The Effect of a Mean Current on Kelvin Waves

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ABSTRACT

The effect of a mean current on Kelvin waves is considered, including the case when the current is not in geostrophic balance. Because the problem is non-separable, a perturbation scheme is developed for weak shear from which the effect of the mean current on the wave phase speed and decay rate are determined. The perturbation scheme also establishes that in general the Kelvin wave contains all vertical modes. The special case of a two-layer fluid is considered in more detail.

1. Introduction

Kelvin waves are particularly simple members of the class of coastal trapped waves. Requiring only the presence of rotation and a vertical coastal wall, they propagate along the coast (with the coast to the right in the Northern Hemisphere) and their amplitudes decay away from the coast. In the absence of any basic mean flow there is an infinite set of Kelvin waves, determined by their vertical modal structure (see, for instance, LeBlond and Mysak, 1978). The barotropic mode is characterized by a pressure field which is approximately depth-independent; its decay scale is the external Rossby radius of deformation and its phase speed is that of an external gravity wave. Since their wavelengths are typically much greater than the continental shelf width, they are thought to contribute a significant component to the semi-diurnal and diurnal tidal amplitudes at the coast. By contrast the baroclinic modes have much shorter length scales characterized by the internal Rossby radii of deformation; their phase speeds are those of internal gravity waves. Internal Kelvin waves have been observed in lakes, estuaries and straits.

The observations which motivated this study are those of Thomson and Huggett (1980) in Johnstone Strait, British Columbia. They identified both barotropic and baroclinic Kelvin waves, the latter being generated by the interaction of the barotropic tide with a topographic feature. A distinctive feature of their observations is the presence of a depth-dependent mean flow, which is not in geostrophic balance. The purpose of this paper is then to explore the effect of such depth-dependent mean flows on Kelvin waves. In Section 2 the problem is formulated for an inviscid, incompressible fluid in the Boussinesq and hydrostatic approximations, and the governing equations are derived. Although our main purpose is to consider Kelvin waves we have included some discussion of Poincaré waves. The principal difficulty caused by the presence of a depth-dependent mean flow is that the system of equations and side-wall boundary conditions are no longer separable. Hence in Section 3 we develop a perturbation theory for small shear in the mean flow. Expressions for the effect of the mean flow on the phase speed and decay rate are derived. However, the most significant conclusion from the perturbation theory is that each Kelvin wave in general contains all vertical modes. In Section 4 we consider the special case of a two-layer fluid, and obtain more specific results than the perturbation theory allows for. Section 5 is a summary.

2. Formulation

We shall consider an inviscid incompressible fluid in a rotating frame of reference contained in a channel of characteristic depth H and width L. We shall make systematic use of the hydrostatic approximation (i.e., $H/L \ll 1$) and the Boussinesq approximation for which $N_1^2 H/g \ll 1$, where N_1^2 is a typical value of the Brunt-Väisälä frequency. The basic flow consists of a current $v_0(x, z)$ in the y direction and a corresponding pressure field $p_0(x, z)$, density $\rho_0(x, z)$ and free surface displacement $\zeta_0(x)$. Using nondimensional variables where x, y are scaled by L, z by H, time t by $|f_1|^{-1}$ where f_1 is the Coriolis parameter, the equations governing the basic flow are

$$-fv_0 + p_{0x} = F_0, \tag{2.1a}$$

$$\mu^2 p_{0z} + \rho_0 = 0, \tag{2.1b}$$

$$p_0 = 0 \quad \text{on} \quad z = \mu^2 \zeta_0. \tag{2.1c}$$

Here the basic current has been scaled by $|f_1 L|$, the

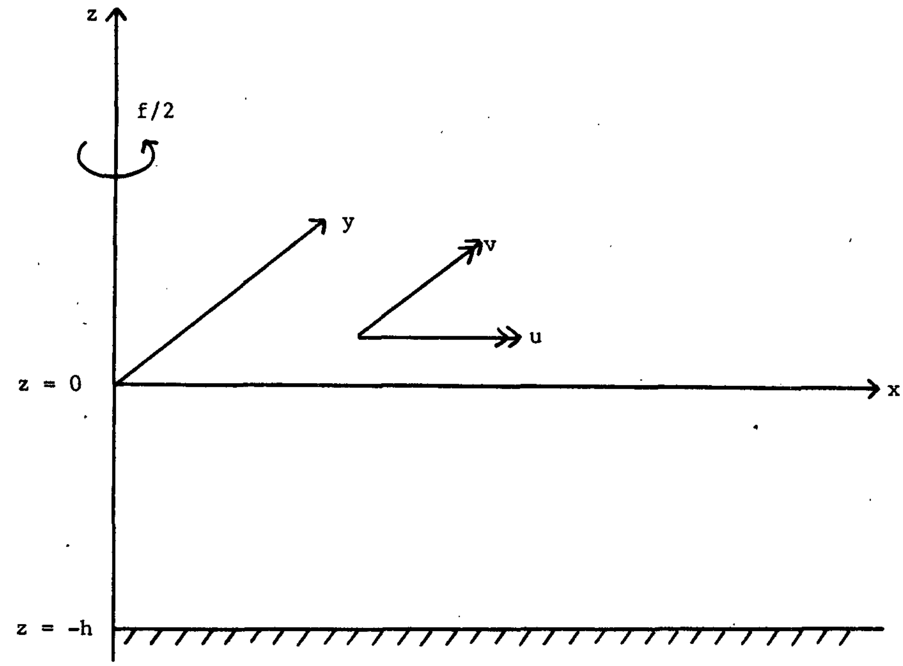

FIG. 1. The coordinate system (x, y, z) where z is vertical; u, v are the horizontal perturbation velocity components.

pressure by $\rho_1 f_1^2 L^2$, the density by ρ_1, and the free surface displacement by $f_1^2 L^2/g$. The non-dimensional parameter μ is defined by

$$\mu^2 = \frac{f_1^2 L^2}{gH}, \tag{2.2}$$

and is the ratio of the length scale L to the external Rossby radius; f is the non-dimensional Coriolis parameter and takes the values +1 (−1) in the Northern (Southern) Hemisphere; and $F_0(x, z)$ is a body force introduced to allow $v_0(x, z)$ to be specified arbitrarily, the case $F_0 \equiv 0$ corresponding to geostrophic mean flow. The non-dimensional Brunt-Väisälä frequency N is defined by

$$\rho_{0z} = -\sigma\rho_0 N^2, \tag{2.3a}$$

where

$$\sigma = N_1^2 H g^{-1}. \tag{2.3b}$$

Eliminating the pressure between (2.1a, b and c) we obtain

$$-f v_{0z} = -SM^2 + F_{0z}, \tag{2.4a}$$

$$f v_0 = \zeta_{0x} - F_0 \quad \text{on} \quad z = \mu^2 \zeta_0, \tag{2.4b}$$

where

$$\rho_{0x} = -\sigma\rho_0 M^2, \tag{2.4c}$$

$$S = \frac{N_1^2 H^2}{f^2 L^2}. \tag{2.4d}$$

The non-dimensional parameter S is the square of the ratio of the internal Rossby radius to the length scale L. Note that $\sigma = \mu^2 S$ and terms of relative order σ are systematically neglected.

