On the optimal target of a macroeconomic growth model

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Abstract

Many questions in macroeconomics lead to random discrete dynamical systems $y_{t+1} = w_i(y_t)$ where the map w_i is contractive and it is chosen in a given set with probability p_i. The aim of this paper is to study the inverse problems for a macroeconomic stochastic growth model. Roughly speaking, solving an inverse problem means to find a model converging to a fixed optimal target. In practical cases, the solution of the inverse problem can be used to know if a given system may converge to a given steady state, forecasting the behaviour of the model.

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§1. A new macroeconomic growth model

In a macroeconomic system it is well known that we have an equilibrium at the time t if the aggregate demand D_t is equal to the its income Y_t, that is $D_t = Y_t$, $\forall t \in \mathbb{N}$. The aggregate demand is the sum of the consumptions C_t, the investments I_t and so the previous relation become

$$Y_t = C_t + I_t. \tag{1.1}$$

The quantity of consumption C_t is a function of the income of the previous year by a linear relation as $C_t = \alpha Y_{t-1}$, where the stochastic coefficient α can take values in a given set $\mathcal{C} = \{s_1, s_2, \dots s_m\} \subset (0,1)$ with probability p_i, $\sum_{i=1}^m p_i = 1$. If the level of investments if fixed, say I_0 (the initial quantity of investments), the previous equation becomes

$$Y_t = \alpha Y_{t-1} + I_0, \tag{1.2}$$

and so the stochastic growth model of the income is described by the system of equations

$$\begin{cases} Y_{t+1} = s_1 Y_t + I_0 & \text{with probability } p_1 \\ Y_{t+1} = s_2 Y_t + I_0 & \text{with probability } p_2 \\ \dots & \\ Y_{t+1} = s_m Y_t + I_0 & \text{with probability } p_m \end{cases} \tag{1.3}$$

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Remark. In the sequel, we think C_t, Y_t and I_0 as elements of $[0,1]$. In other words we are studying the behaviour of the weights (in terms of political and economical decisions) of the involved variables.

To establish if this model converges (and in which sense it converges) and to characterize the eventual attracting point of the system is known as *direct problem*. To find the solution of the direct problem we consider the space $\mathcal{H}([0,1])$, built with all compact subsets of $[0,1]$, and the following metric, known as Hausdorff metric on $\mathcal{H}([0,1])$,

$$h(A,B) = \max\{ \max_{x\in A} \min_{y\in B} |x-y|, \max_{x\in B} \min_{y\in A} |x-y|\}.$$

In [19] is shown that the space $(\mathcal{H}([0,1]),h)$ is a complete metric space. Now let $\mathcal{M}([0,1])$ be the space of all probability measures on $[0,1]$. Define a metric on $\mathcal{M}([0,1])$ as

$$d_H(\mu,\nu) = \sup_{f\in L_1} \int_{[0,1]} f d\mu - \int_{[0,1]} f d\nu, \quad \mu,\nu \in \mathcal{M}([0,1]),$$

where

$$L_1 = \{f : [0,1] \to \mathbb{R} \mid |f(x) - f(y)| \leq |x-y|\}.$$

This is the Monge-Kantorovich metric, referred to in the IFS literature as "Hutchinson metric". The space $(\mathcal{M}([0,1]), d_H)$ is a complete metric space [10]. Now let $w = \{w_1, w_2, \ldots w_N\}$ denote a set of N continuous contraction maps on $[0,1]$, i.e. $w_i : [0,1] \to [0,1]$ and

$$|w_i(x) - w_i(y)| \leq c_i|x-y|, \quad x,y \in [0,1],\ 0 \leq c_i < 1,\ i = 1\ldots N.$$

Definition. The couple $([0,1],w)$ is called Iterated Function Systems (briefly IFS).

