

Iterative approximation of fixed points of asymptotically hemicontractive type mappings in Hilbert spaces

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Abstract: Investigates the convergence criteria of the modified Ishikawa iteration process with errors for the iterative approximation of fixed points of asymptotically hemicontractive type mappings in a real Hilbert space.

Key words: Hilbert space; asymptotically hemicontractive type mapping; modified Ishikawa iteration process with errors; fixed point

CLC number: O177.91 **Document code:** A **Article ID:** 1000-5137(2005)01-0007-06

Throughout this paper, let H be a real Hilbert space with norm $\|\cdot\|$ and K be a nonempty convex subset of H . Let $T:K \rightarrow K$ be a mapping and $F(T)$ denote the set of all fixed points of T in K .

Definition 1 A mapping $T:K \rightarrow K$ is said to be

(i) nonexpansive if

$$\|T^n x - T^n y\| \leq \|x - y\|, \quad n \geq 1, x, y \in K. \quad (1)$$

(ii) an asymptotically hemicontractive type mapping if $F(T) \neq \emptyset$ and

$$\|T^n x - x^*\|^2 \leq \|x - x^*\|^2 + \|x - T^n x\|^2, \quad n \geq 1, x \in H, x^* \in F(T). \quad (2)$$

(iii) uniformly L -Lipschitzian if there exists a positive number $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \quad n \geq 1, x, y \in K.$$

(iv) compact if T is continuous and maps each bounded subset into a relatively compact subset.

It is easy to see that the class of nonexpansive mappings with fixed points is a subclass of the class of asymptotically hemicontractive type mappings.

Definition 2 (i) The Modified Ishikawa Iteration Process with Errors. Let K be a nonempty convex subset of H , $T:K \rightarrow K$ be a mapping, $x_0 \in K$ be a given point, and $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}$ and $\{c'_n\}$ be four sequences in $[0, 1]$. Then the sequence $\{x_n\}$ defined by

Received date: 2004-01-20

Foundation items: The Teaching and Research Award Fund for Outstanding Young Teachers in Higher Education Institutions of MOE, China; The Dawn Program Fund in Shanghai.

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$$\begin{cases} x_{n+1} = a_n x_n + b_n T^n y_n + c_n u_n \\ y_n = a'_n x_n + b'_n T^n x_n + c'_n v_n, n \geq 1 \end{cases}, \quad (3)$$

is called the modified Ishikawa iterative sequence with errors of T , where $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K .

(ii) The Modified Mann Iteration Process with Errors. In (3), if $a'_n = 1$ and $b'_n = c'_n = 0$, then $y_n = x_n$. The sequence $\{x_n\}$ defined by

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n u_n, n \geq 1, \quad (4)$$

is called the modified Mann iterative sequence with errors of T .

Recently, Hu^[1] introduced the concept of p -asymptotically hemicontractive mappings in p -uniformly convex Banach spaces, and studied the problem of approximating fixed points of this class of mappings by the modified Mann and Ishikawa iteration processes. His results include Liu's corresponding results^[8] as special cases. But, unfortunately, his convergence criteria for the modified Mann and Ishikawa iteration processes are hard to be checked since these criteria involve the constant appearing in the norm inequalities in p -uniformly convex Banach spaces.

The purpose of this paper is to investigate the convergence criteria for the modified Ishikawa iteration process with errors for the iterative approximation of fixed points of asymptotically hemicontractive type mappings in a real Hilbert space. In the setting of Hilbert spaces, our results generalize, improve and develop Hu's results^[1] in the following aspects: (i) Our convergence criteria do not involve the constant appearing in the norm inequalities; (ii) Our results extend Hu's result^[1] from the modified Ishikawa iteration process to the modified Ishikawa iteration process with errors; (iii) Our results remove Hu's stronger restriction^[1],

$$0 < \varepsilon \leq \alpha_n \leq \beta_n \leq b < 1,$$

imposed on the iterative parameters $\{\alpha_n\}, \{\beta_n\}$ in the modified Ishikawa iteration process. Moreover, our results also develop and generalize Chidume and Moore's results^[4] since our results extend their method for studying hemicontractive mappings to that for studying asymptotically hemicontractive type mappings. In addition, in the setting of Hilbert spaces, our results are the extension and improvements of the corresponding results of Tan and Xu^[2] and the author^[5-7].

