The Equations for Geostrophic Motion in the Ocean[1]

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ABSTRACT

A multiple-scale approach is used to develop the quasi-geostrophic dynamics for synoptic oceanic scales. This new approach allows the buoyancy frequency N and the Coriolis parameter f to be slowly varying functions of horizontal position. No series expansion of f about some arbitrary central latitude is required. Nor is there a limitation on the geographic extent of the synoptic-scale domain.

Simultaneously, the equation for large-scale geostrophic flow on gyre, or planetary, scales is derived as a solvability or consistency condition on the synoptic-scale dynamics. Thus a single derivation suffices to develop both sets of dynamical equations. In addition, the multiple-scale analysis explicitly describes the interaction terms between the synoptic-scale and gyre-scale fields of motion.

1. Introduction

Since the work of Burger (1958) there has been a general recognition of the existence of two types of geostrophic motion. The first of these, which is now usually termed synoptic-scale quasi-geostrophy, was first systematically derived by Charney (1947) and applied to motions with relatively small scales, on the order of the internal deformation radius. The second class of motion has a truly planetary scale.

In dynamic oceanography the quasi-geostrophic dynamics of type 1 is applicable to the dynamics of midocean eddies, and with some stretching of its Rossby-number limitations, to more intense currents like the Gulf Stream. On the other hand, the motion of type 2 is relevant to thermocline-gyre-scale dynamics whose broad, extensive geometry reaches scales far in excess of the deformation radius.

The traditional derivations of the quasi-geostrophic equations of motion (Phillips, 1963) resort to separate scaling developments for each type of geostrophic motion with no formal hint of the connection between the two types. This leads to a peculiar formal difficulty, especially for the case of the synoptic-scale equations.

In the quasi-geostrophic theory of type 1 the buoyancy frequency N may be a function of height z but formally must be independent of horizontal position and time. This leads to a particularly vexing situation in oceanography where N in fact changes significantly on the gyre scale. Each "patch" of synoptic-scale motion embedded within the gyre scale has a local N used in the theory. The traditional scaling methods give no clue as to how these patches can be stitched together over the gyre. A slavish limitation to formal scaling restrictions leads to unrealistic limitations on the large-scale variations of N. It is clearly unacceptable to be forced to restrict our dynamics to the study of small-scale motions in large-scale gyres whose N is forced to be constant on the gyre scale as well as on the synoptic scale. The essence of the thermocline problem is, in fact, the development of a theory to describe the large-scale variations of density and hence of N.

Equally objectionable is the traditional method of developing the β-plane approximation. In the traditional method, the Coriolis parameter and all spherical metric terms that are functions of latitude are expanded in a Taylor series about some central latitude. It is immediately apparent that the "central" latitude used in these series expansions is rather arbitrary and has no particular physical significance. This troublesome fact is not important *on the synoptic scale* since the differences between one central latitude and another have inconsequential effects on the final theory. Nevertheless, the same difficulty arises as with the determination of the appropriate N to be used: it is not made clear in the traditional theories how well separated domains of synoptic-scale motion smoothly merge on the gyre scale. These questions are of particular importance in oceanography where the scale separation is great between the synoptic (or mesoscale eddy) scale and the gyre scale and yet in which the interaction between motions on these separate scales may be important.

The purpose of the present paper is to present a somewhat more unified approach to these scaling problems. My basic approach is to use the method of multiple space scales. That is, I let the mathematics explicitly recognize the existence of two scales of motion and use the resulting mathematical formalism to

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reproduce what we all believe must be intuitively right, i.e., that slow variations of N and f (the Coriolis parameter) are perfectly allowable. The virtue of the formalism is twofold. It first develops in a systematic way the nature of the alterations of the traditional synoptic-scale dynamics implied by variable N. These alterations will be seen below to be rather obvious and physically plausible. The synoptic-scale equations also contain a variable f which on the synoptic scale is constant, but whose variation on the gyre or planetary scale has the usual profound effect on the synoptic scale.

The second virtue of the formal development is that in the process of deriving the dynamics of type-1 geostrophy, the equations on the gyre scale are also obtained as a solvability (or consistency) condition on the synoptic-scale dynamics. This provides a natural link between the two types of dynamics. In the process, interaction terms between the synoptic and gyre scale motions are clearly exposed.

2. The basic dynamical equations and multiple scaling

Although the method outlined in this section is capable of being extended in an obvious way to more complex situations, I prefer to describe the development of the ideas in the simplest context for greatest clarity. Therefore, I will take as my starting point the hydrostatic and inviscid momentum equations in the Boussinesq and tangent-plane approximations. The reader will easily see how the argument may be immediately generalized to include the full metric complexities of the spherical equations and the meteorologically important consequences of a finite scale height for the density. Similarly, I will ignore the effects of friction and thermal dissipation. I do not wish to imply that any of these approximations is inconsequential, only that the outline of the two-scale development is seen most clearly in the simple system described below. Thus the momentum equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \quad (2.1a)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \quad (2.1b)$$

$$\rho g = -\frac{\partial p}{\partial z}, \quad (2.1c)$$

while mass conservation in the Boussinesq approximation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (2.1d)$$

The thermodynamic equation, in the absence of dissipation, is

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0. \quad (2.1e)$$

In the above equations all variables have their traditional meanings. The Coriolis parameter f is a function of y on the *planetary* scale.