The IFS were born in 1985 ([1]) as applications of the theory of discrete dynamical systems and as useful tools to build fractals and other similar sets. It will be convenient to define the maximum contractivity factor of the IFS as

$$c = \max_{i=1\ldots N} c_i < 1.$$

Associated with these maps is a set of non-zero probabilities $p = \{p_1, p_2, \ldots p_N\}$, $p_i > 0$ and $\sum_{i=1}^N p_i = 1$.

Now for a set $S \in \mathcal{H}([0,1])$, denote $w_i(S) = \{w_i(x), x \in S\}$ and denote the "parallel" action of the set of maps w_i on S as:

$$w(S) = \bigcup_{i=1}^N w_i(S).$$

Also define the iteration sequence $w^{n+1}(S) = w(w^n(S))$ $n = 1,2\ldots.$ Two important results for contractive IFS are given below:

Theorem 1.1. ([19]) i) *There exists a unique compact subset $A \in \mathcal{H}([0,1])$, the attractor of IFS $\{[0,1],w\}$ (indipendent of p) such that*

$$A = w(A) = \bigcup_{i=1}^N w_i(A)$$

and $h(w^n(S), A) \to 0$ *as* $n \to \infty$ *for all* $S \in \mathcal{H}([0,1])$.

ii) *Define the following "Markov operator"* $M : \mathcal{M}([0,1]) \to \mathcal{M}([0,1])$,

$$M(\nu) = \sum_{i=1}^{N} p_i \nu \circ w_i^{-1}.$$

Then there exists a unique measure $\mu \in \mathcal{M}([0,1])$, *termed the invariant measure, which obeys the fixed point condition*

$$M\mu = \mu.$$

Moreover, supp$(\mu) = A$.

If one builds the random sequence

$$\begin{cases} y_{t+1} = w_1(y_t) & \text{with probability } p_1 \\ y_{t+1} = w_2(y_t) & \text{with probability } p_2 \\ \cdots\cdots & \\ y_{t+1} = w_n(y_t) & \text{with probability } p_n, \end{cases}$$

$\forall y_0 \in [0,1]$, $n \in \mathbb{N}$, $i = 1 \ldots \mathbb{N}$, then for almost every code sequence $\{s_1, s_2, \ldots\}$ the set $\cup_{n=1}^{\infty} \{y_i\}$ is dense on the attractor A of the IFS (see [19]). This random walk is called Chaos Game in IFS literature [9]. Furthermore there is a relation between the invariant measure μ and the Chaos Game:

$$\mu(S) = \lim_{n \to +\infty} \frac{1}{n+1} \sum_{k=0}^{n} I_S(y_k), \ \ S \subset [0,1],$$

that is, $\mu(S)$ is the limit of the relative visitation frequency of S during the chaos game.

§2. The inverse problem and applications

In the previous section we have seen that the convergence of the dynamical system (1.3) can be characterized by the invariant measure of the associated Markov operator. So the inverse problem can be formulated as follows: *"given a measure* μ*, find a dynamic model which converges to* μ*"*. In other words, the inverse problem of (1.3) consists, given a target measure μ^*, of finding the maps w_i and the probabilities p_i, $i = 1, 2$, such that the Markov operator built with these parameters has μ^* as fixed point. Referred to the growth model in the previous section, the inverse problem consists of finding the parameters of the model (economical parameters) which allow to reach the fixed goals of economic policy. Anyway, in practical cases one has the maps w_i and the unknown data are only the probabilities p_i. Finding all the solutions of the inverse problem, one can establish if the system may converge to a given *steady state*. Furthermore, following the random sequence and estimating the probabilities through the frequencies, one may forecast the behaviour of the model.

2.1. Formulation of the inverse problem.

Let μ be a given measure. Let $w_i : [0,1] \to [0,1]$ be contractive and affine maps, that is $w_i(x) = s_i x + a_i$, $|s_i| < 1$. Clearly for affine maps the contractivity factor $c_i = |s_i|$.

