On Steady Salinity Distribution and Circulation in Partially Mixed and Well Mixed Estuaries

LI-YAUW OEY

Geophysical Fluid Dynamics Program, James Forrestal Campus, Princeton University, Princeton, NJ 08540

(Manuscript received 16 June 1983, in final form 16 October 1983)

ABSTRACT

Perturbation analysis based on small $\alpha = \text{Ra}^{0.23}F_m{}^{0.9}$, where Ra is the Rayleigh number and F_m is the Froude number, is used to study steady-state circulation and salinity distribution in estuaries. The classical Hansen and Rattray's similarity solution is obtained for the special case of linear variation of longitudinal dispersion coefficient K_H, in a channel of constant width B and depth D. It is argued that K_H, B and D must vary in real estuaries in such a way that the general solution is regular throughout the length of the estuary and shows a salinity structure which resembles that observed in a real estuary.

It is shown that Hansen and Rattray's theory for predicting the importance of upstream salt transport due to the vertical gravitational circulation in estuaries is valid to a good degree of approximation for arbitrary longitudinal variations in width, depth, fresh water discharge, wind stress and various dispersion and mixing coefficients. This finding is checked against available observations in the Mersey estuary, in the channel of Rio Guayas and in the Hudson River. It is also checked against a real-time three-dimensional numerical model's results of New York Harbor.

Finally, Pritchard's classification of estuaries in terms of their principal tidally-averaged advective and diffusive processes is translated on the Hansen–Rattray circulation–stratification diagram. The diagram shows the relative importance of various terms in the salt balance equation.

1. Introduction

The three-dimensional, tidally averaged salt balance equation is

$$uS_x + vS_y + wS_z = (e_1S_x)_x + (e_2S_y)_y + (e_3S_z)_z, \quad (1.1)$$

where x, y, z is a set of right-handed Cartesian axes with the origin on the surface at the landward side of the estuary; x is positive seaward, z positive downward; u, v and w are the fluid velocities in x, y and z directions; S is the salinity in parts per thousand (‰), e_1, e_2 and e_3 are mixing coefficients for S.

Pritchard (1955) has classified estuaries into four different types according to their principal tidally-averaged advective and diffusive processes. The type A estuaries are laterally homogeneous and highly stratified. They are produced by a combination of small tidal action and strong fresh water discharge Q_f. Pritchard showed that (1.1) is then approximated by

$$uS_x + wS_z = 0, \quad (1.2)$$

and the dominant mechanism at work in this type of estuary is therefore purely advective.

In type B estuaries, the tidal action is sufficiently strong in relation to Q_f and the sharp fresh water–saline water interface which occurs in type A estuaries breaks down. Vertical turbulent diffusion is important and the salinity varies continuously with depth, although the top to bottom salinity difference can still be large, of the order of 10‰ at high discharges. Eq. (1.1) is then approximated by

$$uS_x + wS_z = (e_3S_z)_z. \quad (1.3)$$

When the tidal action becomes so strong in relation to Q_f that there is negligible vertical salinity gradient the downstream advective flux becomes balanced by an upstream dispersive flux and (1.1) can be approximated by

$$uS_x + wS_z = (e_1S_x)_x. \quad (1.4)$$

Pritchard called this a type D estuary. One can think of type B and type D estuaries as being subclasses of a more general class of estuaries in which both vertical turbulent diffusion and longitudinal dispersion are important. More precisely, I shall assume an elongated estuary and width-average (1.1) to obtain

$$B(uS_x + wS_z) = (BK_hS_x)_x + (BK_vS_z)_z, \quad (1.5)$$

where $B(x)$ = width of estuary at x, K_v, K_h = vertical and longitudinal dispersion coefficients for S and u, w and S are now interpreted as width-averaged quantities. We can interpret K_v as a turbulent diffusion coefficient. The longitudinal dispersion coefficient K_h, however, is due to a combination of turbulent diffusion, transverse shear and temporal shear dispersions. Fischer (1972) has shown that the transverse shear dispersion can be important in many estuaries. He calculated that the transverse variation of the velocity profile, induced by a combination of transverse variation of bottom topography and longitudinal pressure gradient (density difference plus water slope), gives a transverse dispersion coefficient

$$K_{ht} \approx 0.02\left(D^3 \frac{g}{\rho}\frac{d\rho}{dx} B/32e_3\right)^2 e_2^{-1}, \qquad (1.6)$$

where D is the depth, ρ the density and $g = 9.81$ m s^{-2}. Thus the combination of vertical turbulent diffusion and mean transverse shear is very effective in generating longitudinal dispersion because the width of an estuary is usually large. Other mechanisms which can be important in longitudinal dispersion of salt in an estuary includes tidal "pumping" and "trapping" (Fischer, 1976).

All these facts suggest that I should consider (1.5) rather than (1.4) and (1.3) separately. Clearly, whether a mechanism is important relative to other mechanisms can only be decided by a proper scaling and nondimensionalization of (1.5). This I shall do in Section 2. The question I shall address is "what parameters determine whether an estuary is of type A, B or D?" This problem is similar in spirit to Hansen and Rattray's (1965, 1966) classification of estuaries. Instead of obtaining specific solutions as in their works I shall base my arguments purely on scalings and nondimensionalizations.

It is conceivable that a certain transition zone between the type B and type D estuaries exists within which the estuary is "weakly" stratified. I shall define precisely what "weakly" means in Section 3. I shall also obtain perturbation solution around this weakly stratified state and show that many estuaries can be classified as weakly stratified. These estuaries include the James, Potomac and Hudson Rivers in America, and the Mersey Narrows in England.

Hansen and Rattray use the nondimensional top to bottom salinity difference $\delta S/S_0$ and the ratio of net surface velocity to fresh water discharge velocity u_s/u_{f0} to classify estuaries where in this section S_0 is taken as the cross-sectionally averaged salinity at x. They found that u_s/u_{f0} varies like $F_m^{-3/4}$, where $F_m = u_{f0}/(g\Delta\rho D_0/\rho_0)^{1/2}$, D_0 is depth, ρ_0 density of fresh water, and $\Delta\rho$ is the density difference between fresh and ocean water; they also found that $\delta S/S_0$ varies with both F_m and a mixing parameter $P = u_{f0}/u_T$, where u_T is the tidal rms velocity. Fischer (1976) finds however that if an estuarine Richardson number $\mathrm{Ri}_E = g\Delta\rho Q_f/\rho B u_T^3$ is used instead of P, $\delta S/S_0$ depends primarily on Ri_E. From my perturbation solution, I have found that

$$\delta S/S_0 \sim \mathrm{Ri}_E^{1/3} F_m^{1/15} + \text{hot}, \qquad (1.7a)$$

$$u_s/u_{f0} \sim \mathrm{Ri}_E^{1/6} F_m^{-29/30} + \text{hot}, \qquad (1.7b)$$

where hot denotes higher order term. Both expressions agree with Hansen and Rattray, and Fischers' conclusions that $\delta S/S_0$ is insensitive to F_m and u_s is insensitive to Ri_E. They are more realistic in that the dependencies on both Ri_E and F_m are explicitly displayed: an increase in fresh water discharge and/or a decrease in tidal action must increase the stratification which drives a stronger gravitational circulation and results in larger surface velocity until a new equilibrium state is established. Eq. (1.7a) shows that the seaward–landward density difference will always induce a vertical stratification and from (1.7b), is always required to drive a gravitational circulation. Thus a gravitational circulation cannot exist in a strictly vertically homogeneous estuary.

The perturbation expansion approach allows one to study estuaries of arbitrary variations in depth, width, fresh water discharge and mixing coefficients rather than specific variations considered by Hansen and Rattray in their similarity solution methods. When the depth (and/or width and/or the mixing coefficients) increases seaward at a sufficiently fast rate I have found it possible to construct solutions which vary continuously from the upstream limit of the salinity intrusion to the seaward ocean boundary. It is not necessary, therefore, to separate the estuary into an inner, a middle, and an outer region.

Hansen and Rattray assumed a linear longitudinal salinity distribution and a rectangular channel to obtain equations which relate u_s/u_{f0}, $\delta S/S_0$ and ν (Fig. 1), where $(1 - \nu)$ is the fraction of upstream salt transport by the vertical gravitational circulation. I have shown that their equations are valid, to a good degree of accuracy, to estuaries of arbitrary longitudinal variations in width, depth, fresh water discharge, wind stress and

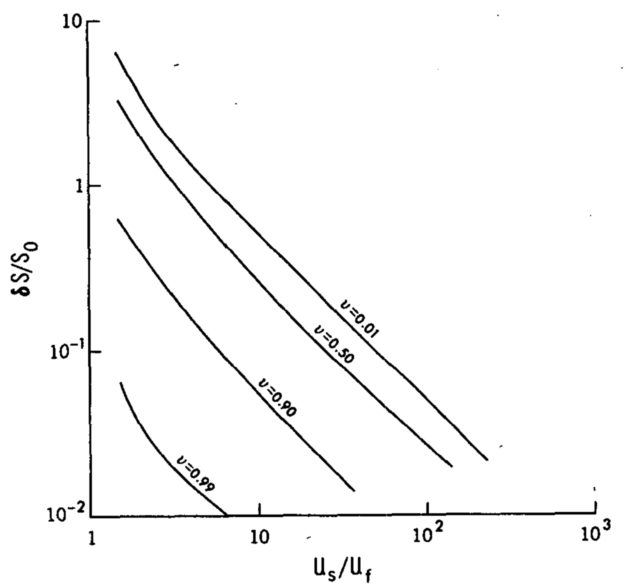

FIG. 1. Hansen-Rattray stratification–circulation diagram. Given $\delta S/S_0$ and u_s/u_f from observations one can predict the fraction ν of upstream salt transport by longitudinal dispersion in an estuary. This relation between ν and $\delta S/S_0$ and u_s/u_f is shown in the text to be independent of longitudinal variations in cross section of the estuary, fresh water discharge rate and the dispersion and mixing coefficients.

various dispersion and mixing coefficients. From observed u_s/u_{f0} and $\delta S/S_0$, one can therefore apply the stratification–circulation diagram with confidence and expect good prediction of ν.

In Section 4, I shall compare the theory with observations in the Mersey, in the channel of the Rio Guayas and in the Hudson River.

In Section 5, I shall compare the theory with a two-dimensional ($xz - t$) model's results in the Hudson River and a three dimensional ($xyz - t$) model's results in New York Harbor. I find to my surprise that the integration time it takes for an estuary to reach an equilibrium state depends very much on its initial state. For an estuary of length of about 250 km, a "bad" initial state can require as many as one hundred days before equilibrium. A "good" initial state is one for which the salinity distribution has an intrusion length which is less than the actual expected (or observed) salt intrusion length.

For the sake of completeness I should mention the type C estuary defined by Pritchard to be one which is homogeneous vertically but is so wide that the centrifugal and/or the Coriolis forces become important and there is a significant salinity difference across its width. We have found in one of our simulations of New York Harbor that this situation occurs at the mouth of the harbor during low discharge (Oey and Mellor, 1983). I shall not consider this type of estuary in this paper.

2. Scaling the governing equations

Consider the following governing equations:

$$B(uu_x + wu_z) + B\frac{P_x}{\rho} = (BA_v u_z)_z, \quad (2.1a)$$

$$\frac{p_z}{\rho} = g, \quad (2.1b)$$

$$(Bu)_x + (Bw)_z = 0, \quad (2.1c)$$

$$B(uS_x + wS_z) = (BK_h S_x)_x + (BK_v S_z)_z, \quad (2.1d)$$

$$\rho = \rho_0(1 + \kappa S), \quad (2.1e)$$

where

- p pressure
- ρ density of water
- A_v turbulent mixing coefficients for momentum;
- ρ_0 density of fresh water
- g ($=9.81$ m s^{-2})
- κ ($=7.5 \times 10^{-4}$),

and I have repeated (1.5) in (2.1d) for convenience. In what follows, I shall assume that the width and the various mixing coefficients are functions of x only.

