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New Convergence Results for Nash Equilibria

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Painleve-Kuratowski convergence results are obtained, under sufficient conditions of minimal character, for approximate Nash equilibria of two sequences of real valued functions. Moreover, an application to a hierarchical game is given.

1. Introduction

Having in mind to obtain existence and stability results for hierarchical games with one leader and two followers playing a non cooperative non zero sum game [10, 3], the aim of this paper is to give new convergence results for a sequence of Nash equilibria. More precisely let V_1 (respectively V_2) be a topological space, Y_1 (respectively Y_2) be a nonempty closed subset of V_1 (respectively V_2) and f_1, f_2 be two real valued functions defined on $Y_1 \times Y_2$. The corresponding Nash Equilibrium problem is the following:

$$\mathcal{N}(f_1, f_2) \begin{cases} \text{find } (\overline{y}_1, \overline{y}_2) \in Y_1 \times Y_2 \text{ such that:} \\ f_1(\overline{y}_1, \overline{y}_2) = \inf\limits_{y_1 \in Y_1} f_1(y_1, \overline{y}_2) \text{ and } f_2(\overline{y}_1, \overline{y}_2) = \inf\limits_{y_2 \in Y_2} f_2(\overline{y}_1, y_2). \end{cases}$$

Let N be the set of solutions to the problem $\mathcal{N}(f_1, f_2)$. One is interested, as in optimization [1, 4, 7, 21], to a general scheme of perturbations which can include penalizations and discretizations. Therefore, let $(f_{1,n})_n$ and $(f_{2,n})_n$ be two sequences of real valued functions defined on $Y_1 \times Y_2$. For all $n \in \mathbb{N}$, let $\mathcal{N}(f_{1,n}, f_{2,n})$ be the corresponding Nash equilibrium problem and let N_n be the solution set of $\mathcal{N}(f_{1,n}, f_{2,n})$. Convergence results for the solutions of $\mathcal{N}(f_{1,n}, f_{2,n})$ to solutions of $\mathcal{N}(f_1, f_2)$ have been given in [6] and, more precisely, sufficient conditions have been given for the condition:

$$\operatorname*{Lim\,sup}_{n \to \infty} N_n \subseteq N \tag{1.1}$$

that is: for any sequence $(y_k)_k$ such that $y_k = (y_{1,k}, y_{2,k})$ converges to $y = (y_1, y_2)$ and y_k is a solution to $\mathcal{N}(f_{1,n_k}, f_{2,n_k})$, for a selection of integers (n_k), we have that y is a solution to $\mathcal{N}(f_1, f_2)$.

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Unfortunatly, even for nice functions and set of constraints, we have not, in general, (see Example 1.1)

$$N \subseteq \operatorname*{Lim\,inf}_{n\to\infty} N_n \tag{1.2}$$

that is: for any $y = (y_1, y_2)$ solution to $\mathcal{N}(f_1, f_2)$ there exists a sequence $(y_n)_n$ such that $y_n = (y_{1,n}, y_{2,n})$ is a solution to $\mathcal{N}(f_{1,n}, f_{2,n})$ and y_n converges to y.

The aim of this paper is to determine a set of approximate solutions to $\mathcal{N}(f_1, f_2)$ (resp. $\mathcal{N}(f_{1,n}, f_{2,n})$), called $N(\epsilon)$ (resp. $N_n(\epsilon)$), such that sufficient conditions of minimal character can be obtained for the condition

$$N(\epsilon) \subseteq \operatorname*{Lim\,inf}_{n\to\infty} N_n(\epsilon) \quad \text{for any } \epsilon > 0 \tag{1.3}$$

that is: for any $y = (y_1, y_2) \in N(\epsilon)$ there exists a sequence $(y_n)_n$ converging to y and such that $y_n = (y_{1,n}, y_{2,n}) \in N_n(\epsilon)$.

