Null scrolls in the 3-dimensional Lorentzian space

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Abstract

In this study, a timelike ruled surface in the 3 - dimensional Lorentzian space $\mathbb{R}_1^3$ which is called null scroll is generated by a null straight line which moves along a null curve with respect to the null frame. In a null scroll, the central point, the curve of striction, pseudo-orthogonal trajectory and some theorems related to these structures are obtained in the 3-dimensional Lorentzian space $\mathbb{R}_1^3$. Results about developable null scrolls are provided as well.

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§1. Introduction

$\mathbb{R}_1^3$ is by definition the 3-dimensional vector space $\mathbb{R}^3$ with the inner product of signature $(1,2)$ given by

$$\langle x,y\rangle = -x_1y_1 + x_2y_2 + x_3y_3$$

for any colomn vectors $x = {}^t(x_1,x_2,x_3)$, $y = {}^t(y_1,y_2,y_3) \in \mathbb{R}^3$. Let $\{e_1,e_2,e_3\}$ be the standart orthonormal basis of $\mathbb{R}_1^3$ given by

$$e_1 = {}^t(1,0,0),\ e_2 = {}^t(0,1,0),\ e_3 = {}^t(0,0,1).$$

A basis $F = \{X,Y,Z\}$ of $\mathbb{R}_1^3$ is called a *(proper) null frame* if it satisfies the following conditions

$$\langle X,X\rangle = \langle Y,Y\rangle = 0,\ \ \langle X,Y\rangle = -1,$$

$$Z = X \wedge Y = \sum_{i=1}^{3} \varepsilon_i \det[X,Y,e_i]\, e_i,$$

where $\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = 1$. Hence we obtain that

$$\langle X,Z\rangle = \langle Y,Z\rangle = 0,\ \ \langle Z,Z\rangle = 1.$$

A vector V in $\mathbb{R}_1^3$ is said to be null if $\langle V,V\rangle = 0$, [2, 4]. A surface in the 3-dimensional Lorentzian space $\mathbb{R}_1^3$ is called a timelike surface if the induced metric on the surface

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is a Lorentzian metric. A ruled surface is a surface swept out by a straight line Y moving along a curve α. The various positions of the generating line Y are called the rulings of the surface. Such a surface, thus has a parametrization in ruled form as follows:

$$\varphi(t,v) = \alpha(t) + vY(t).$$

We call α to be the base curve and Y to be the director curve. If the tangent plane is constant along a fixed ruling, then the ruled surface is called a developable surface. The remaining ruled surfaces are called skew surfaces. If there exists a common perpendicular to two preceding rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the curve of striction [1, 5].

§2. Null Scrolls in $\mathbb{R}^3_1$

Let $\alpha : M \to \mathbb{R}^3_1$ be a null curve, namely, a smooth curve whose tangent vectors $\alpha'(t)$, $\forall t \in I$ are null. For a given smooth positive function $d = d(t)$ let us put

$$X = X(t) = d^{-1}\alpha'. \tag{2.1}$$

Then X is a null vector field along α. Moreover, there exists a null vector field Y along α satisfying $\langle X, Y\rangle = -1$. Here if we put $Z = X \wedge Y$ then we can obtain a (proper) null frame field $F = \{X, Y, Z\}$ along α. In this case the pair (α, F) is said to be a *(proper) framed null curve*.

If the null vector Y moves along α, then the ruled surface is given by the parametrization $(I \times \mathbb{R}, \varphi)$ where

$$\varphi : I \times \mathbb{R} \to \mathbb{R}^3_1$$

is given by

$$(t, v) \to \varphi(t, v) = \alpha(t) + vY(t),\ t \in I,\ v \in J,$$

which can be obtained in the 3-dimensional Lorentzian space $\mathbb{R}^3_1$. Then the ruled surface is called a *null scroll* and denoted by M. It is a timelike surface.

Let α be a (proper) framed null curve and ∇ be Levi-Civita connection on $\mathbb{R}^3_1$. Then a framed null curve α satisfies the following Frenet equations

$$\begin{cases} \nabla_X X = aX + bZ \\ \nabla_X Y = -aY + cZ \\ \nabla_X Z = cX + bY, \end{cases} \tag{2.2}$$

where

$$\begin{cases} a = -\langle \nabla_X X, Y\rangle \\ b = \langle \nabla_X X, Z\rangle \\ c = \langle \nabla_X Y, Z\rangle \end{cases} \tag{2.3}$$

are smooth functions [3, 4].

If we fix the parameter v, then the curve $\varphi_v : I \times \{v\} \to M$ sending (t, v) to $\alpha(t) + vY(t)$ can be obtained on M, the tangent vector field of which is given by

$$A = dX - avY + cvZ.$$

Theorem 2.1. *Let M be a null scroll. Then the tangent planes along a ruling of M coincide if and only if $a = c = 0$.*

Proof. Straightforward computation. □

Then we have following:

Corollary 2.2. *The null scroll M is developable if and only if $a = c = 0$.*

Lemma 2.3. *For the null scroll M we have*

$$a = -\det(Y, Z, \nabla_X X) \tag{2.4}$$

$$c = -\det(X, Y, \nabla_X Y). \tag{2.5}$$

Proof. The equations (2.2) infer the two equalities. □

§3. Position vector of a central point and pseudo-orthogonal trajectory for the null scrolls

If the distance between the central point and the base curve of a null scroll (which is a skew timelike surface), is $\overline{u}$, then the position vector $\overline{\alpha}(t)$ can be expressed by $\overline{\alpha}(t, \overline{u}) = \alpha(t) + \overline{u}Y(t)$, where $\alpha(t)$ is the position vector of the base curve and $Y(t)$ is the directed vector belonging to the ruling. The parameter $\overline{u}$ can be expressed in terms of position vector of the base curve and directed vector of the ruling. Consider three preceding rulings of a null scroll such that the first one is $Y(t)$, and the second one is $Y(t) + dY(t)$. Let P, P' and Q, Q' be the feet on the rulings of the common perpendicular to the two preceding rulings. The common perpendicular to $Y(t)$ and $Y(t) + dY(t)$ is $Y(t) \wedge dY(t)$.