We shall make use of the following results as we go on.

Lemma 1^[2] Suppose that $\{\rho_n\}, \{\sigma_n\}$ are two sequences of nonnegative numbers such that for some real number $N_0 \geq 1, \rho_{n+1} \leq \rho_n + \sigma_n, n \geq N_0$.

(i) If $\sum_{n=1}^{\infty} \sigma_n < \infty$, then $\lim_{n \rightarrow \infty} \rho_n$ exists.

(ii) If $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\{\rho_n\}$ has a subsequence converging to zero, then $\lim_{n \rightarrow \infty} \rho_n = 0$.

We shall also use the following well-known identity for Hilbert spaces H :

$$\|(1-\lambda)x + \lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, x, y \in H, \lambda \in [0, 1]. \quad (5)$$

Lemma 2^[3] Let E be a normed space and K be a nonempty convex subset of E . Let $T: K \rightarrow K$ be a uniformly L -Lipschitzian mapping and $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ be real sequences in $[0, 1]$ with $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$. Let $\{u_n\}, \{v_n\}$ be bounded sequences in K . For an arbitrary $x_1 \in K$, generate a sequence $\{x_n\}$ by the modified Ishikawa iteration process with errors (3). Then for all $n \geq 1$,

$$\begin{aligned} \|x_n - Tx_n\| \leq & \|x_n - T^n x_n\| + L(1+L)^2 \|x_{n-1} - T^{n-1} x_{n-1}\| + L(1+L)c_{n-1} \|u_{n-1} - x_{n-1}\| \\ & + L^2(1+L)c'_{n-1} \|v_{n-1} - x_{n-1}\| + Lc_{n-1} \|x_{n-1} - T^{n-1} x_{n-1}\|. \end{aligned}$$

Now, we prove the following main result of this paper.

Theorem 1 Let K be a nonempty bounded closed convex subset of a real Hilbert space H , and $T:K \rightarrow K$ be a uniformly L -Lipschitzian and asymptotically hemiccontractive type mapping. Let T^n be compact for some $m \geq 1$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}$ and $\{c'_n\}$ be real sequences in $[0, 1]$ satisfying the following conditions:

- (i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, n \geq 1$;
- (ii) $\sum_{n=1}^{\infty} b_n b'_n = \infty, \sum_{n=0}^{\infty} b_n b_n'^2 < \infty$;
- (iii) $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} c'_n < \infty$;
- (iv) $0 \leq \alpha_n \leq \beta_n < 1, n \geq 1$, where $\alpha_n := b_n + c_n, \beta_n := b'_n + c'_n$.

For an arbitrary $x_1 \in K$, define a sequence $\{x_n\}_{n=1}^{\infty}$ iteratively by the modified Ishikawa iteration process with errors (3), where $\{u_n\}, \{v_n\}$ are arbitrary sequences in K . Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T .

Proof Since T is an asymptotically hemiccontractive type mapping, it is known that $F(T) \neq \emptyset$. Let $x^* \in K$ be an arbitrary fixed point of T . Using the identity (5), we obtain the following estimates: For some constants $M_1 \geq 0, M_2 \geq 0$,

$$\begin{aligned} \|y_n - x^*\|^2 &= \|(1 - \beta_n)(x_n - x^*) + \beta_n(T^n x_n - x^*) - c'_n(T^n x_n - v_n)\|^2 \leq \\ &(\|(1 - \beta_n)(x_n - x^*) + \beta_n(T^n x_n - x^*)\| + c'_n\|T^n x_n - v_n\|)^2 \leq \\ &(1 - \beta_n)\|x_n - x^*\|^2 + \beta_n\|T^n x_n - x^*\|^2 \\ &- \beta_n(1 - \beta_n)\|x_n - T^n x_n\|^2 + c'_n M_1, \\ \|y_n - T^n y_n\|^2 &= \|(1 - \beta_n)(x_n - T^n y_n) + \beta_n(T^n x_n - T^n y_n) - c'_n(T^n x_n - v_n)\|^2 \leq \\ &(1 - \beta_n)\|x_n - T^n y_n\|^2 + \beta_n\|T^n x_n - T^n y_n\|^2 \\ &- \beta_n(1 - \beta_n)\|x_n - T^n x_n\|^2 + c'_n M_2. \end{aligned}$$