Define a stream function ψ such that

$$Bw = \psi_x, \quad Bu = -\psi_z;$$

eliminate p and ρ in the above system of equations and obtain

$$\psi_z\psi_{zzx} - 2\psi_z\psi_{zz}\frac{d\ln B}{dx} - \psi_x\psi_{zzz} + B^2 g\kappa S_x + B(A_v\psi_{zz})_{zz} = 0, \quad (2.2a)$$

$$\psi_z S_x - \psi_x S_z + (Bk_h S_x)_x + (BK_v S_z)_z = 0, \quad (2.2b)$$

The boundary conditions are

at $z = 0$:

$$\left.\begin{aligned} \psi &= R(x) \\ A_v\left(\frac{\psi_z}{B}\right)_z &= \tau_w(x) \\ S_z &= 0 \end{aligned}\right\}, \quad (2.3a)$$

at $z = D(x)$:

$$\left.\begin{aligned} \psi &= 0 \\ \psi_x\frac{dD}{dx} - \psi_z &= 0 \\ K_h S_x\frac{dD}{dx} - K_v S_z &= 0 \end{aligned}\right\}, \quad (2.3b)$$

at $x = 0$:

$$\left.\begin{aligned} -\psi_z &= U_0 B_0 \\ S &= S_0 \end{aligned}\right\}, \quad (2.3c)$$

at $x = Li$:

$$S = S_{Li}, \quad (2.3d)$$

where $R(x)$ is the fresh water discharge, $D(x)$ the depth of the estuary and $\tau_w(x)$ the surface (kinematic) wind stress. I shall consider Li to be the salinity intrusion length rather than the physical length L of the estuary. This is important because the dynamics of the density induced flow is determined by $\partial S/\partial x$, which depends on Li, but not on L. The determination of Li is part of the problem. The boundary conditions at $x = 0$ and $x = Li$ should be considered as being tentative. Due to the approximate nature of my solutions part of these boundary conditions may have to be discarded.

Guided by the works of Hansen and Rattray (1965) and Fischer (1972) I define the following nondimensional variables:

$$\Phi = \psi F^{2p}\,\mathrm{Ra}^q/R_0,$$

$$\bar{S} = S/\Delta S,$$

$$F_D = \frac{D}{D_0}, \quad F_B = \frac{B}{B_0}, \quad F_R = \frac{R}{R_0}, \quad F = F_m P_{r0}^{1/2},$$

$$F_A = \frac{A_v}{A_{v0}}, \quad F_h = \frac{K_h}{K_{h0}}, \quad F_v = \frac{K_v}{K_{v0}}, \quad P_{r0} = \frac{A_{v0}}{K_{v0}},$$

$$Z = \frac{z}{d}, \quad X = \frac{x}{L_0},$$

$$d = \frac{D}{F^{2i}\,\mathrm{Ra}^m}, \; L_0 = (K_{h0}/u_{f0})\frac{F^{2j}}{\mathrm{Ra}^{n-1}}, \quad (2.4)$$

where D_0, B_0, A_{v0}, K_{h0} and K_{v0} are some reference values of the corresponding variables, R_0 is the volume rate of fresh water discharge at $x = 0$, ΔS is the salinity difference between fresh water and ocean water, Ra is the estuarine Rayleigh number $\mathrm{Ra} = gk\Delta SD_0{}^3/A_{v0}K_{h0}$, and i, j, m, n, p and q are fixed dimensionless parameters to be defined later. Eq. (2.2) is transformed into:

$$P_{r0}{}^{-1}F^{2(1-2p+3i-j)}\,\mathrm{Ra}^{3m+n-2q}\times\left[\Phi_Z\Phi_{ZZX}-2\Phi_Z\Phi_{ZZ}\frac{d\ln(F_DF_B)}{dX}-\Phi_X\Phi_{ZZZ}\right]+F^{-2j}\,\mathrm{Ra}^nF_D{}^3F_B{}^2\left[S_X-ZS_Z\frac{d\ln F_D}{dX}\right]+F^{8i-2p}\,\mathrm{Ra}^{4m-q}\left(\frac{F_AF_B}{F_D}\right)\Phi_{ZZZZ}=0, \quad (2.5a)$$

$$F_D{}^{-1}F^{2(1-p+i-j)}\,\mathrm{Ra}^{m+n-q}[S_Z\Phi_X-\Phi_ZS_X]-F^{2-4j}\times\mathrm{Ra}^{2n-1}\left\{\left(S_X-ZS_Z\frac{d\ln F_D}{dX}\right)(F_BF_h)_X+(F_BF_h)\times\left[S_{XX}+Z(S_Z+ZS_{ZZ})\left(\frac{d\ln F_D}{dX}\right)^2-ZS_Z\frac{d^2\ln F_D}{dX^2}\right]\right\}-\mathrm{Ra}^{2m}F^{4i}\left(\frac{F_BF_V}{F_D{}^2}\right)S_{ZZ}=0, \quad (2.5b)$$

where for convenience I have omitted the over-bar in S. The solution to (2.1) therefore depends on three dimensionless parameters Ra, F and P_{r0}. Here Ra can be written as

$$\mathrm{Ra}=\left(\frac{u_T}{u_\tau}\right)^2\left(\frac{D_0u_\tau}{A_{v0}}\right)\left(\frac{D_0u_\tau}{K_{h0}}\right)\left(\frac{\mathrm{Ri}_E}{F_m}\right)^{2/3},$$

where u_τ is the friction velocity [$=(|\text{wall shear stress}|/\rho_0)^{1/2}$], and is therefore related to the dimensionless parameters u_T/u_τ, Ri_E, F_m and D_0u_τ/A_{v0} used by Fischer (1972, 1976). In addition, the dimensionless parameter (D_0u_τ/K_{h0}) is introduced. While D_0u_τ/A_{v0} is nearly constant (≈ 0.07) D_0u_τ/K_{h0} depends on physical factors such as irregular coastline and bottom geometry and unsteady winds and its value varies from one estuary to another (see Fischer 1973, Table 2, in which the dispersion coefficient "D" includes the steady vertical circulation contribution which is omitted from my K_{h0}). Henceforth, I shall assume that P_{r0} is of order one and therefore ignore it completely from any scaling considerations. I shall now choose the six parameters i, j, m, n, p and q so that all the derivatives in (2.5) are of order one.

From Pritchard's works (1952, 1954) it is reasonable to assume a balance between the vertical turbulent diffusion term and horizontal salinity gradient term in (2.5a) since it is this balance which gives rise to the density-driven flow structure. Thus

$$4i-p=-j, \quad (2.6a)$$

$$4m-q=n. \quad (2.6b)$$

The length L_0 is now chosen so that it gives a measure of the salinity intrusion length. For this I resorted to the data given in Quirk *et al.* (1971) for the Hudson river. I have plotted the salinity intrusion length versus $(\Delta S)^{0.35}(Q_{f0})^{-0.2}$ in Fig. 2. The fit is fairly good. This gives

$$j = 0.4, \quad n = 0.25. \quad (2.7a)$$

The intrusion length varies more with Q_{f0} than it is with ΔS. The values of ΔS given in Quirk *et al.* range from 19 to 29‰ and I have included the factor $(\Delta S)^{0.35}$ in order that the data points lie closer on the straight line shown in Fig. 2. The original data shows considerable scatter, and the exponent "0.35" may not be accurately determined. Nevertheless, I shall point out shortly that the correlation does give the correct order of magnitude of the scales of the intrusion lengths in other estuaries also. Rigter (1973) and Hamrick (1979) performed a series of laboratory experiments in which they varied ΔS while keeping other flow variables constant. I have examined their data and found that the intrusion length varies like $(\Delta S)^a$, where a ranges from 0.3 to 0.7 for Hamrick's data and >1.0 for Rigter's data. It appears that my value of 0.35 is too low. One should note, however, that the data shows considerable

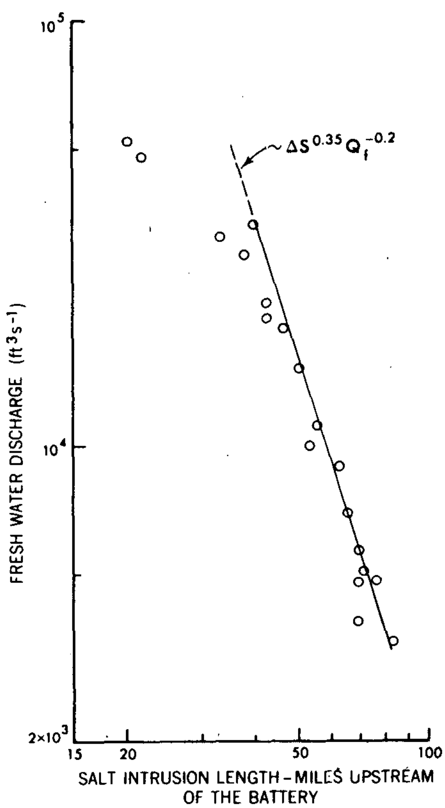

FIG. 2. Correlating the salinity intrusion length L_i in the Hudson River with fresh water discharge rate Q_f and ΔS. L_i depends primarily on Q_f and only slightly on ΔS. The line is the result of applying the given correlation formula to the observations (open circles in the figure).

scatter, and it is uncertain at present what the correct value is. I feel that more data are needed, not only in laboratories, but also in real estuaries.

One might now be tempted to balance the vertical turbulent diffusion of salt with its advection in (2.5b) to obtain another relation. This is suggested by Pritchard's (1954) work in the James River. This will not be useful in the present case, however, because it is the relative balance among terms in (2.5b) which is of interest and this should come out as a product of the analysis rather than its a priori condition. I shall instead require that Φ be of order one and with the data in the James, Hudson and Potomac Rivers and the Mersey Narrows to give estimates of F_m, Ra and $|\psi|/R_0$ as shown in the second through the fourth columns of Table 1, I have chosen

$$p = \frac{1}{5}, \quad q = \frac{-1}{30}. \tag{2.7b}$$

The resulting Φs are given in the fifth column of Table 1. The estimates for $|\psi|R_0^{-1}$ represent some measure of vertical averages of the absolute values of the velocity taken from observations. While it is true that the vertically averaged velocity profile must equal u_{f0} its local value is $\gg u_{f0}$. Since it is the velocity rather than the integral of velocity which is of interest, one must not scale ψ with R_0. The modifying scale $F^{2p}\,\mathrm{Ra}^q R_0^{-1}$ should portray the effects of the density induced current more readily through the dimensionless parameters F and Ra.