The vector $\overrightarrow{PQ}$ coincides with the vector $\overrightarrow{PP'}$ in the limiting position, and $\overrightarrow{PQ}$ will be the tangent vector of the curve of striction. Thus, we have

$$\langle \nabla_X Y, \overrightarrow{PQ} \rangle = 0.$$

Therefore, we get

$$\overline{u} = -ad/c^2. \tag{3.6}$$

Hence the curve of striction is given by

$$\overline{\alpha}(t) = \alpha(t) - \frac{\langle \nabla_X Y, dX \rangle}{\langle \nabla_X Y, \nabla_X Y \rangle} Y(t), \tag{3.7}$$

where $\langle \nabla_X Y, \nabla_X Y \rangle \neq 0$ and ad/c^2 is constant.

Theorem 3.1. *The curve of striction $\overline{\alpha}$ is independent on the choice of the base curve α for the non-developable null scroll M.*

Proof. Let β be a another base curve of the null scroll M, that is, let

$$\varphi(t, v) = \alpha(t) + vY(t)$$

and

$$\varphi(t,s) = \beta(t) + sY(t)$$

be two different base curve for the null scroll M. Then from (3.7) we obtain

$$\overline{\alpha}(t) - \overline{\beta}(t) = 0,$$

thus the proof is complete.

Theorem 3.2. *Let M be a nondevelopable null scroll. Then $\varphi(t, v_0)$ on the ruling through the point $\alpha(t)$ is a central point if and only if $\nabla_X Y$ is a normal vector of the tangent plane at $\varphi(t, v_0)$.*

Proof. Let M be a nondevelopable null scroll and $\nabla_X Y$ be a normal of the tangent plane at $\varphi(t, v_0)$ on the ruling through $\alpha(t)$. The tangent vector field of the curve

$$\varphi_{v_0} : I \times \{v_0\} \to M$$

is $A = dX - av_0Y + cv_0Z$. Thus $\langle \nabla_X Y, A\rangle = 0$. Then we get $v_0 = -ad/c^2$. Therefore $\varphi(t, v_0)$ is a central point of M.

Conversely, let $\varphi(t, v_0)$ be a central point on the ruling through $\alpha(t)$. Then we obtain $\langle \nabla_X Y, Y\rangle = 0$ and $\langle \nabla_X Y, A\rangle = ad + c^2 v = 0$.

Thus $\nabla_X Y$ is a normal vector of the tangent plane at $\varphi(t, v_0)$.

Theorem 3.3. *Let M be a nondevelopable null scroll. The curve of striction*

$$\overline{\alpha}(t) = \alpha(t) - \frac{ad}{c^2} Y(t) \tag{3.8}$$

is a timelike curve in a null scroll M.

Proof. If we use the equation (3.8), we can show easily that the tangent vector field of the curve of striction is a timelike vector field. □

We know that, if there is a curve which meets perpendicularly each of the rulings, then this curve is called an *orthogonal trajectory* of a ruled surface which base curve is non-null. Hence we have

Definition 3.1. Let M be a null scroll in $\mathbb{R}^3_1$. If there exists a curve which makes constant angle with each one of the rulings, the this curve is called a *pseudo-orthogonal trajectory* of M.

Theorem 3.4. *Let M be a null scroll in $\mathbb{R}^3_1$. Then there exists a unique pseudo-orthogonal trajectory of M through each point of M.*

Proof. Let $\varphi : I \times J \to \mathbb{R}^3_1$, defined by

$$\varphi(t,v) = \alpha(t) + vY(t)$$

be a parametrization of M. A pseudo-orthogonal trajectory of M is given by $\beta : \widetilde{I} \to$M, where

$$\beta(t) = \alpha(t) + f(t)Y(t), \quad t \in \widetilde{I}$$

and $\langle \beta', Y\rangle = const$. We may assume that $\widetilde{I} \subset I$.

Now we want to get a curve which passes through the point $p_0 = \varphi(t_0, v_0)$. Thus we can write

$$p = \alpha(t) + f(t)Y(t), \qquad p_0 = \alpha(t_0) + v_0 Y(t_0).$$

Therefore we get $\alpha(t) = \alpha(t_0)$ and $f(t) = v_0$.

If we choose I such that it is one to one, then we have $t = t_0$. Therefore the pseudo-orthogonal trajectory of M through the point p_0 is unique. Since this pseudo-orthogonal trajectory of M makes a constant angle with each of the rulings of M, we have $\widetilde{I} = I$. Thus the proof is complete. □

Theorem 3.5. *Let M be a null scroll in $\mathbb{R}^3_1$. The shortest distance between two rulings is measured only on the curve of striction which is one of the pseudo-orthogonal trajectories.*

Proof. We consider two rulings which pass through the points $\alpha(t_1)$ and $\alpha(t_2)$, where $t_1, t_2 \in I$ and $t_1 < t_2$. We compute the length $\ell(v)$ of an pseudo-orthogonal trajectory between these two rulings

$$\ell(v) = \int_{t_1}^{t_2} \|A\| \; dt = \int_{t_1}^{t_2} (2adv + c^2 v^2)^{\frac{1}{2}} \; dt.$$

To find the value of t which minimizes $\ell(v)$, we notice that

$$\frac{\partial \ell(v)}{\partial v} = 0,$$

which infers $v = -ad/c^2$. This completes the proof. □

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