm{End}(G)$. Then the mapping i_f is an isomorphism from G/ρ_f to I_f.*

Remark 1.2[8] Let $f, g \in End(G)$. If $\rho_f = \rho_g$. then $G/\rho_f = G/\rho_g$. By Proposition 1.1, $G/\rho_f \cong I_f$ under the isomorphism i_f and $G/\rho_g \cong I_g$ under the isomorphism i_g. Thus $I_f \cong I_g$. We denote $i_{f,g} := i_g i_f^{-1}$ and $i_{g,f} := i_f i_g^{-1}$. Then $i_{f,g}$ $(i_{g,f})$ is an isomorphism from I_f to I_g (from I_g to I_f) and $i_{f,g}^{-1} = i_{g,f}$.

The following definitions are based on the book[3]. Let S be a semigroup. An idempotent is an element e of S such that $e^2 = e$. (In this paper we denote by $\mathrm{Idpt}(G)$ the set of all the idempotents in $\mathrm{End}(G)$.) An element a of a semigroup S is called *regular* if there exists x in S such that $axa = a$. A semigroup S is called *regular* if all its elements are regular. The concept of regularity was introduced by von Neumann (1936) in ring theory, where it has also played an important role. An element a of a semigroup S is called *completely regular* if there exists an element $x \in S$ such that $axa = a$ and $ax = xa$. (Trivially, an idempotent is completely regular, and a completely regular element is regular.) A semigroup S is called *completely regular* if all its elements are completely regular. Define a relation $\mathcal{L}$ on S such that $(a, b) \in \mathcal{L}$ if $S^1 a = S^1 b$ (S^1 is the semigroup obtained from S by adjoining an identity if necessary); similarly, define a relation $\mathcal{R}$ on S such that $(a, b) \in \mathcal{R}$ if $aS^1 = bS^1$. $\mathcal{L}$ and $\mathcal{R}$ are equivalence relations on S. $\mathcal{L}$ and $\mathcal{R}$ commute with each other. Define $\mathcal{H} := \mathcal{L} \cap \mathcal{R}$. These equivalence relations are called *Green's relations* on the semigroup S.

We will just say monoid instead of endomorphism monoid for a graph later on. If a graph G possesses a (completely) regular monoid, we also say G is *(completely) endomorphism-regular*. For any graph and semigroup theoretic concepts mentioned which are not defined here, please refer to usual books on graph theory and semigroup theory, eg. [2] and [3]. The following statements quoted from the references will be used later.

Proposition 1.3[8] (1) *Let $f, g \in \mathrm{End}(G)$, then $(f, g) \in \mathcal{L}$ if and only if $\rho_f = \rho_g$ and there exist $h, k \in \mathrm{End}(G)$ such that $h_{I_g} = i_{g,f}, k_{I_f} = i_{f,g}$.*

(2) *Let $f, g \in \mathrm{End}(G)$, then $(f, g) \in \mathcal{R}$ if and only if $I_f = I_g$ and there exist $u, v \in \mathrm{End}(G)$ such that for any $a \in I_f (= I_g)$, $u(f^{-1}(a)) \subseteq g^{-1}(a), v(g^{-1}(a)) \subseteq f^{-1}(a)$.*

Lemma 1.4[7] *Let G be a graph and let $f \in \mathrm{End}(G)$.*

(1) $f \in h\mathrm{End}(G)$ if and only if I_f is an induced subgraph of G.

(2) If f is regular, then $f \in h\mathrm{End}(G)$.

2. Characterization of split graphs with completely regular monoids

In this section, we will give the main results of this paper, namely, Theorems 2.14 and 2.18. First, we need some precedent statements.

Lemma 2.1 *Let G be a graph and let $f, g \in \mathrm{End}(G)$. Suppose f and g are regular.*

(1) If $\rho_f = \rho_g$, then there exist $u, v \in \mathrm{End}(G)$ such that $u_{I_f} = i_{f,g}$, $v_{I_g} = i_{g,f}$.

(2) If $I_f = I_g$, then there exist $u, v \in \mathrm{End}(G)$ such that for any $a \in I_f (= I_g)$, $u(f^{-1}(a)) \subseteq g^{-1}(a), v(g^{-1}(a)) \subseteq f^{-1}(a)$.