Finally, from (2.6)

$$i = \frac{-1}{20}, \quad m = \frac{13}{240}, \tag{2.7c}$$

and (2.5) and (2.3) become

$$P_{r0}^{-1}\alpha\left[\Phi_Z\Phi_{ZZX} - 2\Phi_Z\Phi_{ZZ}\frac{d\ln(F_DF_B)}{dX} - \Phi_X\Phi_{ZZZ}\right] + F_D^3F_B^2\left(S_X - ZS_Z\frac{d\ln F_D}{dX}\right) + \left(\frac{F_BF_A}{F_D}\right)\Phi_{ZZZZ} = 0, \tag{2.8a}$$

$$F_D^{-1}\alpha(S_Z\Phi_X - \Phi_ZS_X) - \alpha\gamma^{-1} \times \Bigg\{(F_BF_h)_X\left(S_X - ZS_Z\frac{d\ln F_D}{dX}\right) + (F_BF_h)\left[S_{XX} + Z(S_Z + ZS_{ZZ})\left(\frac{d\ln F_D}{dX}\right)^2 - ZS_Z\frac{d^2\ln F_D}{dX^2}\right]\Bigg\} - \left(\frac{F_BF_V}{F_D^2}\right)S_{ZZ} = 0, \tag{2.8b}$$

$Z = 0$:

$$\left.\begin{aligned} \Phi &= F^{2p}\,\mathrm{Ra}^qF_R = \tilde{R} \\ \Phi_{ZZ} &= (D_0u_\tau/A_{V0})(u_T/u_\tau)(F^{2p-4i}/\mathrm{Ra}^{2m-q}) \\ &\quad\times(F_BF_D^2/F_A)(\tau_w/u_Tu_{f0}) = \tilde{\tau}_w \\ S_Z &= 0 \end{aligned}\right\}, \tag{2.9a}$$

$Z = Z_1 = \mathrm{Ra}^mF^{2i}$:

$$\left.\begin{aligned} \Phi &= 0 \\ \Phi_Z &= \mathrm{Ra}^{-m}F^{-2i}(D_0/L_0)^2F_DdF_D/dX \\ &\quad\times[\Phi_X - \mathrm{Ra}^mF^{2i}\Phi_Zd\ln F_D/dX] \\ F_vS_Z &= \alpha\gamma^{-1}Z_1F_hF_DdF_D/dX \\ &\quad\times[S_X - Z_1S_Zd\ln F_D/dX] \end{aligned}\right\}, \tag{2.9b}$$

$X = 0$:

$$\left.\begin{aligned} -\Phi_Z &= B_0U_0D(x=0) \\ &\quad\times\mathrm{Ra}^{q-m}F^{2(p-i)}R_0^{-1} = \tilde{U}_0 \\ S &= S_0 \end{aligned}\right\}, \tag{2.9c}$$

$X = X_{Li} = \dfrac{Li}{L_0}$:

$$S = S_{Li}, \tag{2.9d}$$

where

$$\alpha = \mathrm{Ra}^{3m-2q}F^{2(1-2p+3i)} = \mathrm{Ra}^{0.23}F^{0.90}, \tag{2.10a}$$

$$\gamma = \mathrm{Ra}^{1-3m}F^{-6i} = \mathrm{Ra}^{0.84}F^{0.3}. \tag{2.10b}$$

The values of d/D, L_0, L_{in}, α and γ are given in columns six through ten in Table 1, where L_{in} is my

TABLE 1. Determination of the scales for 1) streamfunction Φ, 2) Longitudinal distance $X = x/L_0$ and values of perturbation parameters α and γ from data in four estuaries. The distance from the mouth of the estuary L_{in} is where $S_0 \leqslant 1$‰. (The number in parentheses is the exponent.)

Estuary	F_m	Ra	$\frac{\lvert\psi\rvert}{R_0}$	Φ	$\frac{d}{D}$	L_0 (km)	L_{in} (km)	α	γ
Mersey Narrows	1(−3)	2(+3)	40	2.0	0.5	65	50	0.02	80
Hudson									
(H)	4(−2)	4(+2)	6	1.2	0.5	46	50	0.20	60
(L)	7(−3)	1(+3)	15	1.6	0.4	70	100	0.07	75
James	9(−3)	1(+4)	10	1.1	0.36	57	40	0.16	1500
Potomac	1(−3)	3(+4)	40	1.8	0.3	140	150	0.02	680

estimate of the observed distance from the open ocean boundary, to the point upstream where $S \leqslant 1‰$. My estimates of the relevant quantities used to compute F_m, Ra and L_0 are given in Table 2. One notes the close agreement of L_0 with L_{in} and, because of stratification, d is always less than D, though not by order of magnitude, in agreement with my intuitive picture of the vertical scale of a partially mixed estuary. With all variables properly scaled I can now estimate the order of magnitude of the various terms in (2.8). Eq. (2.8a) gives the expected balance between the horizontal salinity gradient term and the vertical turbulent diffusion term, with the advection term of $O(\alpha)$ which, from Table 1, is <1 for the estuaries listed. Eq. (2.8b) shows that the advection of salt is of $O(\alpha)$ while the horizontal dispersion of salt is of $O(\gamma^{-1})$ smaller. In order for (2.8b) to have a proper balance one must set

$$S_{ZZ} \approx O(\alpha). \quad (2.11)$$

This balance is also suggested by Pritchard's (1954) work in the James River.

Figure 3 shows lines of constant α and β plotted on a Hansen-Rattray stratification circulation diagram. I shall explain the meaning of β more clearly in Section 3 where I shall also show how Fig. 3 is constructed. For now β can be interpreted as being a measure of the longitudinal dispersion term. One notes from the figure that most of the so called partially mixed estuaries (type B in the figure) fall in the region where $\alpha \leqslant 1$. This suggests a perturbation solution to (2.8) in terms of small α. Such a solution can hopefully be useful for $\alpha \leqslant 1$.

3. Approximate solutions to (2.8)

I assume that $\alpha \ll 1$ and place no restrictions on γ. Also, I assume the following perturbation expansions

$$\Phi(X, Z) = \sum_{i=0}^{\infty} \alpha^i \phi_i(X, Z), \quad (3.1a)$$

$$S(X, Z) = \sum_{i=0}^{\infty} \alpha^i \theta_i(X, Z), \quad (3.1b)$$

TABLE 2. Values of relevant quantities used to compute F_m, Ra and L_0 in Table. 1. (The number in parentheses is the exponent).

Estuary	ΔS (‰)	D (m)	$A_{v0} \times 10^3$ (m² s⁻¹)	K_{h0} (m² s⁻¹)	u_{f0} (m s⁻¹)	Q_f (m³ s⁻¹)
Mersey Narrows	30	20	4	200	4(−3)	—
Hudson						
(H)	25	10	1	500	5(−2)	800
(L)	25	10	1	200	1(−2)	200
James	20	7.5	0.25	25	1(−2)	—
Potomac	18	10	0.25	20	1(−3)	70

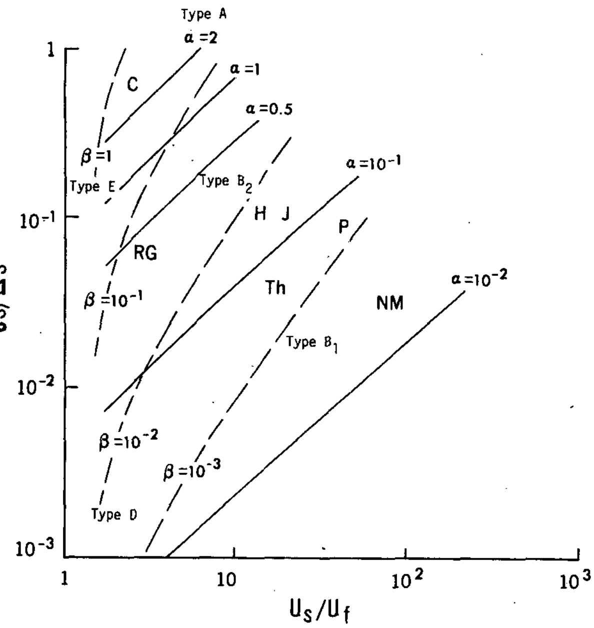

FIG. 3. Pritchard's classification of estuaries interpreted from the present theory by the two parameters: α—a measure of salt advection term and β—a measure of longitudinal salt dispersion. The types A, B, D and E are explained in Table 3. The data points are the approximate positions on this diagram of various estuaries: C—The Columbia River; RG—The Channel of Rio Guayas; H—The Hudson River; J—The James River; Th—The Thames River; NM—The Mersey Narrows; P—The Potomac River.

and let F_B, F_D and F_h vary sufficiently slowly that their first and higher derivatives with respect to X are at most of order one. I substitute (3.1) into (2.8) and (2.9) and obtain the O(1) equations as

$$F_D{}^4 F_B\left(\theta_{0X} - Z\theta_{0Z}\frac{d \ln F_D}{dX}\right) + F_A\phi_{0ZZZZ} = 0, \quad (3.2a)$$

$$\theta_{0ZZ} = 0, \quad (3.2b)$$

$Z = 0$:

$$\phi_0 = \tilde{R}, \quad \phi_{0ZZ} = \tilde{\tau}_w, \quad \theta_{0Z} = 0, \quad (3.3a)$$

$Z = Z_1$:

$$\phi_0 = 0, \quad \phi_{0Z} = 0, \quad \theta_{0Z} = 0, \quad (3.3b)$$

$X = 0$:

$$\left.\begin{aligned} -\phi_{0Z} &= \tilde{U}_0(Z) \\ \theta_0 &= \bar{S}_0 \end{aligned}\right\}, \quad (3.3c)$$

$X = X_{Li}$:

$$\theta_0 = \bar{S}_{Li}, \quad (3.3d)$$

where I have assumed that $D_0/L_0 \leqslant O(\alpha^{1/2})$ in the second of (3.3b) and $\bar{S}_0$ and $\bar{S}_{Li}$ in (3.3c,d) denote vertically integrated averages of S_0 and S_{Li} respectively.

The solutions are

$$\theta_0 = g_0(X), \quad (3.4a)$$

$$\phi_0 = \sum_{i=1}^{3} P_{0i}(X)\phi_{0i}(Z), \quad (3.4b)$$

where

$$\left.\begin{aligned} P_{01}(X) &= F_D{}^4F_BF_A{}^{-1}g_0' \\ P_{02}(X) &= \tilde{\tau}_w(X), \quad P_{03}(X) = \tilde{R}(X) \\ \phi_{01}(Z) &= \frac{-Z^4}{24} + \frac{Z_1Z^3}{16} - \frac{Z_1{}^3Z}{48} \\ \phi_{02}(Z) &= \frac{-Z^3}{4Z_1} + \frac{Z^2}{2} - \frac{Z_1Z}{4} \\ \phi_{03}(Z) &= \frac{Z^3}{2Z_1{}^3} - \frac{3Z}{2Z_1} + 1 \end{aligned}\right\}.$$

To determine the function $g_0(X)$ I need the $O(\alpha)$ salinity equation which, after making use of (3.4a), becomes

$$(F_BF_V/F_D{}^2)\theta_{1zz} + F_D{}^{-1}g_0'\phi_{0Z} + \gamma^{-1}[(F_BF_h)_Xg_0' + (F_BF_h)g_0''] = 0, \quad (3.5)$$

$Z = 0$:

$$\theta_{1Z} = 0, \quad (3.6a)$$

$Z = Z_1$:

$$F_v\theta_{1Z} = \gamma^{-1}Z_1F_hF_D\frac{dF_D}{dX}g_0'. \quad (3.6b)$$

Integrate (3.5) with respect to Z from $Z = 0$ to $Z = Z_1$ and apply (3.6), I obtain

$$g_0'' = \left[\frac{\gamma\tilde{R}}{G} - \frac{d\ln G}{dX}\right]g_0', \quad (3.7a)$$

$$g_0' = C_1\exp\left[\int\left(\frac{\gamma\tilde{R}}{G} - \frac{d\ln G}{dX}\right)dX\right], \quad (3.7b)$$

$$g_0 = C_1\int\exp\left[\int\left(\frac{\gamma\tilde{R}}{G} - \frac{d\ln G}{dX'}\right)dX'\right]dX + C_2, \quad (3.7c)$$

where the constants C_1 and C_2 are to be determined from (3.3c,d) and

$$G(X) = Z_1F_DF_BF_h. \quad (3.8)$$

Hamilton and Rattray (1978) scaled x with K_{h0}/u_{f0}, ψ with R_0 and z with D and used $\mathrm{Ra}F^2 = D_0{}^2u_{f0}{}^2/K_{v0}K_{h0}$ as the small parameter to obtain a perturbation solution of (2.1) and (2.2). In view of the field data and discussions presented above, the magnitudes of the terms in their nondimensional set of equations are not properly ordered. It is therefore difficult to justify the use of a perturbation technique on a mis-scaled set of nondimensional equations. Hamrick (1979) used δ = (depth/intrusion length) as the small parameter to obtain a perturbation solution of (2.1) and (2.2). His set of nondimensional equations is again not properly scaled (see his Eqs. (3.118) through (3.121) and his Table 3.2). In all cases the linearizations which led to (3.2) are the same, and I can obtain solutions which are identical to (3.4). The range of validity of each solution is different in each case, however, and depends on the particular small parameter chosen. Hamilton and Rattray's small parameter is $\approx\mathrm{Ra}^{1/2}\alpha^2$ which from the values of Ra and α given in Table 1 imposes a more restrictive range than simply requiring that α be small. Indeed they indicated that their solution is only valid for well-mixed estuaries in which tidal mixing must be strong enough to reduce the stratification to offset the tendency of the fresh water to strengthen it (i.e., $\mathrm{Ra}F^2 \ll 1$). Hamrick's parameter δ is almost always small and one would have a false impression that the perturbation solution is almost always valid, which is clearly not true. The small parameter α I have chosen here stems from a set of properly scaled equations and therefore gives a more precise indication of the range of validity of the resulting approximate solutions.