The perturbation equations are obtained by linearizing the full nonlinear equations about the basic flow. The horizontal perturbation velocities u, v (see Fig. 1) are scaled by U, and the vertical velocity w by HU/L; neglect of the nonlinear terms requires that the Rossby number $U/|f_1 L|$ be small. The perturbation pressure p is scaled by $\rho_1 U|f_1 L|$, the perturbation density ρ by $\sigma\rho_1 U/|f_1 L|$, the perturbation free surface displacement by $|f_1 L|U/g$, and the vertical displacement η by $HU/|f_1 L|$. Using the hydrostatic and Boussinesq approximations the perturbation equations are

$$u_t + v_0 u_y - fv + p_x = 0, \tag{2.5a}$$

$$v_t + v_0 v_y + fu + v_{0x} u + v_{0z} w + p_y = 0, \tag{2.5b}$$

$$p_z + S\rho = 0, \tag{2.5c}$$

$$\rho_t + v_0 \rho_y - N^2 w - M^2 u = 0, \tag{2.5d}$$

$$u_x + v_y + w_z = 0, \tag{2.5e}$$

$$\eta_t + v_0 \eta_y - w = 0. \tag{2.5f}$$

The boundary conditions are

$$w, \text{ or } \eta = 0 \quad \text{on} \quad z = -h, \tag{2.6a}$$

$$p - \zeta = 0 \quad \text{on} \quad z = \mu^2 \zeta_0, \tag{2.6b}$$

$$\mu^2(\zeta_t + v_0 \zeta_y + \zeta_{0x} u) - w = 0 \quad \text{on} \quad z = \mu^2 \zeta_0. \tag{2.6c}$$

The equations contain two parameters μ and S. For the baroclinic modes which scale with the internal Rossby radius, the divergence parameter μ is small. Neglecting terms of $O(\mu^2)$ replaces the free surface boundary conditions with a rigid lid. For the barotropic mode which scales with the external Rossby radius S is a small parameter; neglecting terms of $O(S)$ shows that the pressure p is depth-independent and equals ζ. The equations are supplemented by side-wall conditions at $x = 0, l$ say, where the x component of velocity u must vanish.

We seek solutions which describe sinusoidal waves propagating in the y direction with frequency ω and wavenumber m. Thus suppose that all variables are of the form

$$p = \mathrm{Re}[\tilde{p}(x, z)\exp(imy - i\omega t)], \text{ etc.} \quad (2.7)$$

For further simplicity and in view of the applications proposed in the Introduction we shall also assume henceforth that $v_0(z)$ is independent of x, and that $\zeta_0 \equiv 0$. From (2.4a) it follows that M^2 is a constant and that F_0 is likewise independent of x. Integration then gives

$$fv_0 = SM^2z - F_0. \quad (2.8)$$

Substituting (2.7) into (2.5a–f) and simplifying yields

$$u(f^2 - \bar{\omega}^2) = i\bar{\omega}(p_x - fm\bar{\omega}^{-1}p + fv_{0z}\eta), \quad (2.9a)$$

$$v(f^2 - \bar{\omega}^2) = fp_x - m\bar{\omega}p + \bar{\omega}^2v_{0z}\eta, \quad (2.9b)$$

$$(f^2 - \bar{\omega}^2)(p_z + SN^2\eta) = SM^2(p_x - fm\bar{\omega}^{-1}p + fv_{0z}\eta), \quad (2.9c)$$

$$p_{xx} - m^2p - (f^2 - \bar{\omega}^2)\eta_z = -fv_{0z}(\eta_x + fm\bar{\omega}^{-1}\eta), \quad (2.9d)$$

where

$$\bar{\omega} = \omega - mv_0. \quad (2.9e)$$

For notational simplicity we have omitted the circumflex. The boundary conditions are

$$\eta = 0 \quad \text{on} \quad z = -h, \quad (2.10a)$$

$$\eta = \mu^2p \quad \text{on} \quad z = 0. \quad (2.10b)$$

The side-wall conditions are

$$p_x - fm\bar{\omega}^{-1}p + fv_{0z}\eta = 0 \quad \text{at} \quad x = 0, l. \quad (2.11)$$

Since the set of equations (2.9a–d) and boundary conditions (2.10a, b) contain no explicit dependence on x, they possess solutions which are proportional to $\exp(ikx)$. For k real these solutions describe Poincaré waves, and for k pure imaginary, Kelvin waves. Substitution into (2.9c, d) yields

$$(f^2 - \bar{\omega}^2)(p_z + SN^2\eta) = SM^2[(ik - fm\bar{\omega}^{-1})p + fv_{0z}\eta], \quad (2.12a)$$

$$(f^2 - \bar{\omega}^2)\eta_z + (k^2 + m^2)p = fv_{0z}(ik + fm\bar{\omega}^{-1})\eta. \quad (2.12b)$$

Imposition of the boundary conditions (2.10a, b) determines a dispersion relation which we shall regard as determining k as a function of m and ω.

In the absence of a basic flow ($v_0 \equiv 0$ and $M^2 \equiv 0$), the equations (2.12a, b) and the boundary conditions (2.10a, b) reduce to

$$\eta_{zz} + SN^2\lambda\eta = 0, \quad (2.13a)$$

$$\eta = 0 \quad \text{on} \quad z = -h, \quad (2.13b)$$

$$\lambda\eta = \mu^2\eta_z \quad \text{on} \quad z = 0, \quad (2.13c)$$

where

$$\lambda = \frac{k^2 + m^2}{\omega^2 - f^2}. \quad (2.13d)$$

Eqs. (2.13a–c) describe a complete set of orthogonal modes $\eta_s(z)$, $s = 0, 1, 2, \cdots$ with real, positive eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots$. The corresponding pressure is given by

$$\lambda_sp_s(z) = \eta_{sz}(z) \quad (2.14)$$

and the orthogonality conditions are

$$\int_{-h}^{0} p_sp_rdz = 0, \quad r \neq s. \quad (2.15)$$

Each eigenvalue λ_s determines a dispersion relation from (2.13d). One mode ($s = 0$) may be identified as the barotropic mode, and the remaining modes as baroclinic modes. Recalling that the Boussinesq approximation has already been invoked, in which terms of relative order $\sigma = \mu^2s$ are systematically neglected, we can show that the barotropic mode is given by

$$p_0 = 1 + O(\sigma), \quad (2.16a)$$

$$\eta_0 = \mu^2\left[\left(\frac{z + h}{h}\right) + O(\sigma)\right], \quad (2.16b)$$

$$\lambda_0 = \mu^2h^{-1}[1 + O(\sigma)]. \quad (2.16c)$$

The baroclinic modes are given by (2.13a, b) with (2.13c) replaced by $\eta = 0$ at $z = 0$, to within an error of $O(\sigma)$ (formally let $\mu^2 \to 0$).

Again assuming that there is no basic shear flow ($v_0 \equiv 0$ and $M^2 \equiv 0$) the side-wall conditions may be satisfied by *each* mode with an appropriate choice of k. The Kelvin waves have $k = -ifm/\omega$ and are given by

$$\eta = \mathrm{Re}[\eta_s(z)\exp(fm\omega^{-1}x + imy - i\omega t)], \quad (2.17a)$$

$$\omega^2 = m^2\lambda_s^{-1}, \quad (2.17b)$$

for each $s = 0, 1, 2, \ldots$. The Poincaré waves have $kl = n\pi$, $n = 1, 2, \ldots$, and are given by

$$\eta = \mathrm{Re}\left[\eta_s(z)\left(\frac{fm}{\omega}\sin\frac{n\pi x}{l} + \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\exp(imy - i\omega t)\right], \quad (2.18a)$$

$$\omega^2 - f^2 = \frac{m^2 + \left(\frac{n\pi}{l}\right)^2}{\lambda_s}, \quad (2.18b)$$

for each $s = 0, 1, 2, \cdots$ (see, for instance, LeBlond and Mysak, 1978). Note that from (2.17b) Kelvin waves can exist for all frequencies, but, from (2.18b), Poincaré waves have a low-frequency cutoff.