Proof (1) Since f is regular, there exists $h \in \mathrm{End}(G)$ such that $fhf = f$. Thus fh is an idempotent and so fh is regular. Then by Lemma 1.4 I_{fh} and I_f are both induced subgraphs of G. As $fh(G) \subseteq f(G)$ and $f(G) = fhf(G) \subseteq fh(G)$, $fh(G) = f(G)$. Therefore $I_{fh} = I_f$. Now define $u := i_{f,g}fh$. As $\rho_f = \rho_g$, u is well defined by Remark 1.2. Let $x, y \in G$ with $\{x, y\} \in E(G)$. Then $fh(x), fh(y) \in I_{fh}$ with $\{fh(x), fh(y)\} \in E(G)$. As I_{fh} is an induced subgraph, $\{fh(x), fh(y)\} \in E(I_{fh}) = E(I_f)$. So $\{u(x), u(y)\} = \{i_{f,g}(fh(x)), i_{f,g}(fh(y))\} \in E(I_g) \subseteq E(G)$. Thus $u \in \mathrm{End}(G)$. Let $a \in I_f$. Then $a = f(b)$ for some $b \in G$ and so $fh(a) = fhf(b) = f(b) = a$. Hence $u(a) = i_{f,g}(fh(a)) = i_{f,g}(a)$, which implies $u_{I_f} = i_{f,g}$. By symmetry, the existence of v can be similarly proved;

(2) Since g is regular, there exists $k \in \mathrm{End}(G)$ such that $gkg = g$. Clearly, $kg(a) = kg(b) \Leftrightarrow g(a) = g(b)$ for any $a, b \in G$ and so $\rho_{kg} = \rho_g$. So, by Remark 1.2 $i_{g,kg}$ is an isomorphism from I_g to I_{kg}. Now set $u := i_{g,kg}f$. We see u is well defined since $I_f = I_g$. Let $x, y \in G$ with $\{x, y\} \in E(G)$. As I_f is an induced subgraph, $\{f(x), f(y)\} \in E(I_f) = E(I_g)$. Thus $\{u(x), u(y)\} = \{i_{g,kg}(f(x)), i_{g,kg}(f(y))\} \in E(I_{kg}) \subseteq E(G)$, i.e. $u \in End(G)$. Now, let $a \in I_f$ and let $b \in f^{-1}(a)$. Since $I_f = I_g$, $a = g(c)$ for some $c \in G$. So, $u(b) = i_{g,kg}f(b) = i_{kg}i_g^{-1}f(b) = i_{kg}i_g^{-1}(a) = i_{kg}i_g^{-1}g(c) = i_{kg}([c]_{\rho_g}) = i_{kg}([c]_{\rho_{kg}}) = kg(c)$. Noticing $gkg(c) = g(c)$, we have $kg(c) \in g^{-1}(g(c))$, and so $u(b) \in g^{-1}(g(c)) = g^{-1}(a)$. Thus $u(f^{-1}(a)) \subseteq g^{-1}(a)$. Also by symmetry, the existence of v can be proved in a similar manner. $\Box$

Theorem 2.2 *Let G be a graph. Suppose $f, g \in \mathrm{End}(G)$ are regular. Then*

(1) $(f, g) \in \mathcal{L} \Leftrightarrow \rho_f = \rho_g$;

(2) $(f, g) \in \mathcal{R} \Leftrightarrow I_f = I_g$;

(3) $(f, g) \in \mathcal{H} \Leftrightarrow \rho_f = \rho_g$ and $I_f = I_g$.

Proof (1) $\{(f, g) \in \mathcal{L} \Rightarrow \rho_f = \rho_g\}$ follows from Proposition 1.3(1); $\{\rho_f = \rho_g \Rightarrow (f, g) \in \mathcal{L}\}$ follows from Lemma 2.1(1) and Proposition 1.3(1).

(2) $\{(f, g) \in \mathcal{R} \Rightarrow I_f = I_g\}$ follows from Proposition 1.3(2); $\{I_f = I_g \Rightarrow (f, g) \in \mathcal{R}\}$ follows from Lemma 2.1(2) and Proposition 1.3(2).

As $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, (3) follows immediately from (1) and (2). $\Box$

Proposition 2.3[3] *The following conditions on an $\mathcal{H}$-class H of a semigroup S are equivalent:*

(1) H contains an idempotent;

(2) Every element of H is completely regular;

(3) H contains a completely regular element.

Now, using the above two statements, we can deduce the following proposition which will be the main idea for the further proofs.

Proposition 2.4 *Let G be a graph. Suppose $f \in \mathrm{End}(G)$ and f is regular. Then the following four statements are equivalent:*

(1) f is completely regular;

(2) $\mathrm{Idpt}(G) \cap [f]_{\mathcal{H}} \neq \emptyset$;

(3) There exists $g \in \mathrm{End}(G)$ such that $g^2 = g$, $I_f = I_g$ and $\rho_f = \rho_g$;

(4) There exists $g \in \mathrm{End}(G)$ such that $g^2 = g$, $f(G) = g(G)$ and $\rho_f = \rho_g$.

Proof (1)$\Rightarrow$ (2). Since $[f]_{\mathcal{H}}$ contains a completely regular element f, then by Proposition 2.3(1,3) $[f]_{\mathcal{H}}$ contains an idempotent, and so $\mathrm{Idpt}(G) \cap [f]_{\mathcal{H}} \neq \emptyset$.