The solution for ϕ_0 is Hansen and Rattray's (1965) solution in which $P_{01}\phi_{01}$ represents the density induced flow, $P_{02}\phi_{02}$ the effect of surface wind shear and $P_{03}\phi_{03Z}$ the parabolic velocity profile due to the fresh water discharge and Z-independence eddy viscosity assumption. With only two arbitrary constants C_1 and C_2 at my disposal I choose to abandon the first of (3.3c). I implicitly assume, therefore, that any realistic salinity profile must be such that $g_0']_{X=0} \ll 1$. A typical longitudinal salinity distribution in an estuary displays the inner, middle and outer zone behavior as suggested by Hansen and Rattray (Fig. 4). I should therefore require a positive g_0'' for $0 < X < X_{Li}/2$ say, and a negative g_0'' for $X_{Li/2} < X$. Thus, as one moves seaward along the estuary, there must exist a point beyond which

$$\frac{dF_DF_BF_h}{dX} > \frac{\gamma\tilde{R}}{Z_1} = \mathrm{Ra}^{1-n}F^{2j}, \quad \text{if} \quad F_R = 1, \quad (3.9a)$$

or

$$dF_DF_BF_h/dx > u_{f0}/K_{h0}. \quad (3.9b)$$

In the James River, for example $u_{f0} \approx 1$ cm s^{-1}, $K_{h0} \approx 10$ m^2 s^{-1} and $x \approx 100$ km. (3.9) suggests that $F_DF_BF_h$ must increase by about 100-fold of its value at the landward side of the estuary in order that, as one moves seaward, the salinity distribution displays the middle and the outer zone behavior envisioned by Hansen and Rattray. Observed salinity distribution in the James River suggest that nearly the whole length of the river lies in the middle zone. Since depth and width of the James River is fairly uniform, the 100-fold increase in $F_DF_BF_h$ in the middle zone must therefore be due to F_h alone. The large increase in F_h can be accounted for, e.g., by the transverse dispersion mechanism proposed by Fischer (1972). It is common for the vertical turbulent diffusivity coefficient near the middle zone of the estuary to have a value which is one-tenth or even one-hundredth of its value at the inner zone of the estuary. This is because of the increased stratification as one moves seaward. From (1.6), one sees that this transverse dispersion mechanism

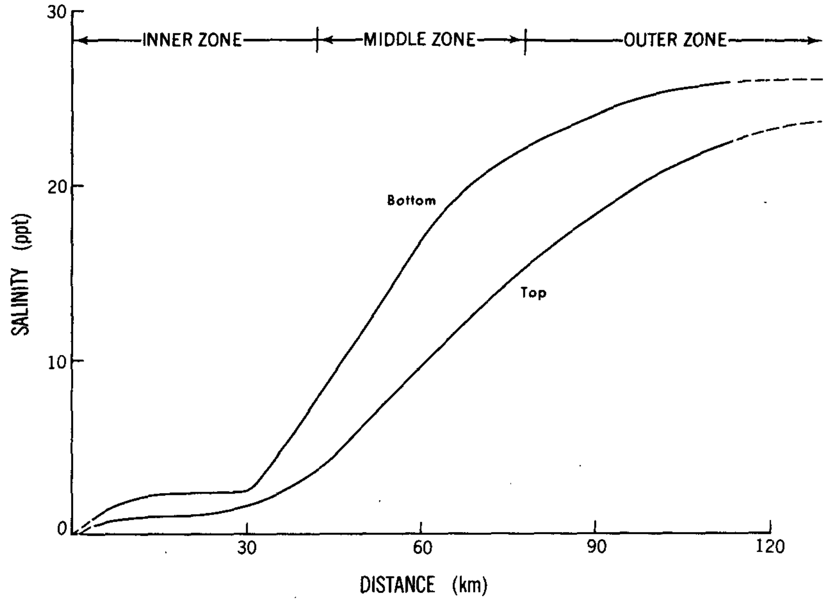

FIG. 4. Typical top and bottom salinity distributions in the Hudson river, numerically calculated by Oey (unpublished results).

alone is capable of producing the required increase in F_h. Near the mouth of the James River where it opens up to Chesapeake Bay, the observed salinity distribution suggests an outer zone behavior, and the large increase in $F_D F_B F_h$ is here provided by large increases in depth and width. Because of these intimate interdependencies between the vertical salinity structure (hence the stratification), the vertical turbulent diffusivity coefficient, the longitudinal dispersion coefficient and the longitudinal salinity structure, it is important in numerical modeling of estuaries to predict well the vertical structure (Oey and Mellor, 1983).

One might question if my analysis is still valid when $dF_h/dX \gg 1$. From (2.8b) I require $\gamma^{-1}dF_h/dX \sim O(1) > \tilde{R}/Z_1 = \mathrm{Ra}^{3m-n}F^{6i+2j}$, from (3.9a). For the James River with $\mathrm{Ra} \approx 10^4$ and $F \approx 9 \times 10^{-3}$, I obtain $\gamma^{-1}dF_h/dX \sim O(1) > O(10^{-1})$, a condition which is easily satisfied. Also, from (2.8), my analysis is valid whenever $d\ln F_D/dX$ (or $d\ln F_B/dX$) $< O(1)$, or $dF_D/dX < O(F_D)$. I can therefore allow fairly large seaward increases in depth and width and still have a correct theory. Such large seaward increases in depth and width occur when a river empties into a large bay as for examples the Raritan and the Hudson rivers emptying into New York Harbor, the James River emptying into Chesapeake Bay and the Delaware River emptying into Delaware Bay. It is clear that such large increases in depth and width must also be accompanied by a large increase in the longitudinal dispersion, as indicated for example by (1.6). It is this combined action of width, depth and dispersion which gives rise to the salinity distributions as observed in real estuaries.

a. Special solutions

To better understand the effects of width, depth and dispersion functions variations upon salinity distribution, density-induced circulation and stratification in estuaries and to see how my theory is related to others' I shall focus on a special power form variation of G:

$$G(X) = (a + bX)^c, \tag{3.10}$$

where $a > 0$, $b > 0$ and c are constants. I assume also constant $R = R_0$ and so obtain from (3.7c)

$$g_0(X) = \begin{cases} (C_1/\gamma\tilde{R}_0)\exp[\gamma\tilde{R}_0(a + bX)^{1-c}/b(1-c)] + C_2, & c \neq 1 \\ (C_1/\gamma\tilde{R}_0)(a + bX)^{\gamma\tilde{R}_0/b} + C_2, & c = 1. \end{cases} \tag{3.11}$$

Setting $c = 1$, $b = \gamma\tilde{R}_0$, $a = Z_1$, $F_D = F_B = 1$ and $P = \gamma\tilde{R}_0/Z_1$, I obtain $F_h = 1 + PX$, or

$$K_h = K_{h0} + u_{f0}x, \tag{3.12a}$$

$$g_0(X) = g_{0l}(X) = C_1X + C_3. \tag{3.12b}$$

This corresponds to the x-variation part of Hansen-Rattray similarity solution in the middle region of the estuary. I shall now show that my perturbation solution reduces to Hansen-Rattray similarity solution in the case when $F_D = F_B = F_R = F_A = F_V = 1$ and $F_h = 1 + PX$. For simplicity of presentation I shall also let $\tilde{\tau}_w = 0$.

The vertical salinity structure comes from the $O(\alpha)$ solution θ_1 which from (3.5) is

$$\theta_1(X, Z) = P_{11}(X) + P_{12}(X)Z^2 + P_{13}(X)\int \phi_{01}dZ + P_{14}(X)\int \phi_{02}dZ + P_{15}(X)\left(\int \phi_{03}dZ - Z\right), \quad (3.13)$$

where

$$\left.\begin{aligned} P_{11}(X) &= \frac{g_1(X)}{F_v} \\ P_{12}(X) &= \frac{-F_D^2(F_BF_hg_0')_X}{2F_vF_B\gamma} \\ P_{13}(X) &= -\left(\frac{F_Dg_0'}{F_BF_v}\right)P_{01}(X) \\ P_{14}(X) &= -\left(\frac{F_Dg_0'}{F_BF_v}\right)\tilde{\tau}_w(X) \\ P_{15}(X) &= -\left(\frac{F_Dg_0'}{F_BF_v}\right)\tilde{R}(X) \end{aligned}\right\}.$$

Here $g_1(X)$ is determined from the $O(\alpha^2)$ salinity equation in the same way as I determined $g_0(X)$ from the $O(\alpha)$ salinity equation

$$g_1(X) = C_4X + C_5, \quad (3.14)$$

where C_4 and C_5 are arbitrary constants. Then

$$S(X, Z) = g_{0l}(X) + \alpha\left[g_1(X) - \left(\frac{PC_1}{2\gamma}\right)Z^2 - C_1^2\int \phi_{01}dZ - C_1\tilde{R}_0\left(\int \phi_{03}dZ - Z\right)\right] + O(\alpha^2), \quad (3.15)$$

after using (3.13). I require that the vertically integrated average of S satisfies (3.3c,d) and obtain

$$\left.\begin{aligned} C_1 &= \frac{\bar{S}_{Li}}{X_{Li}} \\ C_3 &= \bar{S}_0 \\ C_4 &= 0 \\ C_5 &= \frac{C_1PZ_1^2}{6\gamma} - \frac{Z_1^5C_1^2}{576} - \frac{9C_1\tilde{R}_0Z_1}{40} \end{aligned}\right\}.$$

I can now show that $\phi_i = \theta_{i+1} \equiv 0$, $i \geq 1$, so that the perturbation expansions terminate at the zeroth order in α for Φ and at the first order in α for S. The complete solutions are, after casting them in Hansen and Rattray's variables $\xi = xu_{f0}/K_{h0}$, $\eta = z/D_0$ and $S_0 = \Delta S\bar{S}_0$,

$$\frac{S}{S_0} = 1 + \left(\frac{C_1\Delta S}{S_0\,\mathrm{Ra}^{1-n}F^{2j}}\right)\xi + \left(\frac{\Delta S}{S_0}\right)C_1\,\mathrm{Ra}^nF^{2-2j} \times\left[\frac{-7}{120} + \frac{\eta^2}{4} - \frac{\eta^4}{8} + \left(\frac{C_1\,\mathrm{Ra}^nF^{-2j}}{48}\right)\times\left(\frac{-1}{12} + \frac{\eta^2}{2} - \frac{3\eta^4}{4} + \frac{2\eta^5}{5}\right)\right], \quad (3.16a)$$

$$\Psi/R_0 = \frac{1}{2}(2 - 3\eta + \eta^3) - \left(\frac{C_1\,\mathrm{Ra}^nF^{-2j}}{48}\right)(\eta - 3\eta^3 + 2\eta^4). \quad (3.16b)$$

Differentiating (3.16a) with respect to ξ,

$$K_{h0}\frac{dS}{dx} = \nu u_{f0}S_0, \quad (3.17a)$$

where

$$\nu = \frac{C_1\Delta S}{S_0\,\mathrm{Ra}^{1-n}F^{2j}} \quad (3.17b)$$

is the fraction of the total salt flux at $x = 0$ carried upstream by the longitudinal dispersive action caused by transverse dispersion, turbulence, tidal "trapping," etc. (Hansen and Rattray, 1966; Fischer, 1976). If one recalls that Hansen and Rattray define the estuarine Rayleigh number by using S_0 rather than ΔS as used here, one sees that (3.16) and (3.17) reduce identically to Hansen and Rattray's solutions. An equation for ν can also be obtained from its definition:

$$\nu = 1 + \int_0^{D_0} u_vS_vdz/u_{f0}D_0S_0,$$

where u_v and S_v are the deviations of velocity and salinity from their vertical means. The result is

$$\nu = 1 - \frac{\nu F^2\,\mathrm{Ra}}{1680}\left[32 + \frac{76}{48}\left(\nu\,\mathrm{Ra}\frac{S_0}{\Delta S}\right) + \frac{152}{6912}\left(\nu\,\mathrm{Ra}\frac{S_0}{\Delta S}\right)^2\right], \quad (3.18)$$

which is identical with the corresponding formula in Hansen and Rattray (1966).