We will now imagine that there are two separate horizontal scales. The larger one, L, is the gyre scale which it is convenient to assume for present purposes is inconsequentially different from the planetary scale. That is, f varies by O(1) on the scale L. The density, pressure and velocity also vary by O(1) on the scale L. In physical oceanography, this is precisely the variation associated with the thermocline structure and this should clearly lead to variations of N on the same scale where

$$N = \left\{-\frac{g}{\rho_0}\frac{\partial \rho}{\partial z}\right\}^{1/2}. \quad (2.2)$$

At the same time the existence of a synoptic, eddy scale implies that the density, pressure and velocity vary on the smaller, synoptic scale l. The central presumption is that

$$\delta = \frac{l}{L} \ll 1. \quad (2.3)$$

Dimensionless independent variables are introduced as follows. To measure variations on the synoptic scale, I use

$$\left.\begin{aligned}(x', y') &= (x, y)/l\\ t' &= \sigma t\end{aligned}\right\} \quad (2.4)$$

while variations on the gyre scale are measured by

$$\left.\begin{aligned}(X, Y) &= (x, y)/L\\ T &= \sigma t l/L\end{aligned}\right\}. \quad (2.5)$$

In (2.4) and (2.5) a characteristic frequency for the synoptic scale, σ, has been introduced. For most purposes σ can be chosen to be the advective time on the synoptic scale:

$$\sigma = \frac{U}{l}, \quad (2.6)$$

where U is the characteristic magnitude of the synoptic scale velocity field. I assume that the velocity scale for the gyre-scale motion is no larger than U.

The vertical coordinate z is nondimensionalized with the scale D, i.e.,

$$z' = \frac{z}{D}. \quad (2.7)$$

Then each dependent variable may be scaled as follows. For the horizontal velocity:

$$\left.\begin{aligned}u &= Uu'(x', y', z', t', X, Y, T)\\ v &= Uv'(x', y', z', t', X, Y, T)\\ w &= U\frac{D}{l}w'(x', y', z', t', X, Y, T)\end{aligned}\right\}. \quad (2.8)$$

Thus each velocity component is explicitly written as a function of both the synoptic-scale independent variables and the gyre-scale variables.

The Coriolis parameter f is a function only of Y. Thus

$$f = f_0 f'(Y), \tag{2.9}$$

where f_0 is simply twice the earth's rotation rate (2Ω) and f' is the sine of latitude which may be written entirely in terms of Y.

The density field is conveniently written as

$$\rho = \rho_0 + \frac{\rho_0 f U l}{gD}\rho' \tag{2.10}$$

with a corresponding pressure field

$$p = -\rho_0 g z + \rho_0 f U l p'. \tag{2.11}$$

Note that ρ_0 is a constant. The basic vertical density gradient for the synoptic scale motion must be included in ρ'.

The second key parameter is the synoptic-scale Rossby number,

$$\epsilon = \frac{U}{f_0 l}, \tag{2.12}$$

and my basic presumption is that $\epsilon \ll 1$.

The basic equations must now be rewritten in terms of dimensionless variables. The key observation is that each horizontal spatial derivative in (2.1) is transformed as

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{1}{l}\left(\frac{\partial}{\partial x'} + \delta\frac{\partial}{\partial X}\right) \\ \frac{\partial}{\partial y} &= \frac{1}{l}\left(\frac{\partial}{\partial y'} + \delta\frac{\partial}{\partial Y}\right) \end{aligned}\right\}, \tag{2.13a}$$

while similarly

$$\frac{\partial}{\partial t} = \frac{U}{l}\left(\frac{\partial}{\partial t'} + \delta\frac{\partial}{\partial T}\right). \tag{2.13b}$$

Noting this point the set (2.1) may be rewritten

$$\epsilon\left(\frac{\partial u}{\partial t} + \delta\frac{\partial u}{\partial T}\right) + \epsilon\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \epsilon w\frac{\partial u}{\partial z} + \epsilon\delta\left(u\frac{\partial u}{\partial X} + v\frac{\partial u}{\partial Y}\right) - fv = -\frac{\partial p}{\partial x} - \delta\frac{\partial p}{\partial X},$$

$$\epsilon\left(\frac{\partial v}{\partial t} + \delta\frac{\partial v}{\partial T}\right) + \epsilon\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \epsilon w\frac{\partial v}{\partial z} + \epsilon\delta\left(u\frac{\partial v}{\partial X} + v\frac{\partial v}{\partial Y}\right) + fu = -\frac{\partial p}{\partial y} - \delta\frac{\partial p}{\partial Y}, \tag{2.14a,b}$$

$$\rho = \frac{\partial p}{\partial z}, \tag{2.14c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \delta\left(\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y}\right) = 0, \tag{2.14d}$$

$$\epsilon\left(\frac{\partial \rho}{\partial t} + \delta\frac{\partial \rho}{\partial T}\right) + \epsilon\left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right) + \epsilon\delta\left(u\frac{\partial \rho}{\partial X} + v\frac{\partial \rho}{\partial Y}\right) = 0. \tag{2.14e}$$

For typographic simplicity, the primes have been *dropped* from the dimensionless variables. Hence x, y and t correspond to the independent variables that describe the flow on synoptic scale. The goal is now to discuss approximations to (2.14) as ϵ and $\delta \to 0$.