(2)$\Rightarrow$ (3). Let $g \in Idpt(G) \cap [f]_{\mathcal{H}}$. Clearly, $g \in \mathrm{End}(G)$ and $g^2 = g$. Then, as f and g are both regular and $f, g \in \mathcal{H}$, by Theorem 2.2(3) $I_f = I_g$ and $\rho_f = \rho_g$.

(3)$\Rightarrow$ (4). This is clear.

(4)$\Rightarrow$ (1). Since f and g are both regular, by Lemma 1.4 I_f and I_g are induced subgraphs. Hence $I_f = I_g$ follows from $f(G) = g(G)$. Then $f, g \in \mathcal{H}$ by Theorem 2.2(3), i.e. $[f]_{\mathcal{H}}$ contains an idempotent g. Then by Proposition 2.3(1,2), f is completely regular. □

Theorem 2.5[7] *Let $G(V, E)$ be a connected split graph with $V = K \cup S$ and $|K| = n$. Then G is endomorphism-regular if and only if there exists $r \in \{1, 2, \cdots, n\}$ such that $d(x) = r$ for any $x \in S$; or there exists a vertex $a \in S$ with $d(a) = n$ and there exists $r \in \{1, 2, \cdots, n-1\}$ such that $d(x) = r$ for any $x \in S \setminus \{a\}$ (if $S \setminus \{a\} \neq \emptyset$).*

Lemma 2.6[7] *Let $G(V, E)$ be a connected split graph with $V = K \cup S$ and $|K| = n$. Let $f \in \mathrm{End}(G)$. If $\max_{x\in S} d(x) \leq n-2$, then $f_K \in \mathrm{Aut}(K)$.*

Lemma 2.7 *Let $G(V, E)$ be a connected split graph with $V = K \cup S$ such that $|S| = 1$. Let $x \in G$ and $f \in \mathrm{End}(G)$. If $x \notin f(G)$, then $f(G) = G \setminus \{x\}$.*

Proof Let $|K| = n$. Noting K is a complete set, we have $|f(K)| = |K| = n$, and so since $f(K) \subseteq f(G) \subseteq G$, $n = |f(K)| \leq |f(G)| \leq |G| = |K| + |S| = n + 1$. Because $x \notin f(G)$, we see $|f(G)| = n$ and $f(G) = G \setminus \{x\}$. □

The idea for the proof of the next Proposition is mainly based on Proposition 2.4(1,4).

Proposition 2.8 *Let $G(V, E)$ be a connected split graph with $V = K \cup S$ such that $|S| = 1$. Then $\mathrm{End}(G)$ is completely regular.*

Proof Since $|S| = 1$, $\mathrm{End}(G)$ is regular by Theorem 2.5. We further show $\mathrm{End}(G)$ is completely regular. Suppose $S = \{a\}$, $|K| = n$ and $f \in \mathrm{End}(G)$. We consider three cases as follows:

Case 1. $d(a) = n$. In this case, trivially $G = K_{n+1}$ and so $\mathrm{End}(G) = \mathrm{Aut}(G)$ is a group. Thus $\mathrm{End}(G)$ is completely regular.

Case 2. $d(a) = n-1$. Clearly, there exists a unique vertex $b \in K$ such that $\{a,b\} \notin E(G)$.

First assume $|f^{-1}(f(a))| = 1$. Then $f^{-1}(f(a)) = \{a\}$ and so $f(x) \neq f(a)$ for any $x \in K$. For any $x, y \in K$, $f(x) \neq f(y)$ because $\{x,y\} \in E(G)$. Thus f is bijective as G is a finite graph, which implies $f \in Aut(G)$. Hence f is completely regular.

Now assume $|f^{-1}(f(a))| \neq 1$. For any $x \in K \setminus \{b\}$, $f(x) \neq f(a)$ because $\{x,a\} \in E(G)$, and so $x \notin f^{-1}(f(a))$. Then $b \in f^{-1}(f(a))$, because otherwise $|f^{-1}(f(a))| = 1$. Hence $f^{-1}(f(a)) = \{a,b\}$. We further consider the following two possibilities:

(1) $a \in f(G)$. Let g be a mapping from $V(G)$ to itself such that $g(b) = a$ and $g(x) = x$ for any $x \in G \setminus \{b\}$. Let $\{x,y\} \in E(G)$. If $b \notin \{x,y\}$, clearly, $\{g(x), g(y)\} = \{x,y\} \in E(G)$; if $b \in \{x,y\}$, say, $x = b$ and $y \neq b$. Since $\{b,a\} \notin E(G)$, $y \neq a$ and so $y \in G \setminus \{a,b\} = K \setminus \{b\}$. Thus $\{g(x), g(y)\} = \{a,y\} \in E(G)$. So $g \in End(G)$. It is easy to check $g^2(x) = g(x)$ for any $x \in G$, i.e. $g^2 = g$. Now we show $\rho_f = \rho_g$. Let $f(x) = f(y)$ for some $x, y \in G$. Then $\{x,y\} \notin E(G)$ and so $\{x,y\} = \{a,b\}$, say, $x = a$ and $y = b$, which implies $g(x) = g(a) = a = g(b) = g(y)$. Now let $g(x) = g(y)$ for some $x, y \in G$. Then $\{x,y\} \notin E(G)$ and so $\{x,y\} = \{a,b\}$. Noting that $f^{-1}(f(a)) = \{a,b\}$ implies $f(a) = f(b)$, we have $f(x) = f(y)$. Hence $\rho_f = \rho_g$.