When $\mathcal{N}(f_1, f_2)$ and $\mathcal{N}(f_{1,n}, f_{2,n})$ are generalized saddle point problems [2, 8], relation (1.3) has been obtained in [18] for approximate saddle points under conditions of minimal character involving new semicontinuity properties for multifunctions. In this paper, firstly, we extend these results to a non zero sum game and, secondly, we slightly weaken the sufficient conditions given for condition (1.1) in [6]. Let us note that we also extend to nonzero sum games the results on lower semicontinuity of ϵ-solutions to minimum problems, obtained in a sequential setting in [16] and in a topological one in [15]. Finally, to illustrate the usefulness of condition (1.3), an application to a hierarchical game is given, extending the existence and stability results obtained in [20], in the case of uniqueness of the solutions to the lower level problem, to the case in which the set of solutions is not always reduced to a singleton.

Now, let us give an elementary example in which a sequence of functions $(f_{i,n})_n$ converges uniformly to a function f_i for $i = 1, 2$ and such that condition (1.1) is satisfied but condition (1.2) is not.

Example 1.1. Let $Y_1 = Y_2 = [0,1]$ and $f_1(y_1, y_2) = f_2(y_1, y_2) = 0$, $f_{1,n}(y_1, y_2) = f_{2,n}(y_1, y_2) = \frac{y_1 y_2}{n}$ for any $n \in \mathbb{N}$. We have $N = [0,1]\times[0,1]$ and $N_n = ([0,1]\times\{0\})\cup(\{0\}\times[0,1])$ for any $n \in \mathbb{N}$. Then $\text{Lim sup}_{n\to\infty} N_n \subseteq N$ but we have not $N \subseteq \text{Lim inf}_{n\to\infty} N_n$. In fact $\text{Lim inf}_{n\to\infty} N_n = \text{Lim sup}_{n\to\infty} N_n = ([0,1] \times \{0\}) \cup (\{0\} \times [0,1]) \subset N$.

Therefore, in the following, we consider approximate solutions and more precisely:

Definition 1.2. The pair $\overline{y} = (\overline{y}_1, \overline{y}_2) \in Y_1 \times Y_2$ is an ϵ-*approximate Nash equilibrium* for the problem $\mathcal{N}(f_1, f_2)$ if $\overline{y}$ satisfies:

$$\mathcal{N}(f_1, f_2)(\epsilon) \begin{cases} \text{find } (\overline{y}_1, \overline{y}_2) \in Y_1 \times Y_2 \text{ such that:} \\ f_1(\overline{y}_1, \overline{y}_2) + f_2(\overline{y}_1, \overline{y}_2) \le v_1(f_1, \overline{y}_2) + v_2(f_2, \overline{y}_1) + \epsilon \end{cases}$$

where $v_1(f_1, y_2) = \inf_{y_1 \in Y_1} f_1(y_1, y_2)$, $v_2(f_2, y_1) = \inf_{y_2 \in Y_2} f_2(y_1, y_2)$.

Let $N(\epsilon)$ be the set of ϵ-approximate Nash equilibria. We note that, for any $\epsilon > 0$, we have $N \subseteq N(\epsilon)$ and for $\epsilon = 0$: $N(0) = N$.

Remark 1.3. In Example 1.1 we have: $N(\epsilon) = [0,1] \times [0,1] = N$, $N_n(\epsilon) = \{(\overline{y}_{1,n}, \overline{y}_{2,n}) \in Y_1 \times Y_2$ such that $\overline{y}_{1,n}\overline{y}_{2,n} \leq \frac{n\epsilon}{2}\}$ for any $\epsilon > 0$ and $N(\epsilon) \subseteq \operatorname{Lim\,inf}_{n\to\infty} N_n(\epsilon)$ is satisfied for any $\epsilon > 0$.