However, in the presence of a basic flow no such simple theory is generally possible. Although Eqs. (2.12a, b) with the boundary conditions (2.10a, b) still define a set of vertical modes with corresponding dispersion relations, in general there is no longer a complete orthogonal set. Further, it is apparent from the side-wall condition (2.11) that no single vertical mode can alone satisfy such a condition, and a superposition of modes will generally be required. Also the presence of a basic flow leads to the possibility of an instability (i.e., complex values of ω for real k and m), and to critical levels where $\omega^2 = f^2$ and Eqs. (2.12a, b) are singular. We shall not pursue either of these topics here but see Grimshaw (1975) for a comprehensive discussion of the behavior near a critical level when F_0 is zero. The vertical wave action flux is

$$\mathcal{F} = -2\,\mathrm{Im}\{p\eta^*\}. \quad (2.19)$$

It is readily shown from (2.12a, b) that for real m and ω,

$$(f^2 - \omega^2)\mathcal{F}_z = -\mathcal{F}\left(\frac{fm}{\bar{\omega}}F_{0z} + 2SM^2\,\mathrm{Im}k\right) - 2F_{0z}\,\mathrm{Im}\{ik^*p\eta^*\} + 2|p|^2\,\mathrm{Im}(k^{*2}). \quad (2.20)$$

When k is real and $F_0 = 0$ (i.e., the basic flow is geostrophically balanced), the right-hand side of (2.20) is zero and $\mathcal{F}$ is then a wave invariant; this was the case discussed by Grimshaw (1975). $\mathcal{F}$ is then piecewise constant and can change its value only at a critical level. Since the boundary conditions (2.10a, b) show that $\mathcal{F}$ is zero at the boundaries, it follows from the analysis of Grimshaw (1975) that the vertical modes contain no critical levels. When either k is complex or F_0 is non-zero, no such general conclusion concerning critical levels can be made.

Next we observe from (2.12a, b) and the boundary conditions (2.10a, b) that, for real m and ω,

$$\int_{-h}^{0}\left\{SN^2|\eta|^2 + \frac{(k^{*2} + m^2)}{(f^2 - \bar{\omega}^2)}|p|^2 - \frac{SM^2 f v_{0z}}{(f^2 - \bar{\omega}^2)}|\eta|^2 + p\bar{\eta}\frac{[2SM^2\,\mathrm{Im}k - F_{0z}(ik^* - fm\bar{\omega}^{-1})]}{f^2 - \bar{\omega}^2}\right\}dz + \mu^2|p|^2_{z=0} = 0. \quad (2.21)$$

When k is real and $F_0 \equiv 0$, it follows that either $\bar{\omega}^2 > f^2$ or $(f^2 - \bar{\omega}^2)SN^2 < S^2M^4$. When either k is complex or F_0 is non-zero, no such general conclusions can be drawn.

It is difficult to pursue any further the general analytical theory of (2.12a, b). Hence we shall resort to approximation procedures and special cases. In Section 3 we shall develop a perturbation technique when the shear of the basic flow is small. In Section 4 we shall consider a two-layer fluid.

3. Perturbation method for small shear

The equations to be solved are (2.9c) and (2.9d), with the boundary conditions (2.10a, b), and the side-wall conditions (2.11). In this section we shall assume that the shear flow $v_0(z)$ is small, by which we mean that

$$\max_{-h\le z\le 0}\left[\left|\frac{mv_0(z)}{\omega}\right|, \left|\frac{v_{0z}(z)}{f}\right|\right] \ll 1, \quad (3.1)$$

with similar statements for the higher derivatives of $v_0(z)$. The perturbation method proceeds in two steps. First, we determine the structure of solutions which are proportional to $\exp(ikx)$ and so satisfy (2.12a, b), with the boundary conditions (2.10a, b). Using the left-hand side of (3.1) as an implicit small parameter, we introduce the expansions

$$\left.\begin{aligned} p &= p^{(0)} + p^{(1)} + \cdots \\ \eta &= \eta^{(0)} + \eta^{(1)} + \cdots \end{aligned}\right\} \quad (3.2a)$$

and

$$k = k^{(0)} + k^{(1)} + \cdots, \quad (3.2b)$$

while ω and m are held fixed. To leading order, the solutions are those which hold in the absence of a basic shear flow and are described by (2.13a, b, c, d). Thus

$$\eta^{(0)} = \eta_s(z), \quad p^{(0)} = p_s(z) = \lambda_s^{-1}\eta_{sz}(z), \quad (3.3a)$$

$$\frac{k^{(0)2} + m^2}{\omega^2 - f^2} = \lambda_s. \quad (3.3b)$$

Here $s = 0, 1, 2, \cdots$ and is the modal number for the vertical mode $\eta_s(z)$. These modes form a complete, orthogonal set where the orthogonality condition is (2.15). Each mode has an eigenvalue λ_s from which $k^{(0)}$ is determined as a function of ω and m.

At the next order

$$\left.\begin{aligned} p_z^{(1)} + SN^2\eta^{(1)} &= F^{(1)} \\ \eta_z^{(1)} - \lambda_s p^{(1)} &= G^{(1)} \end{aligned}\right\}, \quad (3.4a)$$

where

$$\left.\begin{aligned} F^{(1)}(f^2 - \omega^2) &= SM^2[ik^{(0)} - fm\omega^{-1}]p^{(0)} \\ G^{(1)}(f^2 - \omega^2) &= [-2k^{(1)}k^{(0)} - 2\omega m\lambda_s]p^{(0)} + fv_{0z}[ik^{(0)} + fm\omega^{-1}]\eta^{(0)} \end{aligned}\right\}. \quad (3.4b)$$

The boundary conditions are that $\eta^{(1)} = 0$ at z

$= -h$, and $\eta^{(1)} = \mu^2 p^{(1)}$ at $z = 0$. The compatibility condition for Eqs. (3.4a) is then

$$\int_{-h}^{0} [\eta_s F^{(1)} + p_s G^{(1)}] dz = 0. \quad (3.5)$$

Substituting the expressions (3.4b) for $F^{(1)}$ and $G^{(1)}$ and using (2.8) we find that (3.5) yields the following expression for $k^{(1)}$:

$$2k^{(0)}k^{(1)} \int_{-h}^{0} p_s^2 dz = -2\omega m \lambda_s \int_{-h}^{0} v_0 p_s^2 dz + \int_{-h}^{0} \{2SM^2 ik^{(0)} - F_{0z}[ik^{(0)} + fm\omega^{-1}]\}\eta_s p_s dz. \quad (3.6)$$

For the barotropic mode $s = 0$; invoking the approximations (2.16a, b, c) it may be shown that (3.6) becomes

$$2k^{(0)}k^{(1)}h = [-2\omega m - f(ik + fm\omega^{-1})] \frac{\mu^2}{h} \int_{-h}^{0} v_0 dz + f(ik + fm\omega^{-1})\mu^2 v_0(0). \quad (3.7)$$

For the baroclinic modes $s = 1, 2, \cdots$ it may be shown that

$$2k^{(0)}k^{(1)} \int_{-h}^{0} p_s^2 dz = -2\omega m \lambda_s \int_{-h}^{0} v_0 p_s^2 dz + \frac{1}{2\lambda_s} [ik^{(0)} + fm\omega^{-1}] \int_{-h}^{0} F_{0zz}\eta_s^2 dz. \quad (3.8)$$

It is apparent from either of these expressions that for real ω, m and $k^{(0)}$, $k^{(1)}$ is complex-valued. This implies that the presence of a basic shear flow may cause an instability. Indeed, if instead of keeping ω and m fixed and expanding k [(3.2b)], we had kept k and m real and fixed and expanded $\omega = \omega^{(0)} + \omega^{(1)}$, then on the left-hand side of (3.7) or (3.8) we would replace $k^{(0)}k^{(1)}$ by $-\lambda_s\omega^{(0)}\omega^{(1)}$; $\omega^{(0)}$ is real, but for Im $\omega^{(1)} > 0$ there is an instability. For instance, for the baroclinic modes there is instability whenever $\omega^{(0)2} > f^2$ and

$$\frac{k}{\omega^{(0)}} \int_{-h}^{0} F_{0zz}\eta_s^2 dz < 0. \quad (3.9)$$

The geostrophically balanced flows for which $F_0 \equiv 0$ are thus stable to this order in the expansion, but since $\omega^{(0)}$ [given by (3.3b)] can take either sign a nongeostrophic basic flow is generally unstable.