Since $a \in f(G)$, then $b \notin f(G)$ (because otherwise, there exist $x, y \in G$ with $f(x) = a$ and $f(y) = b$. Since $\{a,b\} \notin E(G)$, $\{x,y\} \notin E(G)$ and so $\{x,y\} = \{a,b\}$. Thus, as $f(a) = f(b)$, $f(x) = f(y)$, i.e. $a = b$, which is a contradiction). Then by Lemma 2.7, $f(G) = G \setminus \{b\}$. On the other hand, by the definition of g, $g(G) = G \setminus \{b\}$, and so $f(G) = g(G)$. Therefore, by Proposition 2.4(1,4) f is completely regular.

(2) $a \notin f(G)$. Let g be a mapping from $V(G)$ to itself such that $g(a) = b$ and $g(x) = x$ for any $x \in G \setminus \{a\} (= K)$. Let $\{x,y\} \in E(G)$. If $x, y \in K$, $\{g(x), g(y)\} = \{x,y\} \in E(G)$; if $x = a$ and $y \in K$, then $y \neq b$ and so $\{g(x), g(y)\} = \{g(a), g(y)\} = \{b,y\} \in E(G)$. So $g \in End(G)$. It is easy to check $g^2 = g$. Let $f(x) = f(y)$ for some $x, y \in G$. Then $\{x,y\} \notin E(G)$ and so $\{x,y\} = \{a,b\}$. Since $g(a) = b = g(b)$, $g(x) = g(y)$. Now let $g(x) = g(y)$, then $\{x,y\} \notin E(G)$ and so $\{x,y\} = \{a,b\}$. So, recalling $f(a) = f(b)$, we have $f(x) = f(y)$. Hence $\rho_f = \rho_g$. Since $a \notin f(G)$, by Lemma 2.7 $f(G) = G \setminus \{a\} = K$. It is easy to see that $g(G) = K$, and so $f(G) = g(G)$. Then f is completely regular by Proposition 2.4(1,4).

Case 3. $d(a) \leq n-2$. In this case, we consider the following two possibilities:

(1) $f(a) \in K$. By Lemma 2.6 $f_K \in \mathrm{Aut}(K)$. Thus $|f_K^{-1}(f(a))| = 1$, and we may let $f_K^{-1}(f(a)) = b$ with $b \in K$. So $f(b) = f_K(b) = f(a)$. Define a mapping g from $V(G)$ to itself such that $g(a) = b$ and $g(x) = x$ for any $x \in K$. Let $\{x,y\} \in E(G)$. If $x, y \in K$, clearly $\{g(x), g(y)\} = \{x,y\} \in E(G)$; otherwise, we may let $x = a$ and $y \in K$, and so $\{g(x), g(y)\} = \{b,y\}$. Noting $f(a) = f(b)$, we see $\{a,b\} \notin E(G)$ and so $y \neq b$. Thus, $\{b,y\} \in E(K) \subseteq E(G)$, i.e. $\{g(x), g(y)\} \in E(G)$. Hence $g \in \mathrm{End}(G)$. It ie easy to check $g^2 = g$.

Let $x, y \in G$ with $f(x) = f(y)$. Then $\{x,y\} \notin E(G)$, and so we may set $x = a$ and $y \in K$. Thus $f(a) = f(x) = f(y) = f_K(y)$ and then $y = f_K^{-1}(f_K(y)) = f_K^{-1}(f(a)) = b$. So

$g(x) = g(a) = b = g(b) = g(y)$. Now let $x, y \in G$ with $g(x) = g(y)$. Then $\{x, y\} \notin E(G)$, and we may also suppose $x = a$ and $y \in K$. Thus $g(x) = g(a) = b$ and $g(y) = y$. So $y = b$ and $f(x) = f(a) = f(b) = f(y)$. Hence we have $\rho_f = \rho_g$. As $f_K \in \mathrm{Aut}(K)$ and $f(a) \in K$, $f(G) = f(K) \cup f(a) = f_K(K) \cup f(a) = K \cup f(a) = K$. By the definition of g, $g(G) = K$ and so $f(G) = g(G)$. Therefore, by Proposition 2.4(1,4) f is completely regular.