Definition 1.4. The pair $\overline{y} = (\overline{y}_1, \overline{y}_2) \in Y_1 \times Y_2$ is a *strict ϵ-approximate Nash equilibrium* for the problem $\mathcal{N}(f_1, f_2)$ if $\overline{y}$ satisfies:

$$\tilde{\mathcal{N}}(f_1, f_2)(\epsilon) \begin{cases} \text{find } (\overline{y}_1, \overline{y}_2) \in Y_1 \times Y_2 \text{ such that:} \\ f_1(\overline{y}_1, \overline{y}_2) + f_2(\overline{y}_1, \overline{y}_2) < v_1(f_1, \overline{y}_2) + v_2(f_2, \overline{y}_1) + \epsilon \end{cases}$$

$\tilde{N}(\epsilon)$ will be the set of solutions to the problem $\tilde{\mathcal{N}}(f_1, f_2)(\epsilon)$.

Remark 1.5. An another well known concept of approximate Nash equilibrium (see for example [3]) for the problem $\mathcal{N}(f_1, f_2)$ is the following:

$\overline{y} = (\overline{y}_1, \overline{y}_2) \in Y_1 \times Y_2$ is an *ϵ-Nash equilibrium* if $\overline{y}$ satisfies:

$$\hat{\mathcal{N}}(f_1, f_2)(\epsilon) \begin{cases} \text{find } (\overline{y}_1, \overline{y}_2) \in Y_1 \times Y_2 \text{ such that:} \\ f_1(\overline{y}_1, \overline{y}_2) \leq v_1(f_1, \overline{y}_2) + \epsilon \ \text{ and } \ f_2(\overline{y}_1, \overline{y}_2) \leq v_2(f_2, \overline{y}_1) + \epsilon \end{cases}$$

Let $\hat{N}(\epsilon)$ be the set of ϵ-Nash equilibria. The two concepts are not equivalent. However, we have the following inclusions:

$$\hat{N}(\epsilon) \subseteq N(2\epsilon) \text{ and } N(\epsilon) \subseteq \hat{N}(\epsilon) \quad \text{for any } \epsilon \geq 0.$$

Through the paper we will consider ϵ-approximate Nash equilibria and a first result concerning strict approximate solutions is the following:

Proposition 1.6. *Assume the following conditions:*

(A1) *for any $(y_1, y_2) \in Y_1 \times Y_2$ there exists a sequence $(\overline{y}_{1,n}, \overline{y}_{2,n})_n$ converging to (y_1, y_2) in $Y_1 \times Y_2$ such that:*

$$\limsup_{n\to\infty}[f_{1,n}(\overline{y}_{1,n}, \overline{y}_{2,n}) + f_{2,n}(\overline{y}_{1,n}, \overline{y}_{2,n})] \leq f_1(y_1, y_2) + f_2(y_1, y_2);$$

(A2) *for any $(y_1, y_2) \in Y_1 \times Y_2$ and any sequence $(y_{1,n}, y_{2,n})_n$ converging to (y_1, y_2) in $Y_1 \times Y_2$ we have:*

$$v_1(f_1, y_2) + v_2(f_2, y_1) \leq \liminf_{n\to\infty}[v_1(f_{1,n}, y_{2,n}) + v_2(f_{2,n}, y_{1,n})].$$

Then, for all $\epsilon > 0$, $\tilde{N}(\epsilon) \subseteq \operatorname{Lim\,inf}_{n\to\infty} \tilde{N}_n(\epsilon)$.

Proof. Let $(y_1, y_2) \in \tilde{N}(\epsilon)$. In light of (A1), there exists a sequence $(\overline{y}_{1,n}, \overline{y}_{2,n})_n$ converging to (y_1, y_2) in $Y_1 \times Y_2$ such that:

$$\limsup_{n\to\infty}[f_{1,n}(\overline{y}_{1,n}, \overline{y}_{2,n}) + f_{2,n}(\overline{y}_{1,n}, \overline{y}_{2,n})] \leq f_1(y_1, y_2) + f_2(y_1, y_2) \tag{1.4}$$