3. The asymptotic expansion of the equations of motion

If we denote dimensional variables by an asterisk, then (2.10) becomes

$$\rho_* = \rho_0 + \rho_0\frac{f_0^2 l^2}{gD}\epsilon\rho, \tag{3.1}$$

while the dimensional, vertical density gradient is given by

$$\frac{1}{\rho_0}\frac{\partial \rho_*}{\partial z_*} = \frac{f_0^2 l^2}{gD^2}\epsilon\frac{\partial \rho}{\partial z}. \tag{3.2}$$

The synoptic scale is characterized by the fact that its scale l is of the order of the Rossby internal deformation radius, i.e.,

$$l \sim \left(-\frac{gD^2}{\rho_0}\frac{\partial \rho_*}{\partial z_*}\right)^{1/2} f_0^{-1}. \tag{3.3}$$

This suggests that ρ in (3.1) should contain, to leading order, a term like ϵ^{-1}.

Thus each variable will be expanded in the following asymptotic series, e.g., for ρ:

$$\rho = \epsilon^{-1}(\rho^{(0)} + \epsilon\rho^{(1)} + \epsilon^2\rho^{(2)} + \cdots). \tag{3.4}$$

Before proceeding, some relationship between ϵ and δ must be specified. The analysis is considerably simplified if we assume that ϵ and δ are both of the same order. The final equations will then contain the ratio δ/ϵ, and it will be possible to vary this ratio to recover interesting limiting orderings between δ and ϵ.

If the asymptotic series (3.4) for each dependent variable is inserted into (2.14) and like orders in ϵ are equated, we obtain a sequence of approximate equations. At $O(\epsilon^{-1})$, we obtain

$$\left(u^{(0)}\frac{\partial u^{(0)}}{\partial x} + v^{(0)}\frac{\partial u^{(0)}}{\partial y}\right) + w^{(0)}\frac{\partial u^{(0)}}{\partial z} - fv^{(0)} = -\frac{\partial p^{(0)}}{\partial x}, \tag{3.5a}$$

$$\left(u^{(0)}\frac{\partial v^{(0)}}{\partial x} + v^{(0)}\frac{\partial v^{(0)}}{\partial y}\right) + w^{(0)}\frac{\partial v^{(0)}}{\partial z} + fu^{(0)} = -\frac{\partial p^{(0)}}{\partial y}, \tag{3.5b}$$

$$\rho^{(0)} = -\frac{\partial p^{(0)}}{\partial z}, \tag{3.5c}$$

$$\frac{\partial u^{(0)}}{\partial x}+\frac{\partial v^{(0)}}{\partial y}+\frac{\partial w^{(0)}}{\partial z}=0, \tag{3.5d}$$

$$u^{(0)}\frac{\partial \rho^{(0)}}{\partial x}+v^{(0)}\frac{\partial \rho^{(0)}}{\partial y}+w^{(0)}\frac{\partial \rho^{(0)}}{\partial z}=0, \tag{3.5e}$$

where, recall, f is independent of x and y.

To preserve geostrophy at lowest order for the velocity field, $u^{(0)}$ and $v^{(0)}$ must be independent of x, y and z, so that the nonlinear terms in (3.5a,b) vanish. In fact, the more natural condition to apply is that $u^{(0)}$ and $v^{(0)}$ themselves identically vanish. It then follows from (3.5d) that

$$\frac{\partial w^{(0)}}{\partial z}=0. \tag{3.6}$$

Hence if $w^{(0)}$ vanishes at some z-level, it is zero for all z. I will assume this is the case so that

$$w^{(0)}=0. \tag{3.7}$$

Thus,

$$u^{(0)}=v^{(0)}=w^{(0)}=0, \tag{3.8}$$

while

$$\left.\begin{aligned}\rho^{(0)}&=\rho^{(0)}(X,Y,z,T)\\ p^{(0)}&=p^{(0)}(X,Y,z,T)\end{aligned}\right\}. \tag{3.9}$$

Thus the lowest-order density and pressure fields are functions only of X and Y and not x and y. The density field $\rho^{(0)}$ reflects the thermocline-gyre scale. It is a function of z as well so that the buoyancy frequency, to lowest order, is

$$N=\left(-f_0^2\frac{l^2}{D^2}\frac{\partial \rho^{(0)}}{\partial z}\right)^{1/2} \tag{3.10}$$

and is a slow function of horizontal coordinates. I have also anticipated a fairly obvious result of the theory by allowing $\rho^{(0)}$ and $P^{(0)}$ to depend only upon the slow time, T.