However, we do not propose to consider the question of stability of the basic flow any further, and instead will proceed to determine the structure of Kelvin and Poincaré waves. Considering first the Kelvin waves, we replace k in (2.12a, b) and (3.2b) by iK_s or $-i\tilde{K}_s$, respectively, and we similarly replace $k^{(0)}$ and $k^{(1)}$ in (3.2b) and other subsequent equations by $iK_s^{(0)}$ and $iK_s^{(1)}$, or $-i\tilde{K}_s^{(0)}$ and $-i\tilde{K}_s^{(1)}$, respectively. We then propose that

$$\eta = \sum_{s=0}^{\infty} \{A_s \exp(-K_s x) + B_s \exp[\tilde{K}_s(x - l)]\}\eta_s(z), \quad (3.10)$$

with a similar expansion for the pressure p; here $\eta_s(z)$ and $p_s(z)$ are defined by (3.2a) and the subscript (s) indicates that the leading term $\eta_s^{(0)}$ is the vertical mode $\eta_s(z)$, while $\eta_s^{(1)}$ is then found from (3.4a). To satisfy the side-wall conditions (2.11) we must also expand either ω or m. We choose the latter and so

$$m = m^{(0)} + m^{(1)} + \cdots, \quad (3.11)$$

We also expand the constants A_s, B_s in a similar way. On substituting (3.10), its counterpart for p [(3.2b)], and (3.11) into the side-wall conditions (2.11) we find that

$$A_s^{(0)} = 0 \quad \text{for} \quad s \neq R, \quad B_s^{(0)} = 0, \quad \text{all} \quad s, \quad (3.12a)$$

$$K_R^{(0)} = -\frac{m^{(0)}f}{\omega}, \quad \left[\frac{m^{(0)}}{\omega}\right]^2 = \lambda_R. \quad (3.12b)$$

The solution to this order is the Kelvin wave in the absence of a shear flow, described by (2.17a, b), and with mode number R. Note that with $m^{(0)}$ given by (3.12b) it follows from (3.3b) that

$$[K_s^{(0)}]^2 = [\tilde{K}_s^{(0)}]^2 = (\lambda_R - \lambda_s)\omega^2 + \lambda_s f^2. \quad (3.13)$$

Since the eigenvalues λ_s are positive and increase with mode number s it follows that $K_s^{(0)}$ is real and positive for $s \leqslant R$, but may be pure imaginary for sufficiently large s if $\omega^2 > f^2$. Also we choose $\tilde{K}_s^{(0)} = -K_s^{(0)}$.

At the next order the side-wall conditions imply that

$$\sum_{s=0}^{\infty} \left\{\left[-K_s^{(0)} - \frac{m^{(0)}f}{\omega}\right]A_s^{(1)}p_s + \left[-K_s^{(1)} - \lambda_R f v_0 + m^{(1)}\frac{(\omega^2 - f^2)}{\omega f}\right]A_s^{(0)}p_s + f v_{0z}A_s^{(0)}\eta_s\right\} + \left[K_s^{(0)} - \frac{m^{(0)}f}{\omega}\right]B_s^{(1)} \exp[-K_s^{(0)}l]p_s = 0, \quad (3.14a)$$

$$\sum_{s=0}^{\infty} \left\{\left[-K_s^{(0)} - \frac{m^{(0)}f}{\omega}\right]A_s^{(1)}p_s + \left[-K_s^{(1)} - \lambda_R f v_0 + m^{(1)}\frac{(\omega^2 - f^2)}{\omega f}\right]A_s^{(0)}p_s + f v_{0z}A_s^{(0)}\eta_s\right\} \times \exp[-K_s^{(0)}l] + \left[K_s^{(0)} - \frac{m^{(0)}f}{\omega}\right]B_s^{(1)}p_s = 0. \quad (3.14b)$$

The compatibility condition for solution of these equations is most readily obtained by first multiplying each equation by p_r and integrating over the depth. Using (3.12a, b) it follows that

$$[K_R^{(0)} - K_r^{(0)}]A_r^{(1)} \int_{-h}^0 p_r^2 dz + \left\{\int_{-h}^0 \left[-K_R^{(1)} - \lambda_R f v_0 + m^{(1)} \frac{(\omega^2 - f^2)}{\omega f}\right] p_r p_R dz + \int_{-h}^0 f v_{0z} \eta_R p_r dz\right\} A_R^{(0)} + [K_R^{(0)} + K_r^{(0)}] B_r^{(1)} \times \exp[-K_r^{(0)} l] \int_{-h}^0 p_r^2 dz = 0, \quad (3.15a)$$

$$[K_R^{(0)} - K_r^{(0)}]A_r^{(1)} \exp[-K_r^{(0)} l] \int_{-h}^0 p_r^2 dz + \left\{\int_{-h}^0 \left[-K_R^{(1)} - \lambda_R f v_0 + m^{(1)} \frac{(\omega^2 - f^2)}{\omega f}\right] p_r p_R dz + \int_{-h}^0 f v_{0z} \eta_R p_r dz\right\} A_R^{(0)} \exp[-K_R^{(0)} l] + [K_R^{(0)} + K_r^{(0)}] B_r^{(1)} \int_{-h}^0 p_r^2 dz = 0, \quad (3.15b)$$

for each $r = 0, 1, 2, \ldots$. Putting $r = R$ gives the compatibility conditions which are

$$\int_{-h}^0 \left[-K_R^{(1)} - \lambda_R f v_0 + m^{(1)} \frac{(\omega^2 - f^2)}{\omega f}\right] p_R^2 dz + \int_{-h}^0 f v_{0z} \eta_R p_R dz = 0, \quad (3.16a)$$

$$B_R^{(1)} = 0. \quad (3.16b)$$

Using (3.6) for $K_R^{(1)}$, (3.12b) for $K_R^{(0)}$ and (2.8), Eq. (3.16a) reduces to

$$m^{(1)} \int_{-h}^0 p_R^2 dz = -\omega \lambda_R \int_{-h}^0 v_0 p_R^2 dz. \quad (3.17)$$

It follows that, to the order considered, the phase speed of the Kelvin wave is given by

$$\frac{\omega}{m} = \frac{\omega}{m^{(0)}} + \frac{\int_{-h}^0 v_0 p_R^2 dz}{\int_{-h}^0 p_R^2 dz} + \cdots, \quad (3.18)$$

and thus the correction to the phase speed due to the presence of a basic flow is simply a weighted Doppler shift. The wave remains non-dispersive to this order of approximation. The decay rate of the dominant mode $r = R$ is K_R, where

$$K_R = K_R^{(0)} + K_R^{(1)} - \omega m^{(1)} f^{-1} + \cdots \quad (3.19a)$$

or

$$K_R = -f m^{(0)} \omega^{-1} + \frac{\int_{-h}^0 f v_{0z} \eta_R p_R dz}{\int_{-h}^0 p_R^2 dz} + \cdots. \quad (3.19b)$$

The last term in (3.19a) comes from substituting (3.11) into the expression (3.3b) for $K^{(0)}$. For the baroclinic modes $R = 1, 2, 3, \ldots$, the correction term in (3.19b) may also be written as

$$\frac{1}{2\lambda_R} \int_{-h}^0 F_{0zz} \eta_R^2 dz \left(\int_{-h}^0 p_R^2 dz\right)^{-1}, \quad (3.20)$$

and is thus present only if the basic flow is not in geostrophic balance.