(2) $f(a) \notin K$, i.e. $f(a) = a$. As $f_K \in \mathrm{Aut}(K)$ by Lemma 2.6, we have $f \in \mathrm{Aut}(G)$. Thus f is also completely regular. □

Proposition 2.9 *Let $G(V, E)$ be a connected split graph with $V = K \cup S$ and $|K| = n$. If $|S| = 2$ such that $S = \{a, b\}$ with $\max\{d(a), d(b)\} = n$, then $\mathrm{End}(G)$ is completely regular.*

Proof We may suppose $d(a) = n$. Let $K_0 = K \cup \{a\}$ and $S_0 = \{b\}$. Then $G = G(V, E)$ is a connected split graph with $V = K_0 \cup S_0$, where K_0 is a complete set with $|K_0| = n + 1$ and S_0 is an stable set with $|S_0| = 1$. then by Proposition 2.8, $\mathrm{End}(G)$ is completely regular. □

Remark 2.10 Let G be a graph and $f \in \mathrm{Idpt}(G)$. then for any $x \in I_f$, $f(x) = x$.

Proof Since $x \in I_f$, there exists $y \in G$ such that $f(y) = x$. Noting $f^2 = f$, we see $f(x) = f(f(y)) = f(y) = x$. □

Lemma 2.11 *Let $G(V, E)$ be a connected split graph with $V = K \cup S$ such that $|K| = n$ and $|S| \geq 2$. Suppose there exists $r \in \{1, 2, \cdots, n-1\}$ such that $d(x) = r$ for any $x \in S$, then $\mathrm{End}(G)$ is not completely regular.*

Proof By Theorem 2.5, $\mathrm{End}(G)$ is regular. We now show there exists $f \in \mathrm{End}(G)$ such that f is not completely regular.

Let $S = \{a_1, a_2, \cdots, a_m\}$ $(m \geq 2)$. Then $N(a_1), N(a_2) \subseteq K$ and $|N(a_1)| = |N(a_2)| = r$. So we may let φ_1 be a bijection from $N(a_1)$ to $N(a_2)$, and let φ_2 be a bijection from $K \setminus N(a_1)$ to $K \setminus N(a_2)$. Further, let φ be a mapping from K to itself such that $\varphi_{N(a_1)} = \varphi_1$ and $\varphi_{K \setminus N(a_1)} = \varphi_2$. Noting φ is a bijection on K, $\varphi \in \mathrm{Aut}(K)$. Since $|K| = n$, $K \setminus N(a_i) \neq \emptyset$ for any $i \in \{1, 2, \cdots, m\}$, and so we may select $k_i \in K \setminus N(a_i)$. Construct a mapping f from $V(G)$ to itself in the following way: $f(x) = \varphi(x)$ if $x \in K$; $f(a_1) = a_2$; $f(a_i) = \varphi(k_i)$ if $i \in \{2, 3, \cdots, m\}$. It is easy to see f is well defined. Let $x, y \in G$ such that $\{x, y\} \in E(G)$. If $x, y \in K$, $\{f(x), f(y)\} = \{\varphi(x), \varphi(y)\} \in E(K) \subseteq E(G)$; if $x = a_1$ and $y \in K$, then $y \in N(a_1)$ and so $f(y) = \varphi(y) = \varphi_1(y) \in N(a_2)$. Thus $\{f(x), f(y)\} = \{f(a_1), f(y)\} = \{a_2, f(y)\} \in E(G)$; now if $x = a_i$ for some $i \in \{2, 3, \cdots, m\}$ and $y \in K$. Then $y \in N(a_i)$. Since $k_i \in K \setminus N(a_i)$, $\{k_i, y\} \in E(K)$ and so $\{f(x), f(y)\} = \{f(a_i), \varphi(y)\} = \{\varphi(k_i), \varphi(y)\} \in E(K) \subseteq E(G)$. Hence $f \in \mathrm{End}(G)$.

Suppose there exists $g \in \mathrm{End}(G)$ such that $g^2 = g$, $\rho_g = \rho_f$ and $I_g = I_f$. By Proposition 2.4(1,4), we only need to yield a contradiction. As $a_2 \in I_f = I_g$, by Remark 2.10 $g(a_2) = a_2$. Noting $f(a_2) = \varphi(k_2) = f(k_2)$ and $\rho_f = \rho_g$, we see $g(a_2) = g(k_2)$ and so $g(k_2) = a_2$. However, on the other hand, since $k_2 \in K$ and $\varphi \in \mathrm{Aut}(K)$, $\varphi^{-1}(k_2) \in K$ and so $f(\varphi^{-1}(k_2)) = \varphi(\varphi^{-1}(k_2)) =$

k_2, which implies $k_2 \in I_f = I_g$. Thus by Remark 2.10 $g(k_2) = k_2 \neq a_2$, which yields a contradiction. $\square$