The next-order momentum balance yields

$$\left.\begin{aligned}fv^{(1)}&=\frac{\partial p^{(1)}}{\partial x}+\frac{\delta}{\epsilon}\frac{\partial p^{(0)}}{\partial X},\\ fu^{(1)}&=-\frac{\partial p^{(1)}}{\partial y}-\frac{\delta}{\epsilon}\frac{\partial p^{(0)}}{\partial Y}.\end{aligned}\right. \tag{3.11a,b}$$

The geostrophic velocities of this order have two components. The relatively rapid variation of the pressure, $p^{(1)}$, varying on the synoptic scale combines with the slower variations of the thermocline-gyre scale pressure, $p^{(0)}$, to yield the O(1) velocity field.

The O(1) continuity equation,

$$\frac{\partial u^{(1)}}{\partial x}+\frac{\partial v^{(1)}}{\partial y}+\frac{\partial w^{(1)}}{\partial z}=0, \tag{3.12}$$

implies that

$$\frac{\partial w^{(1)}}{\partial z}=0; \tag{3.13}$$

since $p^{(0)}$ is independent of x and y. The density equation implies that

$$w^{(1)}\frac{\partial \rho^{(0)}}{\partial z}=0, \tag{3.14}$$

so that (3.13) and (3.14) both imply that

$$w^{(1)}=0. \tag{3.15}$$

The O(ϵ) equations yield

$$\frac{\partial u^{(1)}}{\partial t}+u^{(1)}\frac{\partial u^{(1)}}{\partial x}+v^{(1)}\frac{\partial u^{(1)}}{\partial y}-fv^{(2)}=-\frac{\partial p^{(2)}}{\partial x}-\frac{\delta}{\epsilon}\frac{\partial p^{(1)}}{\partial X}, \tag{3.16a}$$

$$\frac{\partial v^{(1)}}{\partial t}+u^{(1)}\frac{\partial v^{(1)}}{\partial x}+v^{(1)}\frac{\partial u^{(1)}}{\partial y}+fu^{(2)}=-\frac{\partial p^{(2)}}{\partial y}-\frac{\delta}{\epsilon}\frac{\partial p^{(1)}}{\partial Y}, \tag{3.16b}$$

$$\frac{\partial u^{(2)}}{\partial x}+\frac{\partial v^{(2)}}{\partial y}+\frac{\partial w^{(2)}}{\partial z}+\frac{\delta}{\epsilon}\left(\frac{\partial u^{(1)}}{\partial X}+\frac{\partial v^{(1)}}{\partial Y}\right)=0. \tag{3.16c}$$

If (3.16a) is differentiated with respect to y and (3.16b) is differentiated with respect to x and the results are subtracted we obtain

$$\frac{\partial \zeta^{(1)}}{\partial t}+u^{(1)}\frac{\partial \zeta^{(1)}}{\partial x}+v^{(1)}\frac{\partial \zeta^{(1)}}{\partial y}=-f\left(\frac{\partial v^{(2)}}{\partial y}+\frac{\partial u^{(2)}}{\partial x}\right)-\frac{\delta}{\epsilon}\left(\frac{\partial^2 p^{(1)}}{\partial Y\partial x}-\frac{\partial^2 p^{(1)}}{\partial y\partial X}\right), \tag{3.17}$$

where

$$\zeta^{(1)}=\frac{\partial v^{(1)}}{\partial x}-\frac{\partial u^{(1)}}{\partial y}$$

is the synoptic-scale relative vorticity. If (3.16c) is used in (3.17), it follows that

$$\frac{\partial \zeta^{(1)}}{\partial t}+u^{(1)}\frac{\partial \zeta^{(1)}}{\partial x}+v^{(1)}\frac{\partial \zeta^{(1)}}{\partial y}=f\frac{\partial w^{(2)}}{\partial z}+\frac{\delta}{\epsilon}\left[\left(\frac{\partial u^{(1)}}{\partial X}+\frac{\partial v^{(1)}}{\partial Y}\right)f-\frac{\partial^2 p^{(1)}}{\partial Y\partial x}+\frac{\partial^2 p^{(1)}}{\partial y\partial X}\right], \tag{3.18}$$

or, with (3.11a,b),

$$\frac{\partial \zeta^{(1)}}{\partial t}+u^{(1)}\frac{\partial \zeta^{(1)}}{\partial x}+v^{(1)}\frac{\partial \zeta^{(1)}}{\partial y}=f\frac{\partial w^{(2)}}{\partial z}-\frac{\delta}{\epsilon}\left[\frac{1}{f}\frac{df}{dY}\right]\left[\frac{\partial p^{(1)}}{\partial x}+\frac{\delta}{\epsilon}\frac{\partial p^{(0)}}{\partial X}\right]. \tag{3.19}$$

Note that

$$\frac{\delta}{\epsilon}\frac{df}{dY}=\frac{l^2}{U}\frac{f_0}{L}\frac{df}{dY}=\frac{\beta_* l^2}{U}, \tag{3.20}$$

where

$$\beta_*=\frac{df}{dy_*}$$

is the usually defined dimensional β, i.e., the northward gradient of the Coriolis parameter. We see in (3.19) the natural emergence of the important, synoptic-scale parameter

$$\beta = \frac{\beta_* l^2}{U}, \quad (3.21)$$

which we note here is actually varying on the *slow* space scale Y but is constant on the synoptic scale.