In light of (A2), we have:

$$v_1(f_1, y_2) + v_2(f_2, y_1) \leq \liminf_{n\to\infty}[v_1(f_{1,n}, \overline{y}_{2,n}) + v_2(f_{2,n}, \overline{y}_{1,n})] \tag{1.5}$$

and, in light of (1.4), (1.5) and the fact that $(y_1, y_2) \in \tilde{N}(\epsilon)$, we obtain the result. □

Concerning ϵ-approximate solutions we have the following theorem:

Theorem 1.7. *Let $\epsilon > 0$ and assume the following conditions:* (A1), (A2) *and*

(A3) *the function $f_1 + f_2$ is convex;*
(A4) *the function $v_1(f_1, .) + v_2(f_2, .)$ is concave;*
(A5) $\tilde{N}(\epsilon) \neq \emptyset$;
(A6) *let V_1 (respectively V_2) be a first countable topological and real vector space and Y_1 (respectively Y_2) be a convex subset of V_1 (respectively V_2).*

Then $N(\epsilon) \subseteq \mathrm{Lim\ inf}_{n\to\infty} N_n(\epsilon)$.

Proof. Let $(\overline{y}_1, \overline{y}_2) \in N(\epsilon)$ and $(\tilde{y}_1, \tilde{y}_2) \in \tilde{N}(\epsilon)$. We define: $\overline{y}_{i,n} = \frac{1}{n}\tilde{y}_i + (1 - \frac{1}{n})\overline{y}_i$ for $i = 1, 2$ and $n \in \mathbb{N}$. Obviously the sequence $(\overline{y}_{1,n}, \overline{y}_{2,n})$ converges to $(\overline{y}_1, \overline{y}_2)$ and in light of (A3) and (A4) we have:

$$\begin{aligned}
&f_1(\overline{y}_{1,n}, \overline{y}_{2,n}) + f_2(\overline{y}_{1,n}, \overline{y}_{2,n}) \\
&\leq \frac{1}{n}[f_1(\tilde{y}_1, \tilde{y}_2) + f_2(\tilde{y}_1, \tilde{y}_2)] + (1 - \frac{1}{n})[f_1(\overline{y}_1, \overline{y}_2) + f_2(\overline{y}_1, \overline{y}_2)] \\
&< \frac{1}{n}[v_1(f_1, \tilde{y}_2) + v_2(f_2, \tilde{y}_1) + \epsilon] + (1 - \frac{1}{n})[v_1(f_1, \overline{y}_2) + v_2(f_2, \overline{y}_1) + \epsilon] \\
&\leq v_1(f_1, \overline{y}_{2,n}) + v_2(f_2, \overline{y}_{1,n}) + \epsilon.
\end{aligned}$$

Therefore, we proved that $N(\epsilon) \subseteq \mathrm{cl}\, \tilde{N}(\epsilon)$ where $\mathrm{cl}\, A$ is the closure of the set A. Then, in light of Proposition 1.6, we have: $N(\epsilon) \subseteq \mathrm{cl}\, \tilde{N}(\epsilon) \subseteq \mathrm{cl\, Lim\ inf}_{n\to\infty} \tilde{N}_n(\epsilon)$. Since Y_1 and Y_2 are first countable topological spaces, $\mathrm{Lim\ inf}_{n\to\infty} \tilde{N}_n(\epsilon)$ is a sequentially closed subset in $V_1 \times V_2$ and $N(\epsilon) \subseteq \mathrm{Lim\ inf}_{n\to\infty} \tilde{N}_n(\epsilon) \subseteq \mathrm{Lim\ inf}_{n\to\infty} N_n(\epsilon)$. □

Sufficient conditions on the data of the problem for assumptions (A2) and (A4) can be easily obtained:

Corollary 1.8. *Let $\epsilon > 0$. Assume that Y_1 and Y_2 are two sequentially compact subsets and that the following assumptions hold:* (A1), (A3), (A5), (A6) *and*