Now, utilizing the estimates above and the fact that T is an asymptotically hemiccontractive type mapping, we have

$$\begin{aligned} \|T^n x_n - x^*\|^2 &\leq \|x_n - x^*\|^2 + \|x_n - T^n x_n\|^2, \\ \|T^n y_n - x^*\|^2 &\leq \|y_n - x^*\|^2 + \|y_n - T^n y_n\|^2 \leq \\ &(1 - \beta_n)\|x_n - x^*\|^2 + \beta_n\|T^n x_n - x^*\|^2 - \beta_n(1 - \beta_n)\|x_n - T^n x_n\|^2 \\ &+ (1 - \beta_n)\|x_n - T^n y_n\|^2 + \beta_n\|T^n x_n - T^n y_n\|^2 \\ &- \beta_n(1 - \beta_n)\|x_n - T^n x_n\|^2 + c'_n M_3, \end{aligned}$$

where $M_3 = M_1 + M_2$. Therefore, for some constant $M_4 \geq 0$,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \alpha_n)\|x_n - x^*\|^2 + \alpha_n\|T^n y_n - x^*\|^2 \\ &- \alpha_n(1 - \alpha_n)\|x_n - T^n y_n\|^2 + c_n M_4 \leq \\ &(1 - \alpha_n)\|x_n - x^*\|^2 + \alpha_n[\|x_n - x^*\|^2 \\ &- \beta_n(1 - 2\beta_n)\|x_n - T^n x_n\|^2 + (1 - \beta_n)\|x_n - T^n y_n\|^2 \\ &+ \beta_n\|T^n x_n - T^n y_n\|^2 + M_3 c'_n] - \alpha_n(1 - \alpha_n)\|x_n - T^n y_n\|^2 + c_n M_4 = \\ &\|x_n - x^*\|^2 - \alpha_n \beta_n(1 - 2\beta_n)\|x_n - T^n x_n\|^2 \\ &+ \alpha_n \beta_n\|T^n x_n - T^n y_n\|^2 - \alpha_n(\beta_n - \alpha_n)\|x_n - T^n y_n\|^2 M_5(c_n + c'_n), \end{aligned} \quad (8)$$

where $M_5 = \max(M_3, M_4)$, since $\alpha_n \leq 1$. Let $M_6 = \text{diam}(K)$. Then from the uniformly L -Lipschitzian conti-

nulty of T , we obtain

$$\|T^n x_n - T^n y_n\| \leq L \|x_n - y_n\| \leq L b'_n \|T^n x_n - x_n\| + L c'_n \|v_n - x_n\| \leq \beta_n L M_6.$$

Note that $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} c'_n < \infty, \sum_{n=1}^{\infty} b_n b'_n = \infty$ and $\sum_{n=1}^{\infty} b_n b'^2_n < \infty$. Since

$$\begin{aligned} \alpha_n \beta_n &= (b_n + c_n)(b'_n + c'_n) = b_n b'_n + b_n c'_n + b'_n c_n + c_n c'_n, \\ \alpha_n \beta_n^2 &= b_n b'^2_n + (b_n c'_n + b'_n c_n + c_n c'_n) b'_n + \alpha_n \beta_n c'_n, \end{aligned}$$

we deduce that $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n \beta_n^2 < \infty$.

Since $\alpha_n \leq \beta_n, n \geq 1$, it follows from (8) and (9) that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 - \alpha_n \beta_n \|x_n - T^n x_n\|^2 + 2\alpha_n \beta_n^2 M_6^2 \\ &\quad + \alpha_n \beta_n^3 L^2 M_6^2 - \alpha_n (\beta_n - \alpha_n) \|x_n - T^n y_n\|^2 + M_5 (c_n + c'_n) \leq \\ &\quad \|x_n - x^*\|^2 - \alpha_n \beta_n \|x_n - T^n x_n\|^2 + 2\alpha_n \beta_n^2 M_6^2 \\ &\quad + \alpha_n \beta_n^3 L^2 M_6^2 + M_5 (c_n + c'_n), \end{aligned} \quad (*)$$

which hence implies that

$$\alpha_n \beta_n \|x_n - T^n x_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\alpha_n \beta_n^2 M_6^2 + \alpha_n \beta_n^3 L^2 M_6^2 + M_5 (c_n + c'_n). \quad (10)$$