To complete the theory, the density equation must be used to find $w^{(2)}$. The $O(\epsilon)$ density equation yields

$$\frac{\partial}{\partial t}\rho^{(1)} + u^{(1)}\frac{\partial \rho^{(1)}}{\partial x} + v^{(1)}\frac{\partial \rho^{(1)}}{\partial y} + w^{(2)}\frac{\partial \rho^{(0)}}{\partial z} = -\frac{\delta}{\epsilon}\left(\frac{\partial}{\partial T}\rho^{(0)} + u^{(1)}\frac{\partial \rho^{(0)}}{\partial X} + v^{(1)}\frac{\partial \rho^{(0)}}{\partial Y}\right). \quad (3.22)$$

It is now straightforward to use (3.22) to eliminate $w^{(2)}$ in (3.19). However, it is first useful to introduce the following notation. Let

$$(v_S^{(1)}, u_S^{(1)}) = \frac{1}{f}\left(\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}\right)p^{(1)}, \quad (3.23a)$$

$$(v_T^{(1)}, u_T^{(1)}) = \frac{1}{f}\left(\frac{\partial}{\partial X}, -\frac{\partial}{\partial Y}\right)p^{(0)}, \quad (3.23b)$$

so that from (3.11)

$$(v^{(1)}, u^{(1)}) = (v_S^{(1)}, u_S^{(1)}) + \frac{\delta}{\epsilon}(v_T^{(1)}, u_T^{(1)}), \quad (3.24)$$

where the T-subscripted velocities are associated with variations only on the thermocline-gyre scale while S-subscripted velocities vary on the synoptic scale. The $O(1)$ velocity is thus decomposed into motion of synoptic and gyre scale.

It is also useful to introduce

$$q^{(1)} = \zeta^{(1)} + f\frac{\partial}{\partial z}\left(\frac{\rho^{(1)}}{\partial \rho_0/\partial z}\right), \quad (3.25)$$

$$Q^{(1)} = f\frac{\partial \rho^{(0)}}{\partial z} = Q^{(1)}(X, Y, z, T), \quad (3.26)$$

which effects a similar decomposition of potential vorticity between the synoptic and gyre scale.

If (3.22) is now used to evaluate $w^{(2)}$ in (3.19), we obtain, after a little algebra,

$$\left\{\frac{\partial}{\partial t} + u_S^{(1)}\frac{\partial}{\partial x} + v_S^{(1)}\frac{\partial}{\partial y}\right\}q^{(1)} + \beta v_S^{(1)} + \frac{\delta}{\epsilon}\left[v_S^{(1)}\frac{\partial}{\partial z}\left(\frac{\partial \rho^{(0)}/\partial Y}{\partial \rho^{(0)}/\partial z}\right) + u_S^{(1)}\frac{\partial}{\partial z}\left(\frac{\partial \rho^{(0)}/\partial X}{\partial \rho^{(0)}/\partial z}\right) + v_T^{(1)}\frac{\partial q^{(1)}}{\partial y} + u_T^{(1)}\frac{\partial q^{(1)}}{\partial x}\right]$$
$$= -\frac{\delta}{\epsilon}\left[f\frac{\partial}{\partial z}\left(\frac{\partial \rho^{(0)}/\partial T}{\partial \rho^{(0)}/\partial z}\right) + f\frac{\delta}{\epsilon}\left\{u_T^{(1)}\frac{\partial}{\partial z}\left(\frac{\partial \rho^{(0)}/\partial X}{\partial \rho^{(0)}/\partial z}\right) + v_T^{(1)}\frac{\partial}{\partial z}\left(\frac{\partial \rho^{(0)}/\partial Y}{\partial \rho^{(0)}/\partial z}\right)\right\} + \frac{\delta}{\epsilon}\frac{df}{dy}v_T^{(1)}\right], \quad (3.27)$$

where $u_S^{(1)}$, $v_S^{(1)}$, $u_T^{(1)}$, and $v_T^{(1)}$ are given in terms of $p^{(1)}$ and $p^{(0)}$ by (3.23a,b) and where $q^{(1)}$ may be evaluated entirely in terms of $p^{(1)}$ as

$$q^{(1)} = \frac{1}{f}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)p^{(1)} - f^2\frac{\partial}{\partial z}\left(\frac{\partial p^{(1)}/\partial z}{\partial \rho^{(0)}/\partial z}\right)\right]. \quad (3.28)$$