The above discussions serve several purposes. First, it shows the essential correctness of my theory. The fundamental parameter α is a measure of the state of stratification and can be used to classify estuaries as outlined in Fig. 3. Second, the mathematical frame work based on such α-classification becomes very simple and solutions can be obtained by solving a series of linear equations in the routine expansion method. The analysis can be readily extended to include variable depth, width and other mixing coefficients. Finally, divisions of estuaries into inner, middle and outer zones are convenient physically. They are not, however, important mathematically. In practice, the salinity distribution in an estuary varies continuously and smoothly and this variation can be taken into account by selecting suitable variations of width, depth and mixing coefficients in the theory. What I mean by "suitable" is an open question but the important thing is that in every case I should be able to obtain a global solution valid everywhere in the estuary. That such a solution has a good physical basis and is not merely an observational fit is shown by the agreement offered by Hansen and Rattray's theory with observations (Bowden and Gilligan, 1971; Murray and Siripong, 1978; Section 4 of this paper).

The last remark in the previous paragraph brings up some interesting questions. I shall now show that Hansen and Rattray's theory is remarkably general and within the present small-α approximation framework, can be used to determine with confidence the importance of salt transport by vertical gravitational circulations in many estuaries of various shapes.

b. Generalized Hansen-Rattray theory

The O(1) velocity field obtained from (3.4b) gives the classical two-layer density induced circulation with a seaward flowing water in the top layer and a landward flowing water in the bottom layer. The interaction of this circulation with the O(1), vertically homogeneous salinity structure $\theta_0 = g_0(X)$ gives rise to an O(α) vertical salinity structure given by (3.13). This separation of vertical salinity variability from the horizontal salinity variability allows one to derive a simple generalization of Hansen-Rattray theory.

The fraction ν of the total salt flux at any point x along the estuary carried upstream by the longitudinal dispersion satisfies $K_h dS_0/dx = \nu u_f S_0$, where S_0 is the sectionally mean salinity at x. From (3.1b) I obtain

$$K_h \frac{dS_0}{dx} = (g_0' \Delta S\, \mathrm{Ra}^{n-1} F^{-2j} F_h F_B F_D / S_0 F_R) u_f S_0 [1 + O(\alpha)]$$

so that

$$\nu = (g_0' \Delta S\, \mathrm{Ra}^{n-1} F^{-2j} F_h F_B F_D / S_0 F_R)[1 + O(\alpha)]. \tag{3.19}$$

To obtain an equation for ν I need u_v and S_v, which from (3.4b) and (3.13) are

$$u_v = u_f\left[K_1(1 - 9\eta^2 + 8\eta^3) + \frac{1 - 3\eta^2}{2}\right], \tag{3.20a}$$

$$S_v = S_0 K_2\left[\frac{\eta^2}{4} - \frac{\eta^4}{8} + K_1\left(\frac{\eta^2}{2} - \frac{3\eta^4}{4} + \frac{2\eta^5}{5}\right) + \frac{\nu}{2} K_3 \eta^2\right] + \text{constant}, \tag{3.20b}$$

where

$$\left.\begin{aligned} K_1 &= (\nu\, \mathrm{Ra} S_0 / 48 \Delta S F_h F_B F_D)(F_D{}^4 F_B / F_A) \\ K_2 &= \nu\, \mathrm{Ra} F^2 F_R{}^2 / F_v F_h F_B{}^2 \\ K_3 &= d \ln F_D / d \ln S_0 \end{aligned}\right\}$$

and where the constant is inconsequential in the computation of salt flux by vertical gravitational circulation. From (3.20) I obtain

$$\frac{u_s}{u_f} = K_1 + \frac{3}{2}, \tag{3.21a}$$

$$\delta S/S_0 = K_2\left(\frac{1}{8} + \frac{3K_1}{20} + \frac{\nu}{2} K_3\right). \tag{3.21b}$$

By definition $\nu = 1 + \int_0^1 u_v S_v d\eta / u_f S_0$ and hence

$$\nu = 1 - K_2\left[(32 + 76K_1 + \frac{152}{3} K_1{}^2)1680^{-1} + \frac{\nu}{15} K_3(1 + K_1)\right]. \tag{3.22}$$

Thus, apart from the last term in (3.21b) and (3.22) involving K_3, the relationships between u_s/u_f, $\delta S/S_0$ and ν are identical with those obtained from Hansen-Rattray theory (Eq. 3.18). In practice the variation of depth over the whole length of the estuary is usually much smaller than the variation in salinity and therefore $K_3 \ll 1$. I can then conclude that the Hansen-Rattray $\delta S/S_0$ versus u_s/u_f diagram (Fig. 1) for predicting the importance of vertical gravitational circulation in salt transport is valid under arbitrary longitudinal variations of width, depth, fresh water discharge and various mixing and dispersion coefficients. The theoretical and the observed νs in estuaries for which $\alpha \ll 1$ should therefore agree very well. I have not been able to show that my conclusion is valid also for $\alpha >$ O(1). However, the agreement of my theory with Hansen-Rattray theory when S_0 is linear with x suggests that the condition can be somewhat relaxed, perhaps to the case when $\alpha \sim$ O(1). It is important to note that although Hansen and Rattray derived their model by assuming i) a longitudinal dispersion function which increases linearly seaward at a rate equal to the fresh water discharge velocity, ii) straight rectangular channel and iii) the middle zone of the estuary, the relationships between $\delta S/S_0$, u_s/u_f and ν are completely independent of these conditions. Therefore, Hansen and Rattray's theory is very general.

c. Pritchard's classification of estuaries

I mentioned in the Introduction how Pritchard (1955) classified estuaries according to their dominant tidally averaged advective and diffusive processes. I now express his ideas in mathematical terms. It is not possible to do this in full generality however, so I shall restrict my analysis to the case when the estuary is of constant cross section and $F_h = 1 + PX$, the Hansen-Rattray similarity condition on the longitudinal dispersion function [see Eq. (3.12a)].

From (3.16) I obtain

$$\frac{\delta S}{\Delta S} = \left(\frac{C_1}{8}\right) \mathrm{Ra}^{1/4} F^{1.2} + \left(\frac{C_1{}^2}{320}\right) \mathrm{Ra}^{1/2} F^{0.4}, \tag{3.23a}$$

$$\frac{u_s}{u_f} = 1.5 + \left(\frac{C_1}{48}\right) \mathrm{Ra}^{1/4}F^{-0.8}. \quad (3.23b)$$

Therefore,

$$F^2 = (\delta S/\Delta S)b^{-1}, \quad (3.24a)$$

$$\mathrm{Ra} = a^4[(\delta S/\Delta S)/b]^{1.6}C_1^{-4}, \quad (3.24b)$$

where

$$a = 48\left(\frac{u_s}{u_f} - 1.5\right), \quad b = \frac{a}{8} + \frac{a^2}{320}.$$

Here "C_1" measures the dimensionless salinity gradient which should be ≈ 1 if I have chosen the correct scaled variables. This means that $X \approx 1$, or that the salinity intrusion length is close to the x-length scale L_0 chosen in the analysis. One sees from Table 1 that this is approximately so and I shall henceforth set $C_1 \equiv 1$.

From (2.8b), α measures the magnitude of the salt advection term and $\beta = \alpha F_{hX}/\gamma = \alpha P/\gamma$ measures the magnitude of longitudinal salt dispersion term; α and β are functions of Ra and F and are plotted in Fig. 3 after using (3.24). One notes the internal consistencies of the theoretical analysis which require that $\beta \ll \alpha$ when $\nu \approx 0$ and $\beta \approx \alpha$ when $\nu \approx 1$. Since $\beta < \alpha$ always, the perturbation expansion based on $\alpha \ll 1$ and β arbitrary is therefore valid.

When $\alpha \ll 1$ the estuary can be treated as if it were vertically homogeneous. The estuary may not be of type D however because vertical turbulent salt diffusion is still balanced by salt advection and the longitudinal dispersion of salt remains very small. A truly type D estuary is one for which $\alpha \approx \beta \ll 1$ and occupies only a restricted portion in the Hansen-Rattray stratification-circulation diagram. This type of estuarys is not in the same class as the type for which $\alpha \approx \beta \approx O(1)$ although both types are characterized by $\nu \approx 1$. The underlying mathematics (and physics) are different for both types. When $\alpha \approx \beta \geqslant O(1)$ all terms in the salt balance equation are important and I shall call this type E. The velocity profile does not display the classical two-layer estuarine flow because the fresh water discharge rate is very high, not because there is negligible stratification. In this case $\int_0^1 u_v S_v d\eta \approx 0$ because $u_v \approx 0$, not because $S_v \approx 0$. The Columbia River at high discharge belongs to this type (Hansen, 1965).

When $\alpha \gg \beta \approx O(1)$ the salt advection is important and (2.8b) reduces to $\phi_X S_Z - \phi_Z S_X = O(1/\alpha)$ and one has a type A estuary.

When $\beta \ll \alpha < 1$ vertical turbulent salt diffusion is balanced by salt advection and one has a type B estuary. This is the type of estuary for which I constructed the approximate solution and it clearly includes the type D estuary when $\alpha \approx \beta \ll 1$.

The different types are shown in Fig. 3 where I have further subdivided type B into regions for which $\alpha > 0.1$ (types B_2) and $\alpha < 0.1$ (types B_1). Conditions on α and β which distinguish the different types are given in Table 3. One could conceive an estuary which has characteristics intermediate of types D and E. The channel of the Rio Guayas can be classified as this type. I shall not, however, further break down the types.

TABLE 3. Classification of estuaries (see Fig. 3).

Estuary	Conditions on α and β	Common names
A	$\alpha \gg \beta$	Salt wedge
B_1	$\beta \ll \alpha \leqslant 0.1$	Partially mixed
B_2	$\beta \ll 0.1 \leqslant \alpha \leqslant 1$	
D	$\alpha \approx \beta \ll 1$	Well mixed
E	$\alpha \approx \beta \geqslant O(1)$	—

d. *Dependence of circulation and stratification on* Ri_E *and* F_m

Hansen and Rattray (1966) empirically correlated u_s/u_f and $\delta S/S_0$ to u_f/u_T and F_m and found that u_s/u_f depends solely on F_m while $\delta S/S_0$ depends on both F_m and u_f/u_T. Fischer (1972, 1976) considered Ri_E in place of u_f/u_T and showed that $\delta S/S_0$ depends primarily on Ri_E but only slightly on F_m. From (3.23) I obtain

$$\frac{\delta S}{\Delta S} \approx [(D_0^2 u_T^2/K_{h0}A_{v0})^{1/2}\,\mathrm{Ri}_E^{1/3}F_m^{1/15}]320^{-1}, \quad (3.25a)$$

$$\frac{u_s}{u_f} \approx [(D_0^2 u_T^2/K_{h0}A_{v0})^{1/4}\,\mathrm{Ri}_E^{1/6}F_m^{-29/30}]48^{-1} \quad (3.25b)$$

in which I have dropped the first terms in (3.23) (which amounts to assuming that $\alpha \gg \beta$) and have assumed that $C_1 = 1$ and $P_{r0} = 1$.