Returning to (3.15a, b) for $r \neq R$ it follows that

$$\frac{A_r^{(1)}}{A_R^{(0)}} [K_R^{(0)} - K_r^{(0)}] = \left[\frac{\int_{-h}^0 (\lambda_R f v_0 p_r p_R - f v_{0z} p_r \eta_R) dz}{\int_{-h}^0 p_r^2 dz}\right] \times \left[\frac{1 - \exp\{-[K_R^{(0)} + K_r^{(0)}] l\}}{1 - \exp[-2K_R^{(0)} l]}\right], \quad (3.21a)$$

$$\frac{B_r^{(1)}}{A_R^{(0)}} [K_R^{(0)} + K_r^{(0)}] = \left[\frac{\int_{-h}^0 (\lambda_R f v_0 p_r p_R - f v_{0z} p_r \eta_R) dz}{\int_{-h}^0 p_r^2 dz}\right] \times \left[\frac{\exp[-K_R^{(0)} l] - \exp[-K_r^{(0)} l]}{1 - \exp[-2K_R^{(0)} l]}\right]. \quad (3.21b)$$

Since the right-hand sides of these expressions are generally non-zero, we have shown that one of the effects of a basic shear flow on a Kelvin wave is to excite all the vertical modes in addition to the dominant mode. It is interesting to note here that the decay rate $K_r^{(0)}$ [(3.13)] of these other modes for $\omega^2 > f^2$ is greater than $K_R^{(0)}$ for $r < R$, but smaller for $r > R$ and becomes imaginary for large values of r; if $\omega^2 < f^2$, $K_r^{(0)}$ is real but smaller than $K_R^{(0)}$ for $r < R$ and greater for $r > R$. One consequence of this is the difficulty of determining phase speeds by modal analysis of data obtained from vertical sections, as in the presence of a basic mean flow there is no longer a unique vertical mode associated with a given wave (see Thomson and Huggett, 1980).

If $l \to \infty$, so that the channel has infinite width, a slightly modified theory is required. In the expansion (3.10) the coefficients B_s are set equal to zero, and the decay coefficients K_s are chosen so that either K_s is real and positive, or if K_s is pure imaginary, its sign is chosen so that the term represents an outgoing Poincaré wave (Im K_s has the opposite sign to ω). At leading order Eqs. (3.12a, b) follow as before, and at the next order we again obtain (3.17) and (3.19a, b), while $A_r^{(1)}$ is given by (3.21a) with the second factor replaced by unity.

For Poincaré waves in a channel of finite width a similar theory can be developed, which we shall outline briefly. Let

$$\eta_s = \sum_{s=0}^{\infty} [A_s \exp(ik_s x) + B_s \exp(-i\tilde{k}_s x)]\eta_{(s)}(z), \quad (3.22)$$

with a similar expression for the pressure p, where we have replaced k in (2.12a, b) and (3.2b) by k_s or $-\tilde{k}_s$, and similarly we replace $k^{(0)}$ and $k^{(1)}$ in (3.2b) and other subsequent equations by $k_s^{(0)}$ and $k_s^{(1)}$, or $-\tilde{k}_s^{(0)}$ and $-\tilde{k}_s^{(1)}$ respectively. At the leading order

$$A_s^{(0)} = B_s^{(0)} = 0 \quad \text{for} \quad s \neq R, \quad (3.23a)$$

$$\left[ik_R^{(0)} - \frac{m^{(0)}f}{\omega}\right]A_R^{(0)} = \left[ik_R^{(0)} + \frac{m^{(0)}f}{\omega}\right]B_R^{(0)}, \quad (3.23b)$$

$$k_R^{(0)}l = \tilde{k}_R^{(0)}l = n\pi, \quad (3.23c)$$

while

$$\omega^2 = f^2 + \frac{1}{\lambda_R}\left[m^{(0)2} + \left(\frac{n\pi}{l}\right)^2\right]. \quad (3.23d)$$

Here n is an integer, $n = 1, 2, \ldots$. This is just the Poincaré wave in the absence of a basic shear flow described by (2.18a, b), with mode number R. At the next order we find that

$$m^{(1)}\int_{-h}^{0} p_R^2 dz = -\omega\lambda_R \int_{-h}^{0} v_0 p_R^2 dz - \frac{f}{2\omega}\int_{-h}^{0} F_{0z}\eta_R p_R dz, \quad (3.24)$$

which describes the correction to the leading order phase speed. The crosschannel wavenumbers are given by

$$k_R = \frac{n\pi}{l} + \frac{i\int_{-h}^{0} \eta_R p_R(SM^2 - \tfrac{1}{2}F_{0z})dz}{\int_{-h}^{0} p_R^2 dz} + \cdots, \quad (3.25a)$$

$$\tilde{k}_R = \frac{n\pi}{l} - i\frac{\int_{-h}^{0} \eta_R p_R(SM^2 - \tfrac{1}{2}F_{0z})dz}{\int_{-h}^{0} p_R^2 dz} + \cdots. \quad (3.25b)$$

The correction terms are imaginary, and induce an asymmetry in the cross-channel structure. For $s \neq R$,

$$2i \sin k_s^{(0)}l\left[\frac{in\pi}{l} - \frac{m^{(0)}f}{\omega}\right]A_s^{(1)} = H_s[(-1)^n - \exp(-ik_s^{(0)}l)], \quad (3.26a)$$

$$2i \sin k_s^{(0)}l\left[\frac{in\pi}{l} + \frac{m^{(0)}f}{\omega}\right]B_s^{(1)} = H_s\{(-1)^n - \exp[ik_s^{(0)}l]\}, \quad (3.26b)$$

where

$$H_s\int_{-h}^{0} p_s^2 dz = [A_R^{(0)} + B_R^{(0)}] \times \int_{-h}^{0}\left(\frac{m^2 f}{\omega^2} v_0 p_s p_R - f v_{0z}\eta_R p_s\right)dz, \quad (3.26c)$$

$$k_s^{(0)2} = \left(\frac{n\pi}{l}\right)^2 + (\lambda_s - \lambda_R)(\omega^2 - f^2). \quad (3.26d)$$

As for the Kelvin wave all the vertical modes are excited in addition to the dominant mode. Note that the cross-channel wavenumbers $k_s^{(0)}$ for $\omega^2 > f^2$ are real for $s > R$, but may be imaginary for $s < R$; if $\omega^2 < f^2$ then $k_s^{(0)}$ is real for $s < R$, but becomes imaginary for sufficiently large s. Note also that the expansion procedure outlined above fails if $k_s^{(0)}l/\pi$ is an integer; in this case the sth mode must be included at the leading order and there is a coupling between these modes at the next order.