(A2') *for any $(y_1, y_2) \in Y_1 \times Y_2$ and any sequence $(y_{1,n}, y_{2,n})_n$ converging to (y_1, y_2) in $Y_1 \times Y_2$ we have:*

$$f_1(y_1, y_2) \leq \liminf_{n\to\infty} f_{1,n}(y_{1,n}, y_{2,n}) \text{ and } f_2(y_1, y_2) \leq \liminf_{n\to\infty} f_{2,n}(y_{1,n}, y_{2,n});$$

(A4') *the functions $f_1(y_1, .)$ and $f_2(., y_2)$ are concave, respectively, on Y_2 and Y_1.*

Then $N(\epsilon) \subseteq \mathrm{Lim\ inf}_{n\to\infty} N_n(\epsilon)$.

In the following example, conditions (A1) and (A2) are satisfied but condition (A2') is not.

Example 1.9. Let $Y_1 = Y_2 = [0,1]$, $f_1(y_1,y_2) = -(y_1+y_2)$, $f_2(y_1,y_2) = y_1+y_2$ and for $n>1$

$$f_{1,n}(y_1,y_2) = \begin{cases} 1 & \text{if } (y_1,y_2) \in \{\frac{1}{n}\} \times [\frac{1}{n}, 1-\frac{1}{n}] \cup [\frac{1}{n}, 1-\frac{1}{n}] \times \{\frac{1}{n}\} \\ 0 & \text{otherwise} \end{cases}$$

$$f_{2,n}(y_1,y_2) = \begin{cases} 1 & \text{if } (y_1,y_2) \in \{1-\frac{1}{n}\} \times]\frac{1}{n}, 1-\frac{1}{n}] \cup]\frac{1}{n}, 1-\frac{1}{n}] \times \{1-\frac{1}{n}\} \\ 0 & \text{otherwise} \end{cases}$$

Remark 1.10. It is easy to determine functions f_1 and f_2 which are not linear with respect to y_1 and to y_2 but such that conditions (A3) and (A4') are satisfied. For example $f_1(y_1,y_2) = {y_1}^2 - y_1 y_2$ and $f_2(y_1,y_2) = {y_2}^2 + 2y_1y_2$.

However, if f_1 and f_2 are supposed to be convex it can be proved that, under condition (A4'), f_1 and f_2 cannot be quadratic functions.

Remark 1.11. Clearly assumption (A5) is satisfied for all $\epsilon > 0$ if $N \neq \emptyset$. Note that recent existence results for Nash equilibrium, in topological vector spaces and in reflexive Banach spaces, have been obtained in [22, 5, 11, 12, 23, 24].

Remark 1.12. Even if the different statements on ϵ-approximate and strict ϵ-approximate solutions could be surprising, assumptions (A3) and (A4) are crucial for lower semicontinuity property of the ϵ-approximate solutions, as shown by the following example:

Let $Y_1 = Y_2 = [0,1]$, $f_2(y_1,y_2) = f_{2,n}(y_1,y_2) = 0$, for all $n \in \mathbb{N}$,

$$f_1(y_1,y_2) = \begin{cases} \frac{1}{k} & \text{if } y_1 \in]\frac{1}{k+1}; \frac{1}{k}] \\ 0 & \text{if } y_1 = 0 \end{cases}$$

$$f_{1,n}(y_1,y_2) = \begin{cases} \frac{1}{k} + \frac{1}{n}(\frac{1}{k} - \frac{1}{k+1}) & \text{if } y_1 \in]\frac{1}{k+1}; \frac{1}{k}] \\ 0 & \text{if } y_1 = 0 \end{cases}$$

Assumptions (A1),(A2) and (A4) to (A6) are satisfied but assumption (A3) is not. In fact, let $\epsilon = \frac{1}{k}$ with $k \in \mathbb{N}$. Then, $N(\frac{1}{k}) = [0, \frac{1}{k}] \times Y_2$, $N_n(\frac{1}{k}) = [0, \frac{1}{k+1}] \times Y_2$, for all $n \in \mathbb{N}$ and condition (1.3) is not satisfied even if condition $\tilde{N}(\epsilon) \subseteq \text{Lim inf}_{n\to\infty} \tilde{N}_n(\epsilon)$ is satisfied.