Theorem 2.1. *Let $M : \mathcal{M}([0,1]) \to \mathcal{M}([0,1])$ be a contractive Markov operator, associated to an IFS, with contractivity factor $c \in [0,1)$. Let μ^* be the fixed point of M. If $d_H(M\mu,\mu) < \epsilon$ then $d_H(\mu^*,\mu) < \frac{\epsilon}{1-c}$.*

Proof. In fact we have:

$$d_H(\mu,\mu^*) \le d_H(\mu, M\mu) + d_H(M\mu, M\mu^*) \le \epsilon + c d_H(\mu,\mu^*).$$

So $d_H(\mu,\mu^*) \le \frac{\epsilon}{1-c}$. □

The previous theorem states that the inverse problem can be studied through the function $F_M(w,p) := d_H(M\mu,\mu)$. This is a function of the maps w_i and the parameters p_i; however, in the sequel, we will use a fixed family of contractions $\mathcal{W} = \{w_1, w_2, \ldots\}$ with associated probabilities $\{p_1, p_2, \ldots\}$. Associated to each map w_i there is a probability p_i; so the inverse problem consists of finding only the probabilities p_i, putting $p_i = 0$ when the corresponding map is not used in the Markov operator. We will write M_p to put in evidence this fact. It is clear that the optimal solution is to find p^* such that $F(p^*) = 0$, $\sum_{i=1}^N p_i^* = 1$; in fact in this case $\mu = \mu^*$, that is the map M has exactly μ as fixed point. In the other cases Theorem 2.1 gives an estimate of the fixed point.

Theorem 2.2. *Let $\mu \in M([0,1])$ and $\mathcal{W}^N = \{w_1, w_2, \ldots w_N\} \subset \mathcal{W}$ be the subset of the first N maps of $\mathcal{W}$. Then the map $F_M(p) = d_H(M_p\mu,\mu) : \mathbb{R}^n \to \mathbb{R}_+$ is convex.*

Proof. In fact for all $p_1, p_2 \in \mathbb{R}^n$ and $t \in [0,1]$ one has:

$$F_M(tp_1 + (1-t)p_2) = d_H(M_{tp_1+(1-t)}p_2\mu,\mu) =$$

$$\sup_{f\in L_1}\{\int_{[0,1]} f dM_{tp_1+(1-t)p_2}\mu - \int_{[0,1]} f d\mu\} =$$

$$\sup_{f\in L_1}\{\sum_{i=1}^n (tp_{1i} + (1-t)p_{2i}) \int_{[0,1]} f\circ w_i d\mu - \int_{[0,1]} f d\mu\} \le$$

$$\sup_{f\in L_1} t\{\sum_{i=1}^n p_{1i} \int_{[0,1]} f\circ w_i d\mu - \int_{[0,1]} f d\mu\} +$$

$$\sup_{f\in L_1} (1-t)\{\sum_{i=1}^n p_{2i} \int_{[0,1]} f\circ w_i d\mu - \int_{[0,1]} f d\mu\} =$$

$$tF(p_1) + (1-t)F(p_2).$$

□

To solve the inverse problem means to find p^*, $\sum_{i=1}^N p_i^* = 1$, such that $F_M(p^*) = 0$. The function F_M is not differentiable; however, since F_M is convex, then it is locally Lipschitzian and then semismooth [17]. So for solving the nonsmooth equation $F_M(p) = 0$ one can use the nonsmooth version of Newton's method due to Qi and Sun [17]. When the inverse problem has not solution, one can solve the following optimization problem:

$$\min F_M(p),\ 0 \le p_i \le 1,\ \sum_{i=1}^{n} p_i = 1,$$

which gives a lower bound for the approximation. In this setting, this bound can be improved only increasing the number of contractions w_i (that is sending $N \to +\infty$). Necessary and sufficient conditions for the previous optimization problem can be given by Kuhn-Tucker conditions with Clarke's subdifferential. However, in the next section we will show how the previous optimization problem can be reduced to a quadratic optimization problem. This represents a significant simplification since quadratic programming problems can be solved computationally in a finite number of steps.