Let $\liminf_{n \rightarrow \infty} \|x_n - T^n x_n\|^2 = \delta \geq 0$. We claim that $\delta = 0$. Suppose that this is false, that is, $\delta > 0$. Then,

there exists an integer $N_1 \geq 1$ such that $\|x_n - T^n x_n\|^2 \geq \frac{\delta}{2}, n \geq N_1$. Hence, from (10) it follows that

$$\begin{aligned} \frac{\delta}{2} \sum_{k=N_1}^n \alpha_k \beta_k &\leq \sum_{k=N_1}^n \alpha_k \beta_k \|x_k - T^k x_k\|^2 \leq \\ &\quad \|x_{N_1} - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2M_6^2 \cdot \sum_{k=N_1}^n \alpha_k \beta_k^2 \\ &\quad + L^2 M_6^2 \sum_{k=N_1}^n \alpha_k \beta_k^3 + M_5 \cdot \sum_{k=N_1}^n (c_k + c'_k) \leq \\ &\quad \|x_{N_1} - x^*\|^2 + 2M_6^2 \cdot \sum_{k=N_1}^n \alpha_k \beta_k^2 \\ &\quad + L^2 M_6^2 \sum_{k=N_1}^n \alpha_k \beta_k^3 + M_5 \cdot \sum_{k=N_1}^n (c_k + c'_k). \end{aligned}$$

Letting $n \rightarrow \infty$, we get $\sum_{n=N_1}^{\infty} \alpha_n \beta_n < \infty$. This contradicts the fact that $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$. Hence, the claim is

valid. In view of Lemma 2, we get

$$\begin{aligned} \|x_n - T x_n\| &\leq \|x_n - T^n x_n\| + L(1+L)^2 \|x_{n-1} - T^{n-1} x_{n-1}\| \\ &\quad + L(1+L)c_{n-1} \|u_{n-1} - x_{n-1}\| \\ &\quad + L^2(1+L)c'_{n-1} \|v_{n-1} - x_{n-1}\| + Lc_{n-1} \|x_{n-1} - T^{n-1} x_{n-1}\| \end{aligned}$$

This implies that $\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0$.

Next, observe that

$$\begin{aligned} \|T^m x_n - x_n\| &\leq \|T^m x_n - T x_n\| + \|T x_n - x_n\| \leq \\ &\quad \|T^m x_n - T^2 x_n\| + \|T^2 x_n - T x_n\| + \|T x_n - x_n\| \leq \\ &\quad \cdots \leq \sum_{j=1}^{m-1} \|T^{j+1} x_n - T^j x_n\| + \|T x_n - x_n\| \leq \\ &\quad (1 + (m-1)L) \cdot \|T x_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence, $\lim_{n \rightarrow \infty} \|T^n x_n - x_n\| = 0$. Since $\{x_n\}$ is bounded and T^n is compact, $\{T^n x_n\}$ has a convergent subsequence $\{T^{n_j} x_{n_j}\}$. Suppose $\lim_{j \rightarrow \infty} T^{n_j} x_{n_j} = p$. Then

$$\|x_{n_j} - p\| \leq \|x_{n_j} - T^{n_j} x_{n_j}\| + \|T^{n_j} x_{n_j} - p\| \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Thus, $\lim_{j \rightarrow \infty} x_{n_j} = p$. Since

$$\begin{aligned} \|p - Tp\| &\leq \|p - x_{n_j}\| + \|x_{n_j} - Tx_{n_j}\| + \|Tx_{n_j} - Tp\| \leq \\ &(1 + L)\|x_{n_j} - p\| + \|x_{n_j} - Tx_{n_j}\| \rightarrow 0 \text{ as } j \rightarrow \infty, \end{aligned}$$

we obtain $Tp = p$, i.e., $p \in F(T)$.