If (3.10) is used,

$$q^{(1)} = \frac{1}{f}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)p^{(1)} + f\frac{\partial}{\partial z}\left(\frac{\partial p^{(1)}/\partial z}{S}\right)\right], \quad (3.29)$$

where

$$S(X, Y, z, T) = \frac{N^2 D^2}{f_*^2 l^2} \quad (3.30)$$

in which N, given by (3.10), may be a slowly varying function of X and Y while f_* (the dimensional Coriolis parameter, $f_0 f$) is a slowly varying function of Y. The reader will note the fact that $q^{(1)}$ is the usually defined quasi-geostrophic potential vorticity except that traditionally S is taken as a function only of z. Eq. (3.27) is the fundamental equation of our analysis. As it stands, it may be thought of as a predictive equation for $q^{(1)}$ if the gyre-scale structure is completely known. Of course, this is not the case and further consideration is required as described in the following section.

4. Solvability conditions and the final equations

The right-hand side of (3.27) is independent of x, y and t and acts as a uniform source term for the individual rate of change of quasi-geostrophic potential vorticity. Unless suitably restricted, this source term will lead to secular growth in $q^{(1)}$ on the synoptic scale. We, therefore, expect that the right-hand side must separately vanish. This notion can be made more precise if we note that (3.27) may be written

$$\frac{\partial}{\partial t}q^{(1)} + \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} = C(X, Y, z, T), \quad (4.1)$$

where

$$A = u_S^{(1)}q^{(1)} + \beta\frac{p^{(1)}}{f} + \frac{\delta}{\epsilon}\left[\frac{p^{(1)}}{f}\frac{\partial}{\partial z}\left(\frac{\partial \rho_0/\partial Y}{\partial \rho_0/\partial z}\right) + u_T^{(1)}q^{(1)}\right], \quad (4.2a)$$

$$B = v_S^{(1)}q^{(1)} + \frac{\delta}{\epsilon}\left[\frac{p^{(1)}}{f}\frac{\partial}{\partial z}\left(\frac{\partial \rho_0/\partial X}{\partial \rho_0/\partial z}\right) + v_T^{(1)}q^{(1)}\right], \quad (4.2b)$$

and where C is the right-hand side of (3.27).

Now, imagine (4.1) to be integrated over an area that is large compared to l^2 (the area of a synoptic domain) but small compared to L^2 (the gyre area). Let this area be $l^2 A_0$ in size. A_0 is therefore a large number. Then since the second and third terms on the left-hand side of (4.1) combine to form the divergence of the vector (A, B) on the synoptic scale, the size of the integral of these terms is of the order of the perimeter of A_0, i.e., of $O(A_0^{1/2})$. Thus to $O(A_0^{-1/2})$

$$\frac{\partial}{\partial t}\iint_{A_0} q^{(1)}dxdy = \iint_{A_0} C(X, Y, z, T)dxdy$$

$$= A_0 C(X, Y, z, T). \quad (4.3)$$

It is possible that the area average over A_0 of $q^{(1)}$ vanishes. More likely it is finite and order A_0. In the second case, in order that $q^{(1)}$ remain finite, it is necessary that C vanish. Of course, this must also occur if the first case obtains. Hence in either case a necessary condition for the existence of bounded solutions of (3.27) is that the right-hand side of (3.27) vanish, i.e., that

$$\frac{\partial}{\partial z}\left(\frac{\partial\rho^{(0)}/\partial T}{\partial\rho^{(0)}/\partial z}\right) + \frac{\delta}{\epsilon f}\left[\frac{\partial p^{(0)}}{\partial X}\frac{\partial}{\partial z}\left(\frac{\partial\rho^{(0)}/\partial Y}{\partial\rho^{(0)}/\partial z}\right) - \frac{\partial p^{(0)}}{\partial Y}\frac{\partial}{\partial z}\left(\frac{\partial\rho^{(0)}/\partial X}{\partial\rho^{(0)}/\partial z}\right)\right] + \frac{\delta}{\epsilon}\frac{1}{f}\frac{\partial p^{(0)}}{\partial X}\frac{df}{dY} = 0, \quad (4.4)$$

where, recall

$$\rho^{(0)} = -\frac{\partial p^{(0)}}{\partial z}. \quad (4.5)$$

Equation (4.4) is the usual thermocline-gyre-scale equation for a nondissipative fluid (Needler, 1967). In the present case, for $\delta/\epsilon = O(1)$, there are no coupling feedback terms from the synoptic scale. I will return to this point below.