Proposition 1.13. *Let $\epsilon \geq 0$ and assume:*

(A7) *for any $(y_1,y_2) \in Y_1 \times Y_2$ and for any sequence $(y_{1,n}, y_{2,n})_n$ converging to (y_1,y_2) in $Y_1 \times Y_2$ we have:*

$$\liminf_{n\to\infty} [f_{1,n}(y_{1,n}, y_{2,n}) + f_{2.n}(y_{1,n}, y_{2,n})] \geq f_1(y_1,y_2) + f_2(y_1,y_2);$$

(A8) *for any $(y_1,y_2) \in Y_1 \times Y_2$ and for any sequence $(y_{1,n}, y_{2,n})_n$ converging to (y_1,y_2) in $Y_1 \times Y_2$ we have:*

$$\limsup_{n\to\infty} [v_1(f_{1,n}, y_{2,n}) + v_2(f_{2,n}, y_{1,n})] \leq v_1(f_1,y_2) + v_2(f_2,y_1).$$

Then, $\text{Lim sup}_{n\to\infty} N_n(\epsilon) \subseteq N(\epsilon)$ *and* $\text{Lim sup}_{n\to\infty} N_n(\epsilon_n) \subseteq N$ *for any sequence* (ϵ_n) *converging to* 0.

Proof. Let (y_1, y_2) such that there exists a sequence $(y_{1,k}, y_{2,k})_k$ converging to (y_1, y_2) with $(y_{1,k}, y_{2,k}) \in N_{n_k}(\epsilon)$ for a sequence (n_k) of integers. Therefore:

$$\liminf_{k\to\infty}[f_{1,n_k}(y_{1,k}, y_{2,k}) + f_{2,n_k}(y_{1,k}, y_{2,k})] \leq \epsilon + \limsup_{k\to\infty}[v_1(f_{1,n_k}, y_{2,k}) + v_2(f_{2,n_k} y_{1,k})]$$

Moreover, in light of (A7) and (A8), we deduce:

$$\begin{aligned} f_1(y_1, y_2) + f_2(y_1, y_2) &\leq \liminf_{k\to\infty}[f_{1,n_k}(y_{1,k}, y_{2,k}) + f_{2,n_k}(y_{1,k}, y_{2,k})] \\ &\leq \epsilon + \limsup_{k\to\infty}[v_1(f_{1,n_k}, y_{2,k}) + v_2(f_{2,n_k}, y_{1,k})] \\ &\leq \epsilon + v_1(f_1, y_2) + v_2(f_2, y_1) \end{aligned}$$

and the first result follows.

Similarly, we can prove the second part of the thesis. □

Remark 1.14. In [6] the following result is obtained: Assume that the following assumptions are satisfied:

(i) for any $(y_1, y_2) \in Y_1 \times Y_2$ and any sequence $(y_{1,n}, y_{2,n})_n$ converging to (y_1, y_2) in $Y_1 \times Y_2$ we have:

$$\liminf_{n\to\infty} f_{1,n}(y_{1,n}, y_{2,n}) \geq f_1(y_1, y_2) \quad \text{and} \quad \liminf_{n\to\infty} f_{2,n}(y_{1,n}, y_{2,n}) \geq f_2(y_1, y_2);$$

(ii) for any $(y_1, y_2) \in Y_1 \times Y_2$ and any sequence $(y_{2,n})_n$ converging to y_2 in Y_2, there exists a sequence $(\overline{y}_{1,n})_n$ converging to y_1 in Y_1 such that

$$\limsup_{n\to\infty} f_{1,n}(\overline{y}_{1,n}, y_{2,n}) \leq f_1(y_1, y_2);$$

for any $(y_1, y_2) \in Y_1 \times Y_2$, any sequence $(y_{1,n})_n$ converging to y_1 in Y_1, there exists a sequence $(\overline{y}_{2,n})_n$ converging to y_2 in Y_2 such that

$$\limsup_{n\to\infty} f_{2,n}(y_{1,n}, \overline{y}_{2,n}) \leq f_2(y_1, y_2).$$

Then, $\operatorname*{Lim\,sup}_{n\to\infty} N_n \subseteq N$.