Proposition 2.12 *Let $G(V,E)$ be a connected split graph with $V = K \cup S$, where $|K| = n$ and $|S| = 2$ such that $S = \{a,b\}$ with $\max\{d(a), d(b)\} \leq n-1$. Then $\mathrm{End}(G)$ is not completely regular.*

Proof If $d(a) = d(b)$, then by Lemma 2.11, $\mathrm{End}(G)$ is not completely regular; if $d(a) \neq d(b)$, then by Theorem 2.5, $\mathrm{End}(G)$ is not regular, and so is not completely regular. $\square$

Proposition 2.13 *Let $G(V,E)$ be a connected split graph with $V = K \cup S$ such that $|S| \geq 3$. Then $\mathrm{End}(G)$ is not completely regular.*

Proof Clearly, we only need to consider connected split graphs $G(V,E)$ with $V = K \cup S$ such that $\mathrm{End}(G)$ is regular and $|S| \geq 3$. Let $|K| = n$. Then, by Theorem 2.5 there are two cases to be considered:

Case 1. There exists $r \in \{1, 2, \cdots, n\}$ such that $d(x) = r$ for any $x \in S$. If $1 \leq r \leq n-1$, the conclusion follows from Lemma 2.11. Now we suppose $r = n$. Let $a \in S$, and then $d(a) = n$. Let $K_1 = K \cup \{a\}$ and $S_1 = S \setminus \{a\}$. Then $G(V,E)$ is a connected split graph with $V = K_1 \cup S_1$ such that $|K_1| = n+1$, $|S_1| \geq 2$ and $d(x) = n$ for any $x \in S_1$. So by Lemma 2.11, $\mathrm{End}(G)$ is not completely regular.

Case 2. There exists $a \in S$ with $d(a) = n$ and there exists $r \in \{1, 2, \cdots, n-1\}$ such that $d(x) = r$ for any $x \in S \setminus \{a\}$. Also let $K_1 = K \cup \{a\}$ and $S_1 = S \setminus \{a\}$. Then $G(V,E)$ is a connected split graph with $V = K_1 \cup S_1$, where $|K_1| = n+1$, $|S_1| \geq 2$ and there exists $r \in \{1, 2, \cdots, n-1\}$ such that $d(x) = r$ for any $x \in S_1$. Then using Lemma 2.11 once more, we see $\mathrm{End}(G)$ is not completely regular. $\square$

Now we are in the position to present the following characterization of a connected split graph with a completely regular minoid.

Theorem 2.14 *Let $G(V,E)$ be a connected split graph with $V = K \cup S$ where $|K| = n$. Then $\mathrm{End}(G)$ is completely regular if and only if either $|S| = 2$ such that $S = \{a,b\}$ with $\max\{d(a), d(b)\} = n$ or $|S| = 1$.*

Proof Sufficiency is just due to Propositions 2.8 and 2.9. Necessity follows from Propositions 2.12 and 2.13. $\square$

The next corollary follows immediately from the above theorem:

Corollary 2.15 *Let $G(V,E)$ be a complete split graph with $V = K \cup S$. Then $\mathrm{End}(G)$ is completely regular if and only if $|S| = 1$ or $|S| = 2$.*

We now further consider complete endomorphism-regularity of non-connected split graphs. An *isolated vertex* of a graph G is a vertex x with $d(x) = 0$ in G. First we quote some results from the references for further use:

Proposition 2.16 *(1) [7, Theorem 3.3] A non-connected split graph G is endomorphism-regular if and only if G exactly consists a complete graph and several isolated vertices.*

(2) [7, Lemma 2.3(1)] Let G be a graph with a unique maximal clique K. Then for any $f \in \mathrm{End}(G)$, $f_K \in \mathrm{Aut}(K)$.

Lemma 2.17 *Let $G(V, E)$ be a split graph with $V = K \cup S$ such that $d(x) = 0$ for any $x \in S$. Let $f \in \mathrm{End}(G)$. If $f(S) \subseteq K$, then f is completely regular.*

Proof By Proposition 2.16(1), $\mathrm{End}(G)$ is regular. We consider two cases:

(1) $|K| = 1$. Let $K = \{k\}$, and so $f(S) = \{k\}$. If $f(k) = k$, then clearly f is an idempotent, and so f is completely regular. Now we suppose $f(k) \neq k$, then $f(k) \in S$. Let g be a mapping from $V(G)$ to itself such that $g(x) = f(k)$ for any $x \in S$ and $g(k) = k$. It is easy to see g is well defined. Noting $E(G) = \emptyset$, $g \in \mathrm{End}(G)$. Since $f(k) \in S$, $g(f(k)) = f(k)$. So, for any $x \in S$, $g^2(x) = g(f(k)) = f(k) = g(x)$; Obviously, $g^2(k) = g(k)$. Thus $g^2 = g$. Now we show $f(G) = g(G)$ and $\rho_f = \rho_g$. $f(G) = f(S) \cup \{f(k)\} = \{k, f(k)\}$; $g(G) = g(S) \cup \{g(k)\} = \{f(k), k\}$. So $f(G) = g(G)$. Let $x, y \in G$ $(x \neq y)$. It is routine to check that $f(x) = f(y) \Leftrightarrow \{x, y\} \subseteq S \Leftrightarrow g(x) = g(y)$, which implies $\rho_f = \rho_g$. Therefore, by Proposition 2.4(1,4) f is completely regular.