To interpret (4.4) it is useful to reconsider (3.22). If $w^{(2)}$ is split into a synoptic-scale contribution $w_S^{(2)}$ plus a gyre-scale contribution $w_T^{(2)}$ such that

$$\iint_{A_0} w_S^{(2)}dxdy = 0, \quad (4.6)$$

then the condition that $\rho^{(1)}$ remain finite implies that

$$\frac{\partial\rho^{(0)}}{\partial T} + u_T^{(1)}\frac{\partial\rho^{(0)}}{\partial X} + v_T^{(1)}\frac{\partial\rho^{(0)}}{\partial Y} + w_T^{(2)}\frac{\partial\rho^{(0)}}{\partial z} = 0. \quad (4.7)$$

With the aid of (4.7), (4.4) may be rewritten as

$$\boxed{\left(\frac{\partial}{\partial T} + \frac{\delta}{\epsilon}u_T^{(1)}\frac{\partial}{\partial X} + \frac{\delta}{\epsilon}v_T^{(1)}\frac{\partial}{\partial Y} + w_T^{(2)}\frac{\partial}{\partial z}\right)Q^{(1)} = 0,} \quad (4.8)$$

where $Q^{(1)}$ is given by (3.26). That is, the governing equation on the gyre scale is the equation for the conservation of gyre-scale potential vorticity, $f\partial\rho_0/\partial z$, supplemented by geostrophy. When (4.8) is written in terms of $p^{(0)}$, (4.4) is obtained as the equation to be used in calculations. It is a simple matter to show that (4.8) and (4.7) in turn imply the Sverdrup vorticity balance:

$$\frac{df}{dY}\frac{\delta}{\epsilon}v_T^{(1)} = f\frac{\partial w_T^{(2)}}{\partial z}. \quad (4.9)$$

Thus, the usual equations for an ideal-fluid model of the thermocline, or gyre-scale flow, are completely derived as solvability conditions in our attempt to derive equations for the synoptic-scale flow.

For completeness, I note that in *dimensional* units (4.7) and (4.8) are simply

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\rho = 0 \quad &(4.10a)\\ \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)f\frac{\partial\rho}{\partial z} = 0 \quad &(4.10b)\end{aligned}\right\},$$

wherein each variable refers entirely to the gyre-scale variables.

Now let us return to our consideration of the equation for the synoptic-scale motion. Since the right-hand side of (3.27) vanishes, we now obtain directly the equations for the synoptic scale namely, using (3.23a), (3.23b) and (3.5c),

$$\left(\frac{\partial}{\partial t} + \frac{1}{f}\frac{\partial p^{(1)}}{\partial x}\frac{\partial}{\partial y} - \frac{1}{f}\frac{\partial p^{(1)}}{\partial x}\frac{\partial}{\partial x}\right) \times\left[\frac{1}{f}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)p^{(1)} + \frac{\partial}{\partial z}\left(\frac{\partial p^{(1)}}{\partial z}S^{-1}\right) + \beta y\right]$$
$$+ \frac{\delta}{\epsilon}\left(u_T^{(1)}\frac{\partial q^{(1)}}{\partial x} + v_T^{(1)}\frac{\partial q^{(1)}}{\partial y}\right) + \frac{\delta}{\epsilon}\left[\frac{1}{f}\frac{\partial p^{(1)}}{\partial x}\frac{\partial}{\partial z}\left(\frac{\partial u_T^{(1)}}{\partial z}S^{-1}\right) + \frac{1}{f}\frac{\partial p^{(1)}}{\partial y}\frac{\partial}{\partial z}\left(\frac{\partial v_T^{(1)}}{\partial z}S^{-1}\right)\right] = 0. \quad (4.11)$$

In the limit $\delta/\epsilon \to 0$, (4.11) becomes identical to the quasi-geostrophic potential-vorticity equation traditionally derived for synoptic-scale motions (e.g., Pedlosky, 1979). For nonzero values of δ/ϵ there are two additional groups of terms in (4.11) but the presence of these terms is easily understandable and represents the natural extension of the theory. The first of these terms, $u_T^{(1)}(\partial q^{(1)}/\partial x) + v_T^{(1)}(\partial q^{(1)}/\partial y)$, merely represents the advection of the small-scale, synoptic potential vorticity by the large-scale velocity field of gyre scale. The last grouping of terms in (4.11) reflects the advection by the synoptic-scale velocity field of potential vorticity due to the large-scale gradient of the quasi-geostrophic potential vorticity. Both these terms are familiar from studies of quasi-geostrophic stability theory in which these terms represent the contributions to the eddy-potential-vorticity balance by the presence of the basic flow. In the present case, the additional terms arise from our embedding of the synoptic-scale motion in a larger-scale field that has a nonvanishing potential-vorticity gradient. The identification of the last set of terms in (4.11) with the potential-vorticity gradient of the large-scale flow is discussed in detail by Charney and Stern (1962).

Of greater significance is the presence in (4.11) of the functions f and S. Each varies laterally on the gyre scale, but as in the traditional scaling arguments they are formally independent of the synoptic-scale horizontal coordinates. However, it is now no longer necessary to identify N^2 and S (see 3.30) with a gyre-scale average, but merely with an average over areas A_0 that are large compared with l^2 but still small compared with L^2. That is, the deformation radius that appears in S in (4.11) varies continuously from one synoptic region to another. Similarly, the Coriolis parameter actually appearing in (4.11) is constant. Its variation with Y has already led to the beta term in (4.11) without the artificial expansion of f about an arbitrarily defined "central" latitude.