2.2. Moment matching and quadratic optimization problem.

The aim of this section is to show how the previous problem can be reduced to a quadratic optimization problem by moment matching.

Suppose that $\gamma = M\nu$; then for each continuous function $f : [0,1] \to \mathbb{R}$ we have

$$\int_{[0,1]} f d\gamma = \int_{[0,1]} f d(M\nu) = \sum_{i=1}^{N} p_i \int_{[0,1]} f \circ w_i(x) d\mu(x).$$

We now consider the moments of ν and γ defined as

$$g_n = \int_{[0,1]} x^n d\mu, \quad h_n = \int_{[0,1]} x^n d\gamma,$$

where $g_0 = h_0 = 1$. From the previous equation with $f(x) = x^n$, we obtain

$$h_n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \{\sum_{i=1}^{N} p_i s_i^k a_i^{n-k}\} g_k,$$

with $n \in \mathbb{N}$. Let

$$\mathcal{D}([0,1]) = \{g = (g_0, g_1, g_2, \ldots) \mid g_n = \int_{[0,1]} x^n d\nu, n \in \mathbb{N}, \nu \in M([0,1])\}.$$

So it is possible to identify each operator $M : \mathcal{M}([0,1]) \to \mathcal{M}([0,1])$ with a linear map $A : \mathcal{D}([0,1]) \to \mathcal{D}([0,1])$.

Moment matching for the approximation of measure can be justified by the fact that the convergence of moments is equivalent to the the weak convergence of measures. This is summarized in the following theorem.

Theorem 2.3. ([3]) *Let $X = [0,1]$, $\nu, \nu^j \in M(X)$, $j \in \mathbb{N}$, with moments defined by*

$$g_n = \int_X x^n d\nu, \quad g_n^j = \int_X x^n d\nu^j, \quad n \in \mathbb{N}.$$

Then the following assertions are equivalent:

i) $g_n^j \to g_n$ *when* $j \to +\infty$, $\forall n \in \mathbb{N}$;

ii) *the sequence* ν^j *converges weak* to* ν, *i.e. for any continuous function* $f : [0,1] \to \mathbb{R}$, $\int_{[0,1]} f d\mu^{(j)} \to \int_{[0,1]} f d\mu$, *as* $j \to +\infty$;

iii) $d_H(\nu^j, \nu) \to 0$ *when* $j \to +\infty$.

We now consider the sets:

$$\overline{l^2}(\mathbb{N}) = \left\{ c = (c_0, c_1, c_2, \dots) \mid c_i \in \mathbb{R}, \|c\|^2_{\overline{l^2}} = c_0^2 + \sum_{k=1}^{\infty} \frac{c_k^2}{k^2} < \infty \right\}$$

and

$$\overline{l_0^2}(\mathbb{N}) = \{c \in \overline{l^2}(\mathbb{N}) \mid c_0 = 1\} \subset \overline{l^2}(\mathbb{N}).$$

It is easy to prove that $D(X) \subset \overline{l_0^2}(\mathbb{N}) \subset \overline{l^2}(\mathbb{N})$.

Proposition 2.1. *Let* $\nu, \nu^n \in \mathcal{M}([0,1])$, *with moments* $g, g^n \in \mathcal{D}([0,1])$. *Then* $\|g - g^n\|_{\overline{l^2}} \to 0$ *when* $n \to +\infty$ *if and only if* $d_H(\nu, \nu^n) \to 0$ *when* $n \to +\infty$.