(2) $|K| \geq 2$. Noting in this case K is a unique maximal clique of G, $f_K \in \mathrm{Aut}(K)$ by Proposition 2.16(2). Define a mapping g from $V(G)$ to itself in the following rule: $g(x) = x$ for any $x \in K$; $g(x) = f_K^{-1}(f(x))$ for any $x \in S$. As $f(x) \in K$, $|f_K^{-1}(f(x))| = 1$ for any $x \in S$ and so g is well defined.

Let $x, y \in G$ with $\{x, y\} \in E(G)$. Clearly $x, y \in K$ and so $\{g(x), g(y)\} = \{x, y\} \in E(G)$, i.e. $g \in \mathrm{End}(G)$. Since $g(x) \in K$ for any $x \in G$, $g(g(x)) = g(x)$, i.e. $g^2 = g$. Since $f_K \in \mathrm{Aut}(K)$, $f_K(K) = K$. So $f(G) = f(K) \cup f(S) = f_K(K) \cup f(S) = K \cup f(S) = K$; on the other hand, as $g(x) = f_K^{-1}(f(x)) \in K$ for any $x \in S$, $g(S) \subseteq K$, and so $g(G) = g(K) \cup g(S) = K \cup g(S) = K$. Hence $f(G) = g(G)$.

Now we show $\rho_f = \rho_g$. Let $x, y \in G$ with $f(x) = f(y)$. As $f \in \mathrm{End}(G)$, it is impossible that $x, y \in K$ with $x \neq y$. If $x \in S$ and $y \in K$, then $g(x) = f_K^{-1}(f(x)) = f_K^{-1}(f(y)) = f_K^{-1}(f_K(y)) = y = g(y)$; if $x, y \in S$, $g(x) = f_K^{-1}(f(x)) = f_K^{-1}(f(y)) = g(y)$. Now let $g(x) = g(y)$. Similarly, we only need to consider two possibilities: if $x \in S$ and $y \in K$, then $f(x) = f_K(f_K^{-1}(f(x))) = f_K(g(x)) = f_K(g(y)) = f_K(y) = f(y)$; if $x, y \in S$, then $f(x) = f_K(f_K^{-1}(f(x))) = f_K(g(x)) = f_K(g(y)) = f_K(f_K^{-1}(f(y))) = f(y)$. Therefore, by Proposition 2.4(1,4) f is completely regular.□

Now we can characterize a non-connected split graphs with completely regular monoid as follows:

Theorem 2.18 *Let $G(V, E)$ be a non-connected split graph with $V = K \cup S$. Then $\mathrm{End}(G)$ is completely regular if and only if $S = \{a\}$ for some $a \in G$ and $d(a) = 0$.*

Proof Necessity. By Proposition 2.16(1), $d(x) = 0$ for any $x \in S$. Assume $|S| \geq 2$, say, $S = \{a_1, a_2, \cdots, a_m\}$ $(m \geq 2)$, and we will find a contradiction. Define a mapping f from $V(G)$ to itself in the following rule: $f(x) = x$ if $x \in K$; $f(a_i) = a_1$ if $i \in \{2, 3, \cdots, m\}$; $f(a_1) = k_0$ for some $k_0 \in K$. Clearly, f is well defined. Let $x, y \in G$ with $\{x, y\} \in E(G)$. Then $x, y \in K$, and so $\{f(x), f(y)\} = \{x, y\} \in E(G)$. Thus $f \in \mathrm{End}(G)$, and so f is completely regular. Then, by Proposition 2.4(1,4) there exists $g \in \mathrm{End}(G)$ such that $g^2 = g$, $f(G) = g(G)$ and $\rho_f = \rho_g$. Since $g(G) = f(G) = K \cup \{a_1\}$, by Remark 2.10, $g(a_1) = a_1$ and $g(x) = x$ for any $x \in K$. Clearly, $g(a_2) \in K \cup \{a_1\}$. If $g(a_2) = a_1$, $g(a_2) = g(a_1)$ and so $f(a_2) = f(a_1)$ because $\rho_f = \rho_g$. Thus $a_1 = k_0$, which is a contradiction; if $g(a_2) \neq a_1$, then there exists $k \in K$ with $g(a_2) = k$. So, $g(a_2) = g(k)$, which implies $f(a_2) = f(k)$. Thus $a_1 = k$, still a contradiction.