These expressions show the weak dependencies of $\delta S/\Delta S$ on F_m and of u_s/u_f on Ri_E. Eq. (3.25a) shows that the seaward–landward density difference will always induce a vertical stratification and from (3.25b), is always required to drive a gravitational circulation. Thus, a gravitational circulation cannot exist in a strictly vertically homogeneous estuary. If I compare (3.25b) with (3.21a) I can obtain the following formula for $\nu\,\mathrm{Ra}S_0/\Delta S$:

$$\nu\,\mathrm{Ra}\frac{S_0}{\Delta S} = \left(\frac{D_0^2u_T^2}{K_{h0}A_{v0}}\right)^{1/4}\left(\frac{u_f}{u_T}\right)^{1/2}F_m^{-1.3}. \quad (3.26)$$

This suggests that $\nu\,\mathrm{Ra}S_0/\Delta S$ could be correlated with $(u_f/u_T)^{1/2}F_m^{-1.3}$ obtained from observations. I have taken the observational data from Hansen and Rattray's paper (1966, their Fig. 3) and these are plotted in Fig. 5, together with my correlation formula (3.26) and Hansen and Rattray's formula:

$$\nu\,\mathrm{Ra}\frac{S_0}{\Delta S} = 16F_m^{-3/4}. \quad (3.27)$$

My straight-line correlation formula (not drawn in Fig. 5) is

$$\nu\,\mathrm{Ra}\frac{S_0}{\Delta S} = 6.5\left(\frac{u_f}{u_T}\right)^{1/2}F_m^{-1.3}. \quad (3.28)$$

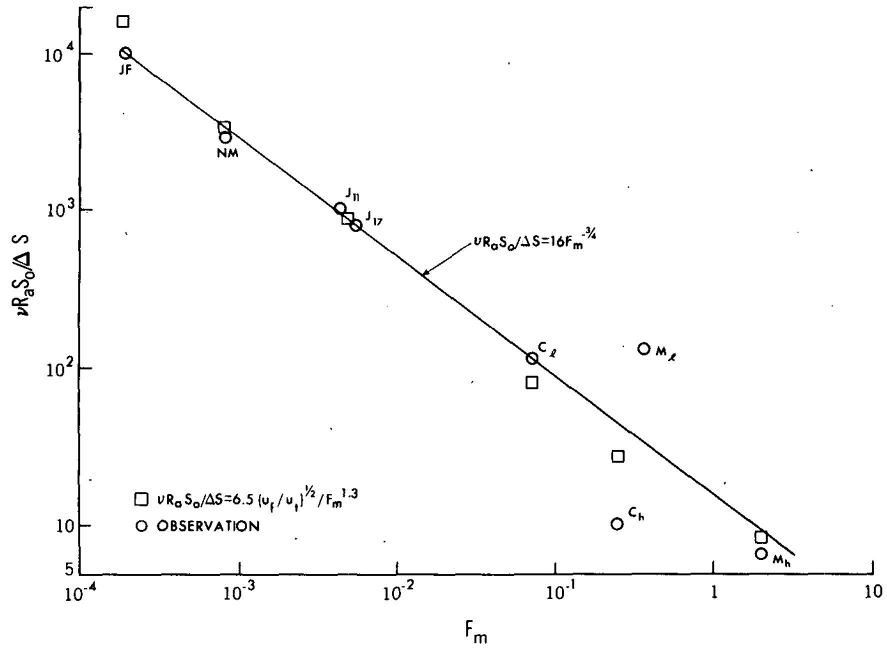

FIG. 5. Correlating ν Ra$S_0/\Delta S$ with F_m and u_f/u_T.

It is difficult to conclude which of the two formulae gives better correlation. I shall show in section 4 that for the channel of Rio Guayas, (3.28) appears to predict u_s/u_f and hence ν more accurately than formula (3.27). More comparison with observations is required to verify (3.27) or (3.28), however.

4. Comparison with observations

I shall compare and interpret the available observations in different estuaries with my theoretical model. I shall consider i) the Mersey Narrows which corresponds very well with Hansen and Rattray middle zone, ii) the channel of the Rio Guayas, a shallow and rigorously mixed estuary and iii) the Hudson River which includes both Hansen and Rattray's middle and outer zones.

a. The Mersey estuary

Bowden and Gilligan (1971) have made extensive measurements of salinity and velocity at four sections along the estuary (Fig. 6). The longitudinal salinity distribution is approximately linear in this region (see their Fig. 4) and corresponds closely to Hansen and Rattray's middle zone. The fourth column in Table 4 shows the observed mean ν given by Bowden and Gilligan. From their $\delta S/S_0$ and u_s/u_f data I have computed ν directly from (3.21) and (3.22) and the results are shown in columns 2 and 3. The range refers to the different discharges which occur during the period of observations. The agreement between theoretical and observed ν's is good. The largest discrepancy is at the Carmmell Lairds section. Two observations in this case were taken at high discharges and I have estimated α to be about 0.8, perhaps not small enough for the theory to apply. At Rock Light the estuary suddenly widens seaward; the original Hansen-Rattray theory clearly breaks down here. The agreement with observation is very good however, as predicted by the general theory.

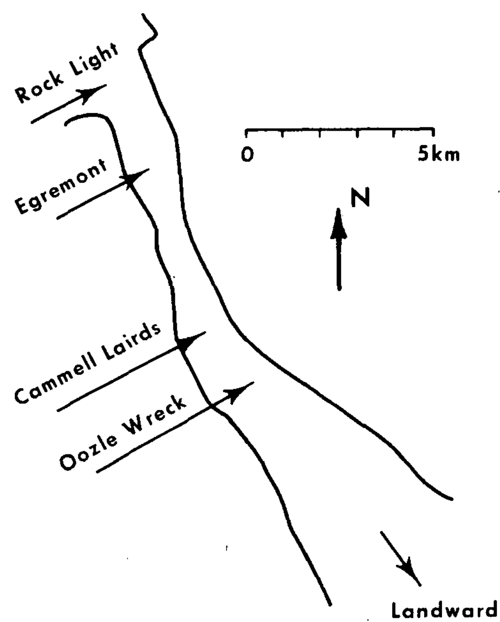

FIG. 6. Location map of the Mersey estuary.

TABLE 4. Comparison with observation in the Mersey (data from Bowden and Gilligan, 1971). The theoretical ν is calculated by using the observed $\delta S/S_0$ and u_s/u_f. $(\delta S/S_0)_{TH}$ and $(u_s/u_f)_{TH}$ are estimated from Hansen and Rattray correlation formulas.

Section in Mersey	ν Theory Range	ν Theory Mean	ν Observed Mean	$\left(\frac{u_s}{u_f}\right)_{OBS}\left(\frac{u_s}{u_f}\right)_{TH}^{-1}$ Mean	$\left(\frac{\delta S}{S_0}\right)_{OBS}\left(\frac{\delta S}{S_0}\right)_{TH}^{-1}$ Mean
Rock Light	0.63–0.95	0.83	0.85	0.42	0.77
Egremont	0.22–0.80	0.53	0.51	1.05	1.02
Carmmell Lairds	0.02–0.47	0.21	0.30	0.97	2.05
Oozle Wreck	0.16–0.93	0.58	0.62	0.95	1.73

A good prediction of ν does not imply a good prediction of $\delta S/S_0$ and u_s/u_f however. If I adopt Hansen and Rattray's (1966) correlation formulas (3.27) and

$$\nu\, \mathrm{Ra} F_m^{\,2} = 20\left(\frac{u_f}{u_T}\right)^{7/5}, \qquad (4.1)$$

I obtain from (3.21) (assume $P_{r0} = 1$)

$$\delta S/S_0 \approx \left[F_m^{-3/4}\left(\frac{u_f}{u_T}\right)^{7/5}\right]\left(\frac{F_D^{\,3}F_R^{\,2}}{F_h^{\,2}F_v F_A F_B^{\,2}}\right), \qquad (4.2a)$$

$$\frac{u_s}{u_f} \approx \left[\frac{1}{3}F_m^{\,3/4}\right]\left(\frac{F_D^{\,3}}{F_h F_A}\right), \qquad (4.2b)$$

where I have also assumed that $K_1 \gg 1$ and $K_3 \ll 1$. The terms in the square brackets in (4.2a,b) are Hansen and Rattray's original forms for $\delta S/S_0$ and u_s/u_f, modified now by the appropriate functions representing the effects of variable width, depth and mixing coefficients, Bowden and Gilligan's values for $(u_s/u_f)_{OBS}/(u_s/u_f)_{TH}$ and $(\delta S/S_0)_{OBS}/(\delta S/S_0)_{TH}$ are reproduced in columns 5 and 6 of Table 4. Here, the subscript OBS denotes observation and TH denotes values obtained from (4.2) without the modifying factors.

Assuming $F_D = F_A = 1$, qualitative effects of increased F_h can be seen from the value of $(u_s/u_f)_{TH}$ at the Rock Light section where $(u_s/u_f)_{TH}$ is too large by about a factor of two and an increased F_h from its value further upstream should remedy this, according to (4.2b). The assumption that $F_D = 1$ throughout the four sections is reasonable. The assumption that $F_A = 1$ is more questionable but F_A can only increase at the Rock Light section due to a more intense mixing there and hence less stratification. This again will reduce $(u_s/u_f)_{TH}$ and improve the agreement. However, F_A should not be sensitive to stratification (Ellison and Turner, 1960). For example, at the Egremont, Carmmell Lairds and Oozle Wreck sections the assumptions that $F_D = F_h = F_A = 1$ lead one to conclude that $(u_s/u_f)_{OBS} \approx (u_s/u_f)_{TH}$.[1]

The overestimation of $(\delta S/S_0)_{TH}$ at the Rock Light can again be corrected by an increase in F_h at this section. The underestimation at Carmmel Lairds is due to the decrease in F_v because of large stratification at this section. The underestimation at Oozle Wreck is more difficult to explain but perhaps can also be due to decrease in F_v. This is because the data points at this section are mostly for higher stratification case where $\delta S/S_0 > 0.1$ and hence the mean is more heavily weighted towards a region of low values of K_v.

I should mention Fischer's (1972) calculations of longitudinal dispersion coefficients in the Mersey. He concluded that the vertical gravitational circulation cannot contribute significantly to upstream salt transport because the transverse shear effects are much more important. My calculations show definitely that this can only be true near the mouth of the Narrows and that in the Narrows both the vertical and transverse shears contribute about equally to the total upstream salt transport.

b. *The channel of the Rio Guayas*

Murray and Siripong (1978) called this a well-mixed estuary, apparently because about 80% of upstream salt transport is carried out by the turbulent diffusive action. The word "well-mixed" is misleading however in this case because of the relatively high value of $\delta S/S_0 \approx 0.14$ over a depth of about 6.5 m. A more exact measure would be to calculate the value of α, which I estimate to be about 0.1, indicating that the vertical stratification is quite significant, as observed.