Proof. The proof follows from the results of Theorem 2.3. □

Proposition 2.2. *Define the following metric on* $\mathcal{D}([0,1])$: *for* $u, v \in \mathcal{D}([0,1])$, $\overline{d_2}(u,v) = \|u - v\|_{\overline{l_2}}$. *Then* $(\mathcal{D}([0,1]), \overline{d_2})$ *is a complete metric space.*

Proof. Let $g^{(n)} = (g_0^{(n)}, g_1^{(n)}, \dots) \in D([0,1])$ be a Cauchy sequence in $\overline{l_2}$. Let $\nu^{(n)} \in \mathcal{M}([0,1])$ be the probability measures whose moments are the components of the $g^{(n)}$. Now consider the sequence $a^{(n)} = (a_0^{(n)}, a_1^{(n)}, \dots)$ where $a_0^{(n)} = g_0^{(n)}$ and $a_k^{(n)} = \frac{g_k^{(n)}}{k}$, $k \in \mathbb{N}$. It is clear that $a^{(n)}$ is a Cauchy sequence in $l^2(\mathbb{N})$ and then, by completeness of $l^2(\mathbb{N})$, there exists an $a \in l^2(\mathbb{N})$ such that $\|a^{(n)} - a\|_{l_2} \to 0$ as $n \to +\infty$. Now let $g = (g_0, g_1, \dots)$ where $g_0 = 1$ and $g_k = k a_k$. Since $|a_k^{(n)} - a_k| \to 0$ as $n \to +\infty$ we obtain that $|g_k^{(n)} - g_k| \to 0$ as $n \to +\infty$. Now $a \in l^2(\mathbb{N})$ and then g is an element of $\overline{l^2}(\mathbb{N})$. We now show that $g \in D([0,1])$. A necessary and sufficient condition for an infinite set of real numbers $c = (c_0, c_1, \dots)$ be the moments of a unique probability measure $\mu \in \mathcal{M}([0,1])$ is that they satisfy the Hausdorff inequalities

$$H_{i,j}(c) = \sum_{m=0}^{j} (-1)^m \frac{j!}{(j-m)!m!} c_{j+m} \geq 0,$$

$i, j \in \mathbb{N}$. Since $g_k^{(n)}$ are the moments of the measures $\nu^{(n)} \in \mathcal{M}([0,1])$, they must satisfy inequalities as the previous one and then, taking the limit as $n \to +\infty$, we obtain:

$$H_{i,j}(g) = \sum_{m=0}^{j} (-1)^m \frac{j!}{(j-m)!m!} g_{j+m} \geq 0,$$

$i, j \in \mathbb{N}$. These are the Hausdorff inequalities for the sequence g and these imply that g_k are the moments of a unique measure $\nu \in \mathcal{M}([0,1])$. Thus $g \in D([0,1])$. □

Recall that for each Markov operator $M : \mathcal{M}([0,1]) \to \mathcal{M}([0,1])$ there exists a linear operator $A : \mathcal{D}([0,1]) \to \mathcal{D}([0,1])$.

Proposition 2.3. *The linear operator A is contractive in $(\mathcal{D}([0,1]), \overline{d_2})$.*

Proof. In the standard basis $\{e_i = (0,0,\ldots,0,1,0,\ldots)\}_{i=0}^{\infty}$ the infinite matrix representation of A is lower triangular. Hence, A has eigenvalues:

$$\lambda_0 = a_{0,0},\ \lambda_n = a_{n,n} = \sum_{i=1}^{N} p_i s_i^n,$$

with $n \geq 1$. Thus, $|\lambda_n| = |a_{n,n}| < c^n < 1$ and, for any $u, v \in D([0,1])$, $\|A(u-v)\|_{\overline{l_2}} \leq c\|u-v\|_{\overline{l_2}}$, which implies the contractivity of A. ☐

From Proposition 2.3 results that there exists an unique solution $g^* \in \mathcal{D}([0,1])$ of the equation $Ag = g$. So the inverse problem on $\mathcal{M}([0,1])$ can be analyzed with the following result.