We note that assumption (A7) is strictly weaker than condition (i) and that (A8) is strictly weaker than condition (ii). In fact, in light of Proposition 3.1.1 in [14], condition (ii) is equivalent to: for all $(y_1, y_2) \in Y_1 \times Y_2$ and for all sequence $(y_{1,n}, y_{2,n})_n$ converging to (y_1, y_2) in $Y_1 \times Y_2$ we have:

$$\limsup_{n\to\infty} v_1(f_{1,n}, y_{2,n}) \leq v_1(f_1, y_2)$$

and

$$\limsup_{n\to\infty} v_2(f_{2,n}, y_{1,n}) \leq v_2(f_2, y_1).$$

As an example satisfying (A7) and (A8) but not (ii), we can consider: $Y_1 = Y_2 = [0, 1]$, $f_i(y_1, y_2) = 0$ and

$$f_{i,n}(y_1, y_2) = \begin{cases} \frac{y_i}{n y_{3-i}} & \text{if } y_{3-i} \in \,]0, 1] \\ 0 & \text{if } y_{3-i} = 0 \end{cases}$$

for any $n \in \mathbb{N}$ and $i = 1, 2$.

In light of Corollary 1.8 and Proposition 1.13, we easily obtain:

Corollary 1.15. *Let $\epsilon > 0$ and assume:* (A3), (A4'), (A5), (A6) *and*

(A9) *for $i = 1, 2$, the sequence $(f_{i,n})_n$ is continuously convergent to f_i [19] that is: for any $(y_1, y_2) \in Y_1 \times Y_2$ and for any sequence $(y_{1,n}, y_{2,n})$ converging to (y_1, y_2) in $Y_1 \times Y_2$ we have: $\lim_{n\to\infty} f_{i,n}(y_{1,n}, y_{2,n}) = f_i(y_1, y_2)$.*

Then the sequence $(N_n(\epsilon))_n$ converges to $N(\epsilon)$ in the Painlevé-Kuratowski sense [9] *that is: $\operatorname*{Lim\,sup}_{n\to\infty} N_n(\epsilon) = \operatorname*{Lim\,inf}_{n\to\infty} N_n(\epsilon) = N(\epsilon)$.*

2. An application

Let X be a topological sequentially compact space and l, f_1, f_2 be three real valued functions defined on $X \times Y_1 \times Y_2$ which rappresent the cost functions of a leader and two followers in a non cooperative and non zero sum game. We consider the following problem:

$$\text{find } \overline{x} \in X \text{ such that: } \inf_{x\in X} \sup_{(y_1,y_2)\in N(x)} l(x, y_1, y_2) = \sup_{(y_1,y_2)\in N(\overline{x})} l(\overline{x}, y_1, y_2) \qquad (\mathcal{P})$$

where $N(x)$ is the set of solutions to the Nash equilibrium problem $\mathcal{N}(f_1(x,.,.), f_2(x,.,.))$ and is called the reaction set of the followers.

Such a problem, called "weak hierarchical Nash equilibrium problem", corresponds to the case in which the leader cannot influence the followers and minimizes the worst. Existence of a solution is not always guaranteed, even for nice functions, as shown by the following example:

Let $X = Y_1 = Y_2 = [0, 1]$ and $l(x, y_1, y_2) = x - (y_1 + y_2)$, $f_1(x, y_1, y_2) = -f_2(x, y_1, y_2) = -y_1(x + y_2)$.

$$\text{The set } N(x) \text{ is: } \begin{cases} [0,1] \times \{0\} & \text{if } x = 0 \\ \{(1,0)\} & \text{if } x \neq 0 \end{cases}$$

and there exists no solution to the problem $(\mathcal{P})$.