We conclude this section by evaluating some of the expressions derived above for a number of special cases. First, for the barotropic mode $R = 0$ which is described by (2.16a, b, c), Eqs. (3.12b), (3.18) and (3.19b) for the Kelvin wave become

$$\left[\frac{m^{(0)}}{\omega}\right]^2 = \frac{\mu^2}{h}, \quad (3.27a)$$

$$\frac{\omega}{m} = \frac{\omega}{m^{(0)}} + \int_{-h}^{0} v_0 dz + \cdots, \quad (3.27b)$$

$$K_0 = -\frac{fm^{(0)}}{\omega} + \frac{\mu^2}{h}\left[\int_{-h}^{0} F_0 dz - F_0(0)\right] + \cdots. \quad (3.27c)$$

Second, to discuss the baroclinic modes, we shall suppose that N^2 is a constant, whence it follows that for $R = 1, 2, \cdots$

$$SN^2\lambda_R = \left(\frac{R\pi}{h}\right)^2, \quad (3.28a)$$

$$\eta_R = \sin\frac{R\pi}{h}(z + h),$$

$$p_R = \frac{SN^2 h}{R\pi}\cos\frac{R\pi}{h}(z + h). \quad (3.28b)$$

If we also suppose that

$$v_0(z) = V_0 + V\frac{(z + h)}{h} + \frac{1}{2}W\frac{(z + h)^2}{h^2}, \quad (3.29)$$

then the phase speed and decay rate for the Kelvin wave are given by (3.12b), (3.18) and (3.19b), where

$$\frac{\int_{-h}^{0} v_0 p_R^2 dz}{\int_{-h}^{0} p_R^2 dz} = V_0 + \frac{1}{2}V + \frac{1}{6}W + \left(\frac{h}{2R\pi}\right)^2 W, \quad (3.30a)$$

$$\frac{\int f v_{0z}\eta_R p_R dz}{\int_{-h}^{0} p_R^2 dz} = -\frac{fW}{2SN^2h}. \qquad (3.30b)$$

The first three terms in (3.30a) are just the mean value of $v_0(z)$ and only the last term contains the effect of the vertical structure of the basic flow; note that this last term is not present unless the basic shear flow contains a non-trivial curvature. In particular, this last term is not present for a geostrophically balanced basic flow. Similarly (3.30b) shows that the decay rate is affected by the basic flow only when there is a non-trivial curvature.

As an illustration of the application of these results we shall use the data obtained by Thomson and Huggett (1980) in Johnstone Strait, British Columbia. Putting $L = 3$ km, $H = 0.3$ km, $N_1 = 5 \times 10^{-3}$ s^{-1}, $f = 1.1254 \times 10^{-4}$ s^{-1}, we find that $S = 19.74$. In the non-dimensional variables $h = 1$, $N = 1$, $f = 1$ and we find from (3.28a) that $\gamma_R^{1/2} = 0.707R$. Thus in the absence of a basic flow the phase speed (3.12b) of the first mode baroclinic Kelvin wave is ± 1.414, and its decay rate $K_R^{(0)}$ is ∓ 0.707. Using the data provided by Thomson and Huggett (1980) we estimate that $W = -3.328$ and $V = 0.555$, with V_0 chosen to make the mean value of $v_0(z)$ zero. It then follows from (3.18), (3.19b) and (3.30a, b) that the correction to the phase speed is -0.084, and the correction to the decay rate is 0.084. Note that our estimate for W is mainly based on the curvature of the observed flow near the bottom of the channel where there is considerable scatter in the data, and the value quoted above is probably rather too large in absolute value. Even so, we see that the corrections obtained from (3.30a, b) are quite small relative to the values obtained at leading order.

4. Two-layer fluid

In this section we shall consider the two-layer fluid for which the basic density and velocity are given by

$$\rho_0 = 1 + \sigma H(-z - d), \qquad (4.1a)$$

$$v_0 = v_1 H(z + d) + v_2 H(-z - d). \qquad (4.1b)$$

Here v_1, v_2 are constants and $H(\cdot)$ is the Heaviside function; the interface between the two fluids is at $z = -d$ in the basic flow, and at $z = -d + \alpha$ in the perturbed flow. Since the basic density is constant in each fluid, N^2 and M^2 are both zero in each fluid. The equations to be satisfied in each fluid are again (2.5a–f), with boundary conditions (2.6a–c) and sidewall conditions (2.11). They are now supplemented by conditions at the interface which are

$$[p]_-^+ + S\alpha = 0, \quad \text{and} \quad \eta = \alpha \quad \text{at} \quad z = -d, \qquad (4.2)$$

where $[\cdot]_-^+$ denotes a discontinuity across the interface. Since the perturbation density ρ is now zero in each fluid, integration of (2.5c) and use of the boundary conditions (2.6b) and (4.2) show that

$$p = \zeta + S\alpha H(-z - d). \qquad (4.3)$$

Next, using (2.9d) and the boundary conditions (2.10a, b) the vertical displacement η in each fluid is determined. Finally, again using the boundary condition (4.2) we obtain the following pair of coupled equations for ζ and α:

$$d(\zeta_{xx} - m^2\zeta) - \mu^2\zeta(f^2 - \bar{\omega}_1^2) + \alpha(f^2 - \bar{\omega}_1^2) = 0, \qquad (4.4a)$$

$$h(\zeta_{xx} - m^2\zeta) - \mu^2\zeta(f^2 - \bar{\omega}_1^2) + S(h - d) \times (\alpha_{xx} - m^2\alpha) - \alpha(\bar{\omega}_1^2 - \bar{\omega}_2^2) = 0, \qquad (4.4b)$$

where

$$\bar{\omega}_i = \omega - mv_i. \qquad (4.4c)$$

The side-wall conditions (2.11) become

$$\left.\begin{aligned} &\zeta_x - fm\bar{\omega}_1^{-1}\zeta = 0 \\ &\zeta_x - fm\bar{\omega}_2^{-1}\zeta + S(\alpha_x - fm\bar{\omega}_2^{-1}\alpha) = 0 \end{aligned}\right\} \quad \text{at} \quad x = 0, l. \qquad (4.5)$$

The vertical modes are obtained from (4.4a, b) by seeking solutions proportional to $\exp(ikx)$. There are only two such modes; the barotropic mode is given by

$$k^2 + m^2 = \frac{\mu^2(\bar{\omega}_1^2 - f^2)(\bar{\omega}_2^2 - f^2)}{(h - d)(\bar{\omega}_1^2 - f^2) + d(\bar{\omega}_2^2 - f^2)} \qquad (4.6a)$$

and

$$\alpha = \mu^2\zeta\left[\frac{(h - d)(\bar{\omega}_1^2 - f^2)}{(h - d)(\bar{\omega}_1^2 - f^2) + d(\bar{\omega}_2^2 - f^2)}\right], \qquad (4.6b)$$

while the baroclinic mode is given by

$$k^2 + m^2 = \frac{(h - d)(\bar{\omega}_1^2 - f^2) + d(\bar{\omega}_2^2 - f^2)}{Sd(h - d)} \qquad (4.7a)$$

and

$$\zeta = -S\alpha\left[\frac{(h - d)(\bar{\omega}_1^2 - f^2)}{(h - d)(\bar{\omega}_1^2 - f^2) + d(\bar{\omega}_2^2 - f^2)}\right].$$