Theorem 2.4. *Let $\mu \in \mathcal{M}([0,1])$ with moment $g \in \mathcal{D}([0,1])$. Let (w,p) be a contractive IFS map with contractivity factor $c \in [0,1)$, such that $\overline{d_2}(g,h) = \|g - h\|_{\overline{l_2}} < \epsilon$, where $h \in \mathcal{D}([0,1])$ is the moment of the measure $\gamma = M\nu$. Then*

$$\overline{d_2}(g, g^*) < \frac{\epsilon}{1-c},$$

where g^* is the moment of the invariant measure ν^*.

The proof is trivial and we omit it. Let $\mathcal{W}^N = \{w_1, w_2, \ldots w_N\}$ and

$$\Omega_N = \{p^N = (p_1, p_2, \ldots, p_N) : p_i \geq 0, \sum_{i=1}^{N} p_i = 1\}.$$

Obviously $\Omega_N \subset \mathbb{R}^N$ is compact. Let $\mu \in \mathcal{M}([0,1])$ be the target measure with moments $g \in \mathcal{D}([0,1])$. For a given let $p \in \Omega_N$ and M_p be the Markov operator corresponding to the IFS $(\{w_1, w_2, \ldots w_N\}, p)$. Furthermore $\gamma = M\mu$ with vector of moments $h \in \mathcal{D}([0,1])$. The distance between the moments of ν and γ is

$$\Delta(p) = \|g - h\|_{\overline{l_2}}.$$

Let $A : \mathcal{D}([0,1]) \to \mathcal{D}([0,1])$ the linear operator associated to M. Then $h = Ag$, where $h_n = \sum_{i=1}^{N} A_{n,i} p_i$ and

$$A_{n,i} = \int_{[0,1]} (w_i x + a_i)^n d\mu = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} s_i^k a_i^{n-k} g_k.$$

Then if $S(p) = (\Delta)^2(p)$, with trivial calculus, one can show that $S(p) = p^T Q p + b^T p + c$, $p, b \in \mathbb{R}^N$. The element of the symmetric matrix Q are given by

$$q_{i,j} = \sum_{n=1}^{\infty} \frac{A_{n,i} A_{n,j}}{n^2},\ i,j = 1,2,\ldots N.$$

Furthermore,

$$b_i = -2 \sum_{n=1}^{\infty} \frac{g_n A_{n,i}}{n^2}, \ i = 1, 2, \dots N$$

and

$$c = \sum_{n=1}^{\infty} \frac{g_n^2}{n^2}.$$

Since $0 \leq A_{n,i} \leq 1$, then the infinite sums converge.

Thus, given a target measure μ with moment vector g, the inverse problem becomes the one of finding an IFS such that the collage distance $\overline{d_2}(g, h) = 0$, where $h = Ag$. When this problem has not solution one may solve the following quadratic programming problem with linear constraints

$$\min S^N(p), \quad \sum_{i=1}^{N} p_i = 1, \quad 0 \leq p_i \leq 1.$$

This represents a significant simplification, since quadratic programming problems can be solved computationally in a finite number of steps. Furthermore, this minimum represents a lower bound for the approximation and it can be improved only increasing the number of maps w_i (that is, sending $N \to +\infty$).

§3. Conclusions

Starting from some classical macroeconomic models introduced by Ramsey, Domar, Nardini [2, 4, 16] we have built a new model in which the public expenditure is part of a shock factor and the level of investments is a linear function of the income. For this model we have formulated and studied the inverse problem; roughly speaking, it means to find a set of parameters such that the dynamic model converges to a fixed optimal target of the public decision makers. Finding all the solutions of the inverse problem, one can establish if the system may be converge to a given steady state, forecasting the behaviour of the economical phenomena. The inverse problem is a nonsmooth equation and when it has not solution, it is possible to solve a nondifferentiable convex optimization problem on $\mathbb{R}^n$ which gives a lower bound of the approximation. Necessary and sufficient conditions for this problem can be given with Clarke subdifferential calculus. Furthermore, by moment matching we have shown that this problem can be reduced to a constrained quadratic programming problem with linear constraints. This represents a significant simplification since quadratic programming problems can be solved computationally in a finite number of steps.

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