Therefore, for any $\epsilon > 0$ we introduce the following regularized problem:

$$\text{find } \overline{x} \in X \text{ such that: } \inf_{x\in X} \sup_{(y_1,y_2)\in N(x,\epsilon)} l(x, y_1, y_2) = \sup_{(y_1,y_2)\in N(\overline{x},\epsilon)} l(\overline{x}, y_1, y_2) \qquad (\mathcal{P}(\epsilon))$$

where $N(x, \epsilon)$ is the set of *ϵ-approximate Nash equilibria*, that is the set of solutions to the problem:

$$\mathcal{N}(f_1(x,.,.), f_2(x,.,.))(\epsilon) \begin{cases} \text{find } (\overline{y}_1, \overline{y}_2) \in Y_1 \times Y_2 \text{ such that:} \\ f_1(x, \overline{y}_1, \overline{y}_2) + f_2(x, \overline{y}_1, \overline{y}_2) \leq \\ v_1(f_1(x,.,.), \overline{y}_2) + v_2(f_2(x,.,.), \overline{y}_1) + \epsilon \end{cases}$$

Moreover, let $\tilde{N}(x, \epsilon)$ be the set of solutions to the problem:

$$\begin{cases} \text{find } (\overline{y}_1, \overline{y}_2) \in Y_1 \times Y_2 \text{ such that :} \\ f_1(x, \overline{y}_1, \overline{y}_2) + f_2(x, \overline{y}_1, \overline{y}_2) < \inf_{y_1\in Y_1} f_1(x, y_1, \overline{y}_2) + \inf_{y_2\in Y_2} f_2(x, \overline{y}_1, y_2) + \epsilon \end{cases}$$

For the sake of simplicity we put:

$$u_1(x,y_2) = \inf_{y_1\in Y_1} f_1(x,y_1,y_2), \quad u_2(x,y_1) = \inf_{y_2\in Y_2} f_2(x,y_1,y_2).$$

Then, we can obtain the following result:

Proposition 2.1. *Let $\epsilon > 0$. Assume that:*

Y_1 and Y_2 satisfy assumption (A6);
The function l is sequentially lower semicontinuous on $X \times Y_1 \times Y_2$;
For any $(x, y_1, y_2) \in Y_1 \times Y_2$ and any sequence (x_n) converging to x, there exists a sequence $(\overline{y}_{1,n}, \overline{y}_{2,n})_n$ converging to (y_1, y_2) in $Y_1 \times Y_2$ such that:

$$\limsup_{n\to\infty}[f_{1,n}(x_n,\overline{y}_{1,n},\overline{y}_{2,n}) + f_{2,n}(x_n,\overline{y}_{1,n},\overline{y}_{2,n})] \le f_1(x,y_1,y_2) + f_2(x,y_1,y_2);$$

The function $u_1 + u_2$ is sequentially lower semicontinuous on $X \times Y_1 \times Y_2$;
The function $f_1(x,.,.) + f_2(x,.,.)$ is convex on $Y_1 \times Y_2$, for all $x \in X$;
The function $u_1(x,.) + u_2(x,.)$ is concave on $Y_1 \times Y_2$ for all $x \in X$;
$\tilde{N}(x,\epsilon) \neq \emptyset$.

Then there exists a solution to the problem $(\mathcal{P}(\epsilon))$.

Proof. In light of Theorem 1.7, the multifunction N is sequentially lower semicontinuous on X and, in light of Proposition 3.2.1 in [14], the marginal function w defined by $w(x,\epsilon) = \sup_{(y_1,y_2)\in N(x,\epsilon)} l(x,y_1,y_2)$ is sequentially lower semicontinuous. □

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