Sufficiency. By Proposition 2.16(1), $\mathrm{End}(G)$ is regular. Let $f \in \mathrm{End}(G)$. We show f is completely regular. We consider two cases: (1) $|K| = 1$, say $K = \{k\}$. If $f(a) = k$, using Lemma 2.17 we see f is completely regular; if $f(k) = a$, noting $G = \overline{K}_2$, similarly we see f is completely regular; if $f(a) = a$ and $f(k) = k$, trivially f is completely regular. (2) $|K| \geq 2$. By Lemma 2.16(2) $f_K \in \mathrm{Aut}(K)$. If $f(a) \in K$, by Lemma 2.17, f is completely regular. Now assume $f(a) \notin K$, and then $f(a) = a$. Take g as the identity function of G (i.e. $g(x) = x$ for any $x \in G$). Clearly $g \in Idpt(G)$ and $f(G) = f(K) \cup \{f(a)\} = f_K(K) \cup \{a\} = K \cup \{a\} = g(G)$. We further show $\rho_f = \rho_g$. Let $x, y \in G$ with $g(x) = g(y)$, then $x = y$ and so $f(x) = f(y)$; Now let $x, y \in G$ with $f(x) = f(y)$. Then $\{x, y\} \notin E(G)$. Thus, if $x \neq y$, we may suppose $x = a$ and $y \in K$. So $f(x) = f(a) = a$ and $f(y) = f_K(y) \in K$, which contradicts $f(x) = f(y)$. Hence $x = y$ and so $g(x) = g(y)$. Therefore, by Proposition 2.4(1,4), f is completely regular. $\square$

Note. The above theorem can also be proved directly without using Lemma 2.17. However, since Lemma 2.17 gives a large class of completely regular endomorphisms of a non-connected split graph, it would be appropriate to derive the result independently as a lemma.

The *full transformation semigroup* $\mathcal{T}(X)$ consists of all mappings from a set X to itself with respect to the composition of mappings. It is well known that $\mathcal{T}(X)$ is regular (cf. [3, p54, Exercise 9]), however, $\mathcal{T}(X)$ is not completely regular except for trivial cases. By Theorem 2.18, this may be derived easily as follows:

Corollary 2.19 *Let X be a finite set and denote by $\mathcal{T}(X)$ the full transformation semigroup of X. Then, $\mathcal{T}(X)$ is completely regular if and only if $|X| \leq 2$.*

Proof Let $|K| = 1$. Then the assertion follows from Theorem 2.18 immediately. $\square$

Acknowledgement I would like to thank Professor Dr. U.Knauer and Dr. E.Wilkeit for helpful discussions during my study in Germany.

References:

[1] FOLDES F, HAMMER P H. *Split graphs* [C]. in: Proc. 8th South-eastern Conf. on Combinatorics, Graph Theory and Computing, F.Hoffman *et al*. (eds), Louisiana, Baton Rouge, 1977, 311–315.
[2] HARARY F. *Graph Theory* [M]. Addison-Wesley, Reading, 1969.
[3] HOWIE J M. *An Introduction to Semigroup Theory* [M]. Academic Press, New York-London, 1976.
[4] KLAVZAR S. *Absolute retracts of split graphs* [J]. Discrete Math., 1994, **134**: 75–84.
[5] KNAUER U, BÖTTCHER M. *Endomorphism spectra of graphs* [J]. Discrete Math., 1992, **109**: 45–57.
[6] KNAUER U, NIEPORTE M. *Endomorphisms of graphs I, The monoid of strong endomorphisms* [J]. Arch. Math., 1989, **52**: 607–614.

[7] LI Wei-min, CHEN Jian-fei. *Endomorphism–regularity of split graphs* [J]. European J. Combin., 2001, **22**: 207–216.
[8] LI Wei-min. *Green's relations on the endomorphism monoid of a graph* [J]. Math. Slovaca, 1995, **45**: 335–347.
[9] LI Wei-min. *A regular endomorphism of a graph and its inverses* [J]. Mathematika, 1994, **41**: 189–198.
[10] MARKI L. *Problems raised at the problem session of the Colloquium on Semigroups in Szeged, August 1987* [J]. Semigroup Forum, 1988, **37**: 367–373.
[11] MOLCHANOV V A. *Semigroups of mappings on graphs* [J]. Semigroup Forum, 1983, **27**: 155–199.
[12] WILKEIT E. *Graphs with a regular endomorphism monoid* [J]. Arch. Math., 1996, **66**: 344–352.

具有完全正则自同态半群的分裂图

李为民
(上海交通大学数学系，上海 200240)

摘要: 本文给出了具有完全正则自同态半群的分裂图的结构特征. 其证明方法有望应用于其他图族自同态半群的正则性及完全正则性的研究.

关键词: 自同态；正则性；分裂图.