Here we have again applied the Boussinesq approximation and neglected terms of $O(\sigma)$. The analytical advantages in treating this two-layer fluid model can now be seen. First, explicit expressions can be obtained for the vertical modes without the approximation of small shear, here represented by $(v_1 - v_2)$. Second, there are only two vertical modes and consequently the side-wall conditions can also be satisfied without the approximation of small shear. However, before proceeding to discuss the side-wall conditions, the dispersion relations (4.6a) and (4.7a) can be used to derive conditions for instability. This is achieved by keeping k and m real, and determining the conditions for Im $\omega > 0$. For the barotropic mode, it can be shown from (4.6a) that there is stability for

$$m^2(v_1 - v_2)^2 \leqslant \frac{f^2h^2}{d(h-d)}. \quad (4.8)$$

Otherwise the flow is unstable for sufficiently large values of $(k^2 + m^2)/\mu^2$. For the baroclinic mode, it can be shown from (4.7a) that there is stability whenever (4.8) is satisfied. Otherwise the flow is unstable whenever

$$(k^2 + m^2)Sh < m^2(v_1 - v_2)^2 - \frac{f^2h^2}{d(h-d)}. \quad (4.9)$$

To determine the structure of Kelvin waves, we shall suppose that the dispersion relations (4.6a) and (4.7a) determine k as a function of m and ω, both real-valued. Considering first the Kelvin waves, we replace k in (4.6a) and (4.7a) by iK_0 and iK_1, respectively. We can propose that [see (3.10)]

$$\zeta = A_0 \exp(-K_0x) + B_0 \exp[K_0(x - l)] - SQ\{A_1 \exp(-K_1x) + B_1 \exp[K_1(x - l)]\}, \quad (4.10a)$$

$$\alpha = \mu^2Q\{A_0 \exp(-K_0x) + B_0 \exp[K_0(x - l)]\} + A_1 \exp(-K_1x) + B_1 \exp[K_1(x - l)], \quad (4.10b)$$

where

$$Q = \frac{(h-d)(\bar\omega_1{}^2 - f^2)}{(h-d)(\bar\omega_1{}^2 - f^2) + d(\bar\omega_2{}^2 - f^2)}. \quad (4.10c)$$

Substitution of these expressions into the side-wall conditions (4.5) yield four homogeneous equations for A_0, A_1, B_0 and B_1, whose determinant gives the dispersion relation relating ω and m. To keep the analysis as simple as possible we shall suppose that the channel has infinite width ($l \to \infty$), in which case we can set B_0 and B_1 equal to zero. The sidewall condition (4.5) at $x = 0$ then yields

$$\left(K_0 + \frac{fm}{\bar\omega_1}\right)\left(K_1 + \frac{fm}{\bar\omega_2}\right) + Q\left(\frac{fm}{\bar\omega_1} - \frac{fm}{\bar\omega_2}\right)(K_0 - K_1) = 0, \quad (4.11a)$$

$$\left(K_0 + \frac{fm}{\bar\omega_1}\right)A_0 = SQ\left(K_1 + \frac{fm}{\bar\omega_1}\right)A_1. \quad (4.11b)$$

For the baroclinic mode we may let $\mu^2 \to 0$ and so $K_0 = m$ (>0). It can be verified *a posteriori* that the error so induced has relative order σ. Eq. (4.11a) then simplifies to

$$K_1m^{-1}[(h-d)\bar\omega_1(\bar\omega_1 - f) + d\bar\omega_2(\bar\omega_2 - f)] + f[(h-d)(\bar\omega_1 - f) + d(\bar\omega_2 - f)] = 0. \quad (4.12)$$

With $k = iK_1$ in (4.7a), we may substitute for K_1 in (4.12) and further simplification then gives

$$m^2Sd(h-d)[(h-d)(\bar\omega_1 - f)^2 + d(\bar\omega_2 - f)^2] = [(h-d)\bar\omega_1(\bar\omega_1 - f) + d\bar\omega_2(\bar\omega_2 - f)]^2. \quad (4.13)$$

This is a quartic equation for ω; only those solutions which give a positive value for K_1 in (4.12) can be chosen. The solutions determine ω as a function of m, and in general the waves are dispersive, in contrast to the case when there is no shear. To analyze these expressions we write them in the form

$$m^2Sd\frac{(h-d)}{h}f(f - \bar\omega) = (\bar\omega - \omega_a)(\bar\omega - \omega_b)(\bar\omega - \omega_{a'})(\bar\omega - \omega_{b'}), \quad (4.14a)$$

$$K_1m^{-1} = \frac{h(\bar\omega - \omega_{a'})(\bar\omega - \omega_{b'})}{m^2Sd(h-d)}, \quad (4.14b)$$

where

$$\omega_{a,b} = \tfrac{1}{2}f \pm \tfrac{1}{2}\left[f^2 - \frac{4d(h-d)}{h^2}m^2(v_1 - v_2)^2\right]^{1/2}, \quad (4.14c)$$

$$\omega_{a',b'} = \tfrac{1}{2}f \pm \tfrac{1}{2}\left[f^2 - \frac{4d(h-d)}{h^2}m^2(v_1 - v_2)^2 + 4m^2S\frac{d(h-d)}{h}\right]^{1/2}, \quad (4.14d)$$

$$\bar\omega = \omega - mv_1\frac{(h-d)}{h} - \frac{mv_2d}{h}. \quad (4.14e)$$

Note that $\omega_{a,b}$ are the roots for $\bar\omega$ of the right-hand side of (4.13). Also $\bar\omega$ is just the frequency with the same Doppler shift as that given by (3.18), if, in that formula, the Doppler shift term is evaluated for the special case of a two-layer fluid. For sufficiently small shear $[m^2(v_1 - v_2)^2 \ll f^2]$, $\omega_a \approx f$, $\omega_b \approx 0$ if $f > 0$ and the roots of (4.14a) are

$$\bar\omega \approx \omega_0 + \frac{d(h-d)}{h^2}m^2(v_1 - v_2)^2\frac{(\omega_0 - 2f)}{(\omega_0 - f)^2} + O\left[\frac{m^4(v_1 - v_2)^4}{f^4}\right], \quad (4.15a)$$

$$\frac{K_1}{m} \approx -\frac{f}{\omega_0} + \frac{d(h-d)}{h^2}\frac{m^2(v_1 - v_2)^2}{\omega_0{}^2(\omega_0 - f)^2} \times [(\omega_0 - f)^2 - \omega_0f] + O\left[\frac{m^4(v_1 - v_2)^4}{f^4}\right], \quad (4.15b)$$

where

$$\omega_0{}^2 = m^2S\frac{d(h-d)}{h}. \quad (4.15c)$$

In the limit $m^2(v_1 - v_2)^2f^{-2} \to 0$, these expressions agree completely with (3.18) and (3.19b); for small but finite values of $m^2(v_1 - v_2)^2 f^{-2}$ they give the continuation to the second order of the approximation scheme developed in Section 3. Eq. (4.15a) shows clearly the dispersive nature of the waves. If the shear is increased further but $\omega_{a,b}$ and hence $\omega_{a',b'}$ are all real, with $\omega_{a'} > \omega_a > \omega_b > \omega_{b'}$ then we must choose those solutions of (4.14a) which make K_1 (4.14b) positive; hence either $\bar{\omega} < \omega_{b'}$ or $\bar{\omega} > \omega_{a'}$. From (4.14a) it is readily shown that there is a unique root for which $\bar{\omega} < \omega_{b'}$ for all m; this root is just the continuation of (4.15) as the shear increases. If

$$(v_1 - v_2)^2 > Sh, \tag{4.16}$$

then there is also a second root for which $\bar{\omega} > \omega_{a'}$ for all m. These results remain valid if $\omega_{a,b}$ cease to be real, but $\omega_{a',b'}$ remain real. However, if both $\omega_{a,b}$ and $\omega_{a',b'}$ become complex-valued [i.e., $f^2 + 4m^2Sd(h - d)h^{-1} < 4d(h - d)m^2(v_1 - v_2)^2h^{-2}$], then both solutions will continue to exist provided that $m^2(v_1 - v_2)^2$ is not too large. Exact analytical expressions other than (4.15a) are not readily obtained for these roots. However, for

$$m^2S\frac{d(h-d)}{h} \ll f^2 - \frac{4d(h-d)}{h^2}m^2(v_1 - v_2)^2, \tag{4.17}$$

the following approximate expressions can be derived:

$$\bar{\omega} \approx \omega_{b'} - \left[m^2S\frac{d(h-d)}{h}\frac{f\omega_{a'}}{(\omega_{a'} - \omega_{b'})^2}\right]^{1/2} + O\left[m^2S\frac{d(h-d)}{h}\right], \tag{4.18a}$$

$$\bar{\omega} \approx \omega_{a'} + \left[m^2S\frac{d(h-d)}{h}\frac{f\omega_{b'}}{(\omega_{a'} - \omega_{b'})^2}\right]^{1/2} + O\left[m^2S\frac{d(h-d)}{h}\right]. \tag{4.18b}$$