With $\delta S/S_0 = 0.14$ and $u_s/u_f = 3.0$ I obtain from (3.21) and (3.22) $\nu = 0.938$, which agrees well with the observed ν of 0.92. Murray and Siripong also estimated $\delta S/S_0$ and u_s/u_f using (3.27) and (4.1) which give $u_s/u_f = 5.65$ and $\nu = 0.79$. If I use formula (3.28) I obtain $u_s/u_f = 3.85$ and $\nu = 0.915$, which agree better with the observed values. This good agreement could be fortuitous however because Eq. (4.2) indicate that the estimated u_s/u_f and $\delta S/S_0$ must depend on the geometric shape and mixing coefficients variations along the channel.

c. *The Hudson River estuary*

The portion of the Hudson River estuary that I shall be discussing extends from about 50 km upstream of

[1] Note that in Hansen-Rattray theory K_h must increase linearly seaward at a rate equal to u_{f0}. This increase is small however over the three sections under discussion.

the Battery to the Narrows in the New York Harbor (Fig. 8). The longitudinal salinity distributions for a typical low discharge period and a typical high discharge period are shown in Fig. 7. One sees that at high discharge the region of interest lies wholly in the inner zone where the vertically integrated averaged and tidally averaged salinity distribution is linear. My theory reduces to Hansen-Rattray theory and I expect good prediction of ν. For the low discharge case the outer zone starts just downstream of the Battery due to the more intense mixing in the widened area (Fig. 8) Hansen-Rattray theory is no longer applicable here but my analysis shows that I should still obtain good prediction of ν.

Hunkins (1981) has made salinity and velocity measurements at Yonkers, some 30 km upstream of the Battery and lies wholly in the middle zone at low discharge, and at the Narrows. The fresh water discharges for the three surveys are given in column 2 of Table 5. Based on these numbers, I assume that the salinity distributions correspond to top of Fig. 7 for the low discharge survey and to bottom of Fig. 7 for the high discharge survey. I use the observed $\delta S/S_0$ and u_s/u_f in columns 3 and 4 to compute ν from (3.21) and (3.22). The results are shown in column 6 and can be compared with the observed ν's directly obtained from Hunkins' measurements of salt fluxes across the sections. The agreements are not as good as I have expected and are only slightly improved when the theoretical ν's are corrected for wind effects. Hunkins noted that the cross-channel differences in current and salinity

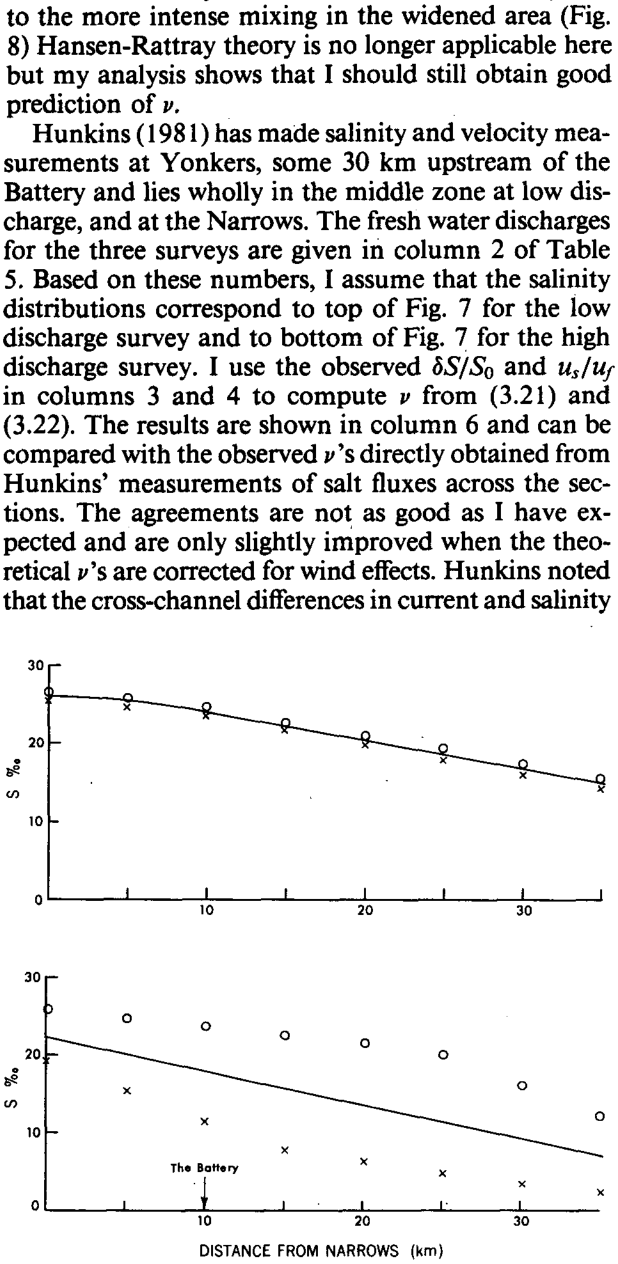

FIG. 7. Salinity distributions in the Hudson (compiled from Bowman, 1977). The upper figure is for Q_f = 190 m^3 s^{-1} and the lower figure is for Q_f = 1400 m^3 s^{-1}: surface salinity (crosses), bottom salinity (open circles), and mean salinity (solid line).

measurements were some five times larger than could be accounted for by Coriolis and centrifugal forces. I think that this anomaly is in some sense spurious and is due to the short sampling period (25 h) and possibly time-dependent effects (Hunkins, p. 733; Oey and Mellor, 1983). These spurious effects result in larger measured dispersion due to the transverse shear and hence larger values of ν.

5. Comparison with numerical simulations

I shall compare my theoretical results with two sets of finite difference numerical model results in the New York Harbor. I have described details of the numerical method in Oey and Mellor (1983). One set of results is from the two-dimensional $xz - t$ simulation of the Hudson River. The other set is from the three-dimensional $xyz - t$ simulation of the New York Harbor (Fig. 8).

a. The xz − t Hudson River model

The model region extends from the Battery where the river empties into New York Harbor, to the dam at Troy, some 250 km north of the Battery. I have set Q_f = 200 m^3 s^{-1} and $\eta = 0.7\sin(2\pi t/T)$, where T is the M$_2$ tidal period ($\approx$12.42 h), and η is the free surface elevation at the open boundary at the Battery. These values represent approximately the conditions of the river in dry seasons. The salinity at the open boundary is computed during ebb using one-sided difference and is assigned a constant 26‰ during flood (Thatcher and Harleman, 1972). The grid spacings are ΔX = 3 km and $\Delta Z \approx$ 1 m. I have also performed a separate calculation with ΔX = 1.5 km and the results agree very well with the coarse grid calculation. This ensures small numerical diffusivity which may give incorrect salinity intrusion length. Since I shall compare model's results with steady-state theory I have also ensured that the calculation has reached an "equilibrium" state defined by

$$\langle \bar{S} \rangle = \frac{1}{T_0}\int_0^{T_0}\left\{\iiint S dz dx dy \times \left[\iint (H+\eta) dx dy\right]^{-1}\right\} dt = \text{constant},$$

where T_0 is an averaging period.

Fig. 9 shows $\langle \bar{S} \rangle$ and $\langle \bar{V} \rangle$, where $\langle \bar{V} \rangle = \frac{1}{T_0}\int_0^{T_0} \times \iint \eta dx dy dt [\iint dx dy]^{-1}$, H is water depth below mean tide level as functions of T_0 for two calculations with different initial conditions set I1 and set I2. The linear, vertically homogeneous initial salinity distributions for both sets are:

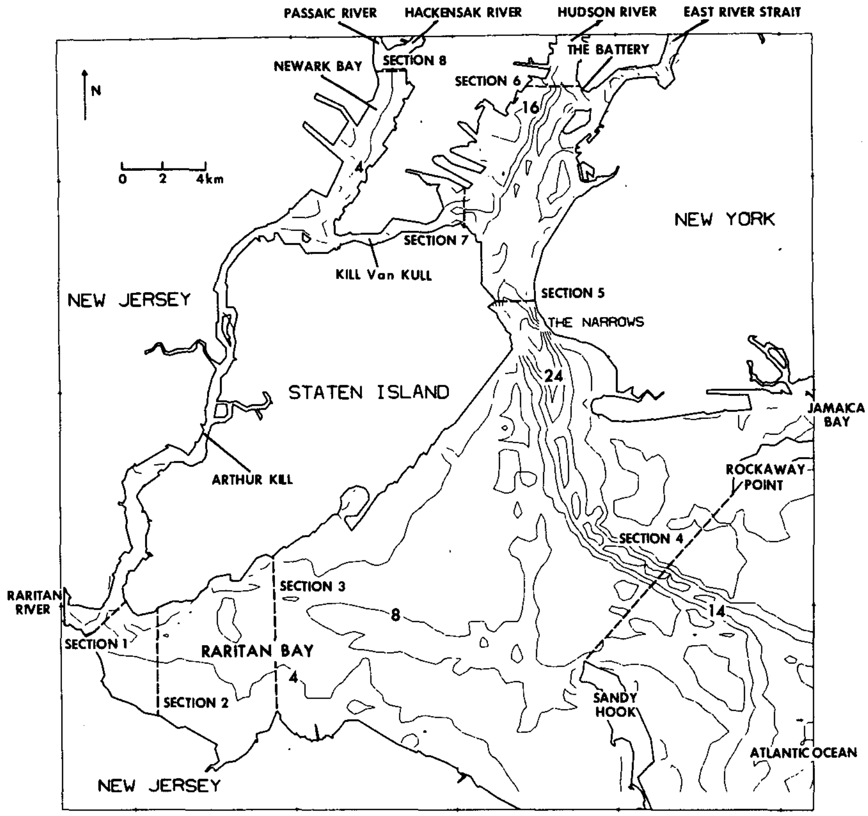

FIG. 8. Map of New York Harbor. The depth contours are in meters below mean tidal level.

Set I1:

$$S_0(x) = 28(12 - x)/12, \quad 12 \geq x \geq 0$$
$$= 0, \quad x > 12,$$

Set I2:

$$S_0(x) = 28, \quad 60 \geq x \geq 0$$
$$= 28(72 - x)/12, \quad x > 60,$$

where x is in km and is zero at the ocean boundary (at the Battery). The final equilibrium salinity intrusion length L_i, defined as the x-length at which $S_0 = 0.1‰$ is 51 km. Thus the initial salinity intrusion length for the set I1 calculation is less than L_i while that for the set I2 calculation is greater than L_i. One sees from Fig. 9 that, although the volume storage $\langle \bar{V} \rangle$ for both sets I1 and I2 quickly come to an equilibrium state in about 30 days, the salt storage $\langle \bar{S} \rangle$ takes a much longer time to settle. For set I1, $\langle \bar{S} \rangle$ reaches an equilibrium state in about 60 days, whereas set I2 has not settled down. For set I1 the absolute value of $d\langle \bar{S} \rangle / dT_0$ is about 10^{-3} ‰ day^{-1} near end of 60 days while the value is 10^{-2} ‰ day^{-1} for set I2. In set I2, the estuary has to empty its excess salt storage and, because of the large ratio of tidal volume to fresh water volume of about 100:1, this process is very slow. The opposite is true for the set I1 calculation, in which the salt storage must increase from its initially low value. The large mixing action of the tide now allows a much

TABLE 5. Comparison with observation in the Hudson River at the Narrows (in New York Harbor) and at Yonkers (~45 km upstream of the Narrows) (data from Hunkins, 1981).