5. Synoptic-scale feedback on the gyre-scale

If δ/ϵ is O(1), the analysis given above shows that equating like orders in ϵ yields no effect of the synoptic scale on the gyre dynamics. In fact, some rectified synoptic eddy effect can be anticipated at higher order. That is, at higher order we can anticipate that not all the nonlinear terms involving quadratic products of synoptic-scale fields can be written in terms of a *synoptic-scale* divergence. It is generally rather complicated to determine these higher-order effects. Their importance, however, may be significant out of proportion to their size. As Rhines and Young (1981) have emphasized, the effect of residual rectified eddy activity on a basically advective dynamics can sometimes be crucial in resolving certain ambiguities in the ideal-fluid model.

Consider equation (3.11b). Let us imagine the interesting situation where the synoptic-scale motion is larger in size than the gyre-scale velocity. That is, consider the case where δ/ϵ is small. To keep things as simple as possible, I will suppose that although $\delta < \epsilon$, it is also true that $\delta > \epsilon^2$. In that case, the next-order contribution to the vorticity and density balances beyond the terms we have so far considered would come from the terms of $O(\epsilon\delta)$ in (2.14a,b,e). It is then straightforward to show that (3.27) would be modified by the addition of the terms

$$R = \left[\frac{\partial}{\partial X}(u_S^{(1)}q^{(1)}) + \frac{\partial}{\partial Y}(v_S^{(1)}q^{(1)})\right] \qquad (5.1)$$

to the left-hand side. The areal average over A_0 of R will not vanish if there is a rectified flux of synoptic-scale potential vorticity. If this term is added to (4.4), we obtain

$$\frac{\partial}{\partial z}\left(\frac{\partial\rho^{(0)}/\partial T}{\partial\rho^{(0)}/\partial z}\right) + \frac{\delta}{\epsilon f}\left[\frac{\partial p^{(0)}}{\partial X}\frac{\partial}{\partial z}\left(\frac{\partial\rho^{(0)}/\partial Y}{\partial\rho^{(0)}/\partial z}\right) - \frac{\partial p^{(0)}}{\partial Y}\frac{\partial}{\partial z}\left(\frac{\partial\rho^{(0)}/\partial X}{\partial\rho^{(0)}/\partial z}\right)\right] + \frac{\delta}{\epsilon}\frac{1}{f}\frac{\partial p^{(0)}}{\partial X}\frac{df}{dY}$$
$$= -\epsilon\left[\frac{\partial}{\partial X}\langle u_S^{(1)}q^{(1)}\rangle + \frac{\partial}{\partial Y}\langle v_S^{(1)}q^{(1)}\rangle\right], \qquad (5.2)$$

where angle brackets on the terms on the right-hand side of (5.2) denote spatial averaging over the regional scale A_0. The resulting terms are then independent of x and y, and I assume such regional averages will also be independent of the fast time t. Since ϵ^2/δ is presumptively small, the additional terms in (5.2) are, in principle, numerically small, but they may have a significant role to play in the solution of (5.2) as noted above.

6. Discussion

Through the use of multiple-space-scale analysis, we have been able to derive, in a single, uniform way, equations for both the large-scale and synoptic-scale geostrophic dynamics. In the limit $\delta/\epsilon \to 0$, the equations (4.11) and (5.2) reduce to the traditional equations first described by Charney (1947) and Burger (1958). The advantage of the present treatment is its explicit recognition that the large-scale, slowly varying fields that enter into the synoptic-scale dynamics have a dynamical context of their own. This allows N^2 in the synoptic-scale dynamics to vary slowly and continuously on the gyre scale. The use of the multiple-scale analysis restricts only the *scale* of variation of synoptic fields but *not* the geographic extent of the synoptic domain as do the traditional treatments. It is also unnecessary to expand f about a central latitude which in the traditional theory is unsatisfactorily arbitrary.

For finite δ/ϵ the theory also describes the interaction terms between the two scales of motion. This allows us to consider problems in which both scales are simultaneously present. One such obvious problem involves the instability of thermocline-scale motions. Starting with a solution of (5.2) without the eddy terms, the linear version of (4.11) and its finite-amplitude extensions could be used to calculate the synoptic eddy field and then this information can be used to calculate the changes in thermocline structure found by the eddy-flux terms. Note that the present formalism allows a systematic method within the geostrophic theory of calculating the slow temporal evolution of static stability which will affect the stability in the synoptic scale.

At the present time numerical experiments that want to describe accurately the order-one variations of isopycnal depth and that at the same time are able to resolve the synoptic scale have been forced to use the relatively costly primitive equations. This has been so not because the motion is not geostrophic, but only to account for isopycnal variations too large to be encompassed by the traditional quasi-geostrophic dynamics. It may be possible to implement the two-scale dynamics developed in this paper to describe such situations while retaining the simplicity of geostrophy.

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