Correspondingly, K_1 [(4.14b)] is given by

$$K_1 \approx \left[\frac{f\omega_{a'}h}{m^2Sd(h-d)}\right]^{1/2}, \tag{4.19a}$$

$$K_1 \approx \left[\frac{f\omega_{b'}h}{m^2Sd(h-d)}\right]^{1/2}. \tag{4.19b}$$

Note that (4.18b) and (4.19b) apply only when (4.16) holds.

As an illustration of these results we again use the data obtained by Thomson and Huggett (1980). With the same scaling employed at the end of Section 3, we choose $(v_1 - v_2) = -1.2$, and $d = 1/3$. In the absence of a shear flow the phase speed of the Kelvin wave is -2.094 (our sign convention in this section requires that $m > 0$, and since $K_1 > 0$, $\omega < 0$); the decay rate is 0.478. Using the approximate expressions (4.15a, b) with the frequency $\omega = -1.25$ [the semi-diurnal (M_2) frequency], we find that the shear flow alters the phase speed to -2.224, and the decay rate to 0.530. [These calculations do not include the effect of the Doppler shift given by (4.14a).] As was the case in Section 3, the corrections due to the shear flow are quite small.

For the barotropic mode, we anticipate that K_0/K_1 is $O(\sigma^{1/2})$; then to within an error of relative order $\sigma^{1/2}$ (4.11a) simplifies to

$$K_0 + \frac{fm}{\bar{\omega}_1} = Q\left(\frac{fm}{\bar{\omega}_1} - \frac{fm}{\bar{\omega}_2}\right). \tag{4.20}$$

With k equal to iK_0 in (4.6a), we may substitute for K_0 in (4.20). The result is

$$\frac{m^2h}{\mu^2\bar{\omega}_1{}^2\bar{\omega}_2{}^2}(\bar{\omega}^2 - \bar{\omega}_a{}^2)(\bar{\omega}^2 - \omega_b{}^2) = [\bar{\omega}^2 - f^2 + m^2(v_1 - v_2)^2d(h-d)h^{-2}], \tag{4.21a}$$

$$\frac{K_0}{mf} = -\frac{\bar{\omega}}{\bar{\omega}_1\bar{\omega}_2} + \frac{m(v_1 - v_2)}{\bar{\omega}_1\bar{\omega}_2}\frac{d(h-d)}{h} \times \frac{(\bar{\omega}_1{}^2 - \bar{\omega}_2{}^2)}{\{\bar{\omega}^2 - f^2 + m^2(v_1 - v_2)^2d(h-d)h^{-2}\}}. \tag{4.21b}$$

Here $\omega_{a,b}$ and $\bar{\omega}$ are defined in (4.14c) and (4.14e), respectively. This is a sixth-order equation for ω; however, for sufficiently small shear $[m^2(v_1 - v_2)^2 \ll f^2]$ the only root of interest here is

$$\omega = \hat{\omega}_0 + m\left[\frac{v_1d}{h} + v_2\frac{(h-d)}{h}\right] + m^2(v_1 - v_2)^2\frac{d(h-d)}{h^2}\frac{(3\hat{\omega}_0 - 2f^2)}{2\hat{\omega}_0(\hat{\omega}_0{}^2 - f^2)} + O[m^3(v_1 - v_2)^3/f^3], \tag{4.22a}$$

$$\frac{K_0}{mf} = -\frac{1}{\hat{\omega}_0}\left\{1 + m^2(v_1 - v_2)^2\frac{d(h-d)}{h^2} \times \frac{3}{2(\hat{\omega}_0{}^2 - f^2)} + O\left[\frac{m^3(v_1 - v_2)^3}{f^3}\right]\right\}, \tag{4.22b}$$

where

$$\hat{\omega}_0{}^2 = \frac{m^2h}{\mu^2}. \tag{4.22c}$$

In the limit $m^2(v_1 - v_2)^2f^{-2} \to 0$ these expressions agree completely with the formulas (3.18) and (3.19b); for small but finite values of $m^2(v_1 - v_2)^2f^{-2}$ they give the continuation to the second order of the approximation scheme developed in Section 3. Note that the second term on the right-hand side of (4.22a) is just the Doppler shift term defined by (3.18), where for the barotropic mode $p_0 = 1$ [see (2.16a)].

5. Summary

This paper has examined the effects of a depth-dependent mean flow on Kelvin waves, and to a lesser extent, on Poincaré waves. In Section 2 the problem was formulated for an inviscid, incompressible fluid in the hydrostatic and Boussinesq approximations. The governing equations are (2.9c, d), together with boundary conditions (2.10a, b) at the rigid bottom ($z = -h$) and free surface ($z = 0$), respectively (see Fig. 1). Although these equations and boundary conditions have modal solutions in which the cross-channel dependence is represented by the factor $\exp(ikx)$, it is apparent that in general no single such mode can satisfy the side-wall conditions (2.11) at $x = 0, l$. Thus Kelvin and Poincaré waves in the channel in general are represented by a superposition of vertical modes; only in the absence of a depth-dependent mean flow can a single vertical mode represent a Kelvin or Poincaré wave. An important consequence of this is the difficulty of determining phase speeds by modal analysis of data obtained from vertical sections, as in the presence of a basic mean flow there is no longer a unique vertical mode associated with a given wave.

Because the effect of the side-wall conditions is to render the problem non-separable, we proceeded in Section 3 to develop a perturbation scheme for small shear [see (3.1)]. This scheme describes the superposition of vertical modes required to define a Kelvin wave and leads to expressions for the effect of the mean shear flow on the wave phase speed (3.18) and decay rate (3.19b). To the order considered the phase speed correction is simply a weighted Doppler shift. Similar expressions (3.24) and (3.25a, b) were obtained for Poincaré waves. As an illustration of the application of our results we used the data obtained by Thomson and Huggett (1980) in Johnstone Strait, British Columbia. Although the observed shear flow was quite strong [using the criterion (3.1)] the calculated corrections to the phase speed and decay rate were relatively small, thus suggesting that the formulas obtained from our perturbation scheme may be quite useful.

In order to get some results beyond the perturbation scheme of Section 3 we considered in Section 4 the special case of a two-layer fluid [see (4.1a, b)]. However, even in this special case the analysis was very complicated and some approximations were made. For small shear the results obtained extended the perturbation scheme to second order. Otherwise only qualitative statements could be made except in the limit of weak stratification (4.17).

In this paper we have not considered the stability of the mean flow although some results were obtained from the perturbation scheme, and for the special case of a two-layer fluid. In particular, the results from the perturbation scheme suggest that non-geostrophic basic flows are generally unstable. Also, critical level effects have not been considered here although it should be noted that the analysis of Grimshaw (1975) shows that for geostrophically balanced flows the vertical modes contain no critical levels. In general we would expect critical level effects to be associated with transience and an inability to establish vertical modes.

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