Section	Q_f (m^3 s^{-1})	$\frac{\delta S}{S_0}$	$\frac{u_s}{u_f}$	ν Observed	ν Theory	ν Theory corrected for winds
The Narrows	845	0.58	8	0.19	0.17	0.18
	238	0.14	21	0.53	0.43	0.43
Yonkers	182	0.36	12	0.46	0.18	0.20

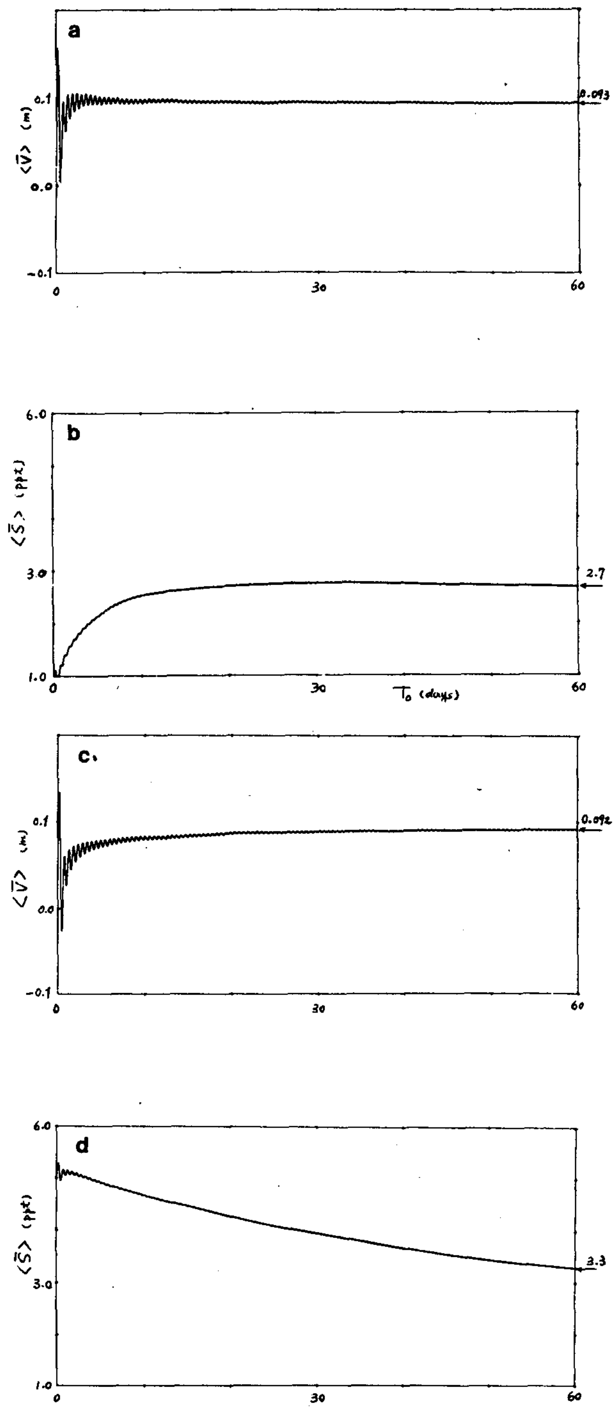

FIG. 9. Time-averaged (a) volume and (b) salt storage for the Set I1 calculation and time averaged (c) volume and (d) salt storage for the Set I2 calculation.

more rapid convergence to an equilibrium state. I have carried out the set I2 calculation for another 60 days before an equilibrium state is reached.

The results of these numerical calculations and comparisons with observations will be reported separately. For the purpose of comparing the numerical model's results with theoretical results I have chosen the simplest case of constant depth and width. The value of α is equal to 6.5×10^{-2} for this calculation when K_H is chosen to be equal to 300 $m^2 s^{-1}$ and the S_0 is shown in Fig. 10. The agreement is good. The numerically calculated S_0 shows the general exponential form predicted by (3.7c) for $F_D = F_h = F_B = 1$.

b. The xyz − t *New York harbor model*

Oey and Mellor (1983) has computed this real-time, three-dimensional model of New York Harbor for ninety days. I shall use the results from the last thirty days' integration. The model is driven by real-time data of tides, fresh water and sewage discharges and winds. Because of its fine grid size (0.5 km in a 30 km × 30 km domain and 1 m in depths of about 10 m) the model resolves both the horizontal and vertical circulations and salinity structures rather well. Oey and Mellor have calculated the various components of salt fluxes according to the format given by Fischer (1972). Fig. 8 shows the various sections at which these fluxes have been computed. From these I have calculated ν_{NUM} shown in the sixth column of Table 6. I have also computed the averages of $\delta S/S_0$ and u_s/u_f across each section and these are shown in columns 3 and 4. From these averages I calculated ν_{TH} from (3.21) and (3.22) with $K_3 = 0$. This is shown in column 5. The agreements are remarkably good, despite the very complex coastline geometry and bottom topography which exist in the harbor.

In Section 7 ν_{NUM} is negative. I interpret this and other discrepancies as being mainly caused by unsteady subtidal oscillations which occur during the simulation period. This and other discussions on the numerical model's results are discussed in Oey and Mellor (1983). Nevertheless, negative ν_{NUM} signifies very strong vertical gravitation circulation, as predicted also by the theoretical value of $\nu_{TH} = 0.10$.

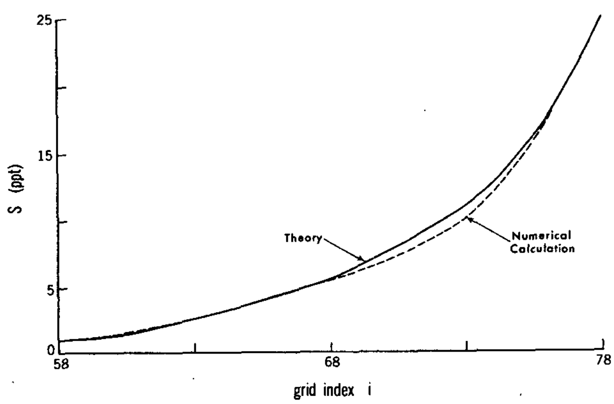

FIG. 10. Comparison of numerical model with theoretical longitudinal salinity distributions in a straight channel of constant geometry.

TABLE 6. Comparison of theoretical ν and numerical model's ν in New York Harbor. Refer to Fig. 8 for locations of the cross sections. (The number in parentheses is the exponent).

Section	Q_f ($m^3 s^{-1}$)	$\frac{\delta S}{S_0}$	$\frac{u_s}{u_f}$	ν_{TH}	ν_{NUM}	F_m	$\frac{S_0}{\Delta S}$	$\frac{\delta S}{S_0}$ from (5.1)
1	17	3.0(−2)	38.2	0.77	0.72	8.0(−4)	0.86	9.0(−3)
	77	1.3(−1)	23.0	0.43	0.48	4.0(−3)	0.59	8.3(−2)
2	9	6.1(−3)	64.0	0.92	0.85	5.0(−4)	0.86	8.0(−3)
	48	3.8(−2)	24.3	0.80	0.70	3.0(−3)	0.58	5.0(−2)
3	9	5.0(−3)	121.0	0.88	0.70	2.0(−4)	0.90	6.0(−3)
	48	2.2(−2)	40.2	0.83	0.80	1.0(−3)	0.65	2.5(−2)
4	185	3.7(−3)	14.0	0.99	0.98	2.0(−3)	0.97	3.6(−3)
	1000	4.0(−2)	6.3	0.92	0.92	9.0(−3)	0.85	2.5(−2)
5	80	2.0(−2)	25.4	0.90	0.91	2.0(−3)	0.92	1.4(−2)
	700	1.8(−1)	6.2	0.87	0.84	2.0(−2)	0.68	8.7(−2)
6	765	3.8(−1)	8.0	0.45	0.43	2.0(−2)	0.54	3.3(−1)
7	36	1.6(−1)	29.0	0.10	<0	2.0(−3)	0.59	2.0(−2)
8	6	3.2(−2)	26.8	0.83	0.85	1.0(−3)	0.61	1.5(−2)
	33	2.2(−1)	12.0	0.50	0.40	7.0(−3)	0.19	2.8(−1)

Finally, I assume $\alpha \gg \beta$, $P_{r0} = 1$ and obtain from (3.16)

$$\delta S/S_0 = 7.2 F_m{}^2 \left(\frac{u_s}{u_f}\right)^2 \left(\frac{\Delta S}{S_0}\right). \tag{5.1}$$

This estimated $\delta S/S_0$ is given in the last column of Table 6 and should be compared with the model's $\delta S/S_0$ of column 3. Again, the agreement is quite good, despite the various assumptions that have been made in Hansen-Rattray theory.

Acknowledgments. I should like to thank one of the referees for bringing Hamilton and Rattray and Hamrick's works to my attention. Thanks also to Johann Callan for her typing. The figures were drafted by P. Tunison and processed by J. Conner. The study was sponsored in part by the New Jersey Sea Grant Program, the Northeast Office of the Office of Marine Pollution Assessment of NOAA.

REFERENCES

Bowden, K. F., and R. M. Gilligan, 1971: Characteristic features of estuarine circulation as represented in the Mersey Estuary. *Limnol. Oceanogr.*, **16**, 490–502.

Bowman, M. J., 1977: Hydrographic Properties. MESA New York Bight Atlas Monogr. 1, New York Sea Grant Institute, Albany, 78 pp.

Ellison, T. H., and J. S. Turner, 1960: Mixing of dense fluid in a turbulent pipe flow, Part 2. Dependence of transfer coefficients on local stability. *J. Fluid Mech.*, **8**, 529–544.

Fischer, H. B., 1972: Mass transport mechanisms in partially stratified estuaries. *J. Fluid Mech.*, **53**, 671–687.

——, 1973: Longitudinal dispersion and turbulent mixing in open channel flow. *Annual Reviews in Fluid Mechanics,* Vol. 5, Annual Reviews, 59–78.

——, 1976: Mixing and dispersion in estuaries. *Annual Reviews in Fluid Mechanics,* Vol. 8, Annual Reviews, 107–133.

Hamilton, P., and M. Rattray, Jr., 1978: Theoretical aspects of estuarine circulation. *Estuarine Transport Processes,* B. Kjerfve, Ed., University of South Carolina Press, 37–73.

Hamrick, J. M., 1979: Salinity intrusion and gravitational circulation in partially stratified estuaries. PhD thesis. University of California, Berkeley, 451 pp.

Hansen, D. V., 1965: Currents and mixing in the Columbia River estuary. *Ocean Science and Ocean Engineering. Trans. Joint Conf. Mar. Tech. Soc., Amer. Soc. Limnol. Oceanogr.*, Washington, 943–955.

——, and M. Rattray, 1965: Gravitational circulation in straits and estuaries. *J. Mar. Res.*, **23**, 104–122.

——, and ——, 1966: New dimensions in estuary classification. *Limnol. Oceanogr.*, **11**, 319–325.

Hunkins, K., 1981: Salt dispersion in the Hudson Estuary. *J. Phys. Oceanogr.*, **11**, 729–738.

Murray, S. P., and A. Siripong, 1978: Role of lateral gradients and longitudinal dispersion in the salt balance of a shallow, well mixed estuary. *Estuarine Transport Processes,* B. Kjerfve, Ed., Univ. of South Carolina Press, 113–124.

Oey, L. Y., and G. L. Mellor, 1983: Real time, 3-D simulation of the Hudson-Raritan Estuary, Part I and II. Manuscripts under revision.

Pritchard, D. W., 1952: Salinity distribution and circulation in the Chesapeake Bay estuarine system. *J. Mar. Res.*, **11**, 106–123.

——, 1954: A study of the salt balance in a coastal plain estuary. *J. Mar. Res.*, **13**, 133–144.

——, 1955: Estuarine circulation patterns. *Proc. ASCE,* Sep. No. 717, Vol. 81.

Quirk, Lawler and Matusky Engineers, 1971: Environmental effects of Bowline generating station on Hudson River. QL and M Project No. 169-1.

Rigter, B. P., 1973: Minimum length of salt intrusion in estuaries. J. Hydraul. Div., ASCE, **99**, (HY9), 1475–1496.

Thatcher, M., and D. R. F. Harleman, 1972: A mathematical model for the prediction of unsteady salinity intrusion in estuaries. R. M. Parsons Lab. Rep. No. 144, MIT, 232 pp.