On the other hand, it follows from (*) that for any element $x^* \in F(T)$,

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + 2\alpha_n \beta_n^2 M_5^2 + \alpha_n \beta_n^3 L^2 M_6^2 + M_5(c_n + c'_n).$$

By Lemma 1, we conclude that $\lim_{n \rightarrow \infty} \|x_{n+1} - x^*\|^2$ exists. In particular, $\lim_{n \rightarrow \infty} \|x_n - p\|^2$ exists. Since $\lim_{n \rightarrow \infty} \|x_n - p\|^2 = \lim_{j \rightarrow \infty} \|x_{n_j} - p\|^2 = 0$, we know that, $\{x_n\}$ converges strongly to the fixed point $p \in F(T)$. The proof is completed.

Remark 1 The conclusion of Theorem 1 remains true if condition (ii) in Theorem 1 is replaced by the condition: $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n \beta_n^2 < \infty$. A prototype of our parameters is as follows:

$$a_n = 1 - \frac{1}{\sqrt{(n+1)}} = a'_n; b_n = \frac{1}{\sqrt{(n+1)}} - \frac{1}{(n+1)^2} = b'_n; c_n = \frac{1}{(n+1)^2} = c'_n$$

for all integers $n \geq 1$. Observe that

$$\sum_{n=1}^{\infty} \alpha_n \beta_n = \sum_{n=1}^{\infty} \frac{1}{(n+1)} = \infty \text{ and } \sum_{n=1}^{\infty} \alpha_n \beta_n^2 = \sum_{n=1}^{\infty} \frac{1}{(n+1)^{3/2}} < \infty.$$

Corollary 1 Let K be a nonempty compact convex subset of a real Hilbert space H , and $T:K \rightarrow K$ be a uniformly L -Lipschitzian and asymptotically hemiccontractive type mapping. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}$ and $\{c'_n\}$ be real sequences in $[0, 1]$ satisfying the following conditions:

- (i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, n \geq 1$;
- (ii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty, \sum_{n=0}^{\infty} \alpha_n \beta_n^2 < \infty$;
- (iii) $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} c'_n < \infty$;
- (iv) $0 \leq \alpha_n \leq \beta_n < \infty, \forall n \geq 1$, where $\alpha_n = b_n + c_n, \beta_n = b'_n + c'_n$.

For an arbitrary $x_1 \in K$, define a sequence $\{x_n\}_{n=1}^{\infty}$ iteratively by the modified Ishikawa iteration process with errors (3), where $\{u_n\}, \{v_n\}$ are arbitrary sequences in K . Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T .

Proof The existence of a fixed point of T follows from Schauder's fixed point theorem. So, $F(T) \neq \emptyset$. It is easy to see that $T:K \rightarrow K$ is compact. Utilizing Remark 1, we can deduce that the conclusion of Corollary 1 is true.

Corollary 2 Let $K, H, \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}, \alpha_n, \beta_n$ be what we mentioned in Corollary 1. Let $T:K \rightarrow K$ be a nonexpansive mapping. For an arbitrary $x_1 \in K$, define a sequence $\{x_n\}_{n=1}^{\infty}$ iteratively by the modified Ishikawa iteration process with errors (3), where $\{u_n\}, \{v_n\}$ are arbitrary sequences in K . Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T .

Proof The existence of a fixed point of T follows from Schauder's fixed point theorem. So, $F(T) \neq \emptyset$.

Since T is nonexpansive, T is a uniformly 1-Lipschitzian and asymptotically hemicontractive type mapping. Therefore, the conclusion of Corollary 2 follows from Corollary 1.

Remark 2 Clearly, from Theorem 1, Corollaries 1 and 2, we can immediately obtain the corresponding results on the modified Mann iteration process with errors, respectively.

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Hilbert 空间中渐近半压缩型映象的不动点的迭代逼近

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摘要: 研究实 Hilbert 空间中用于迭代逼近渐近半压缩型映象不动点的带误差的修正的 Ishikawa 迭代程序的收敛判据.

关键词: Hilbert 空间; 渐近半压缩型映象; 带误差的修正的 Ishikawa 迭代程